Revisiting the flocking transition using Active Spins

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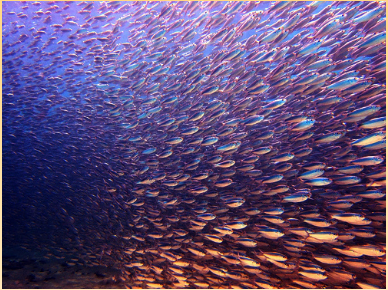
Models from Statistical Mechanics in Applied Sciences

- Energy consumption at the microscopic scale ⟶ Self-propulsion
- Aligning interactions

⟶ Collective motion (with long range-order?)

The Vicsek model [Vicsek et al. PRL 75, 1226 (1995)]

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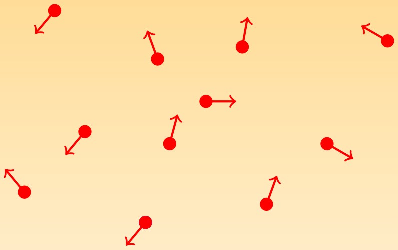

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- Local alignment rule

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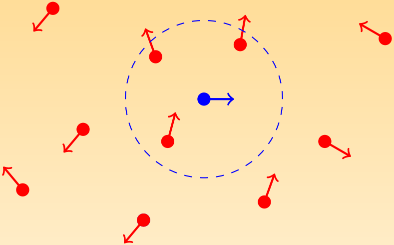

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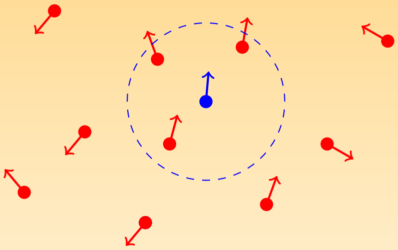

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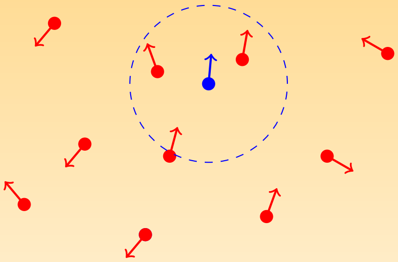

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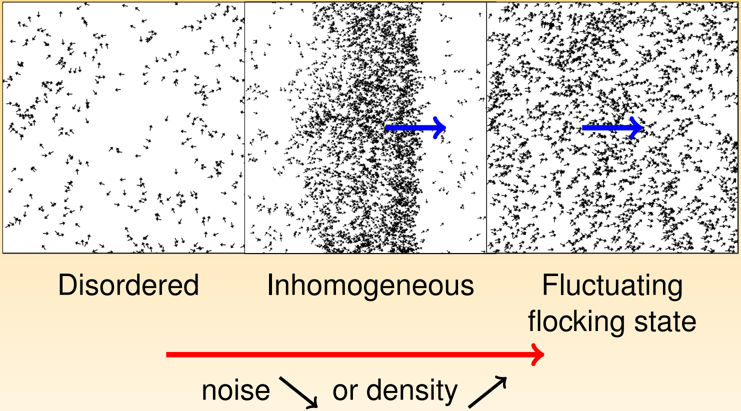
Flocking transition



- Non-equilibrium transition to long-range order in $d = 2$

A long-standing debate

- Simulations are simple but strong finite size effects
- Second-order (1995) vs First-order (2004) [Gregoire and Chate, PRL 2004]
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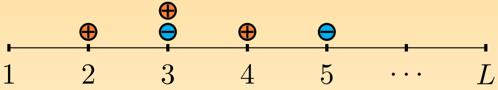
- Use a much simpler model: active spins
  ⟶ on lattice
  ⟶ discrete symmetry

Active Ising model



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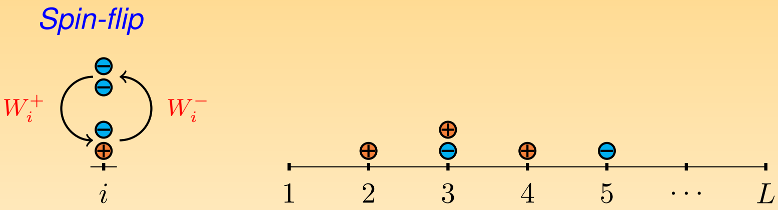

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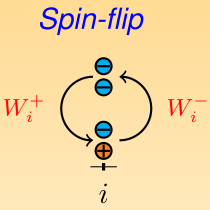
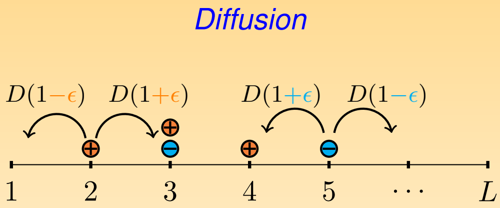
Active Ising model





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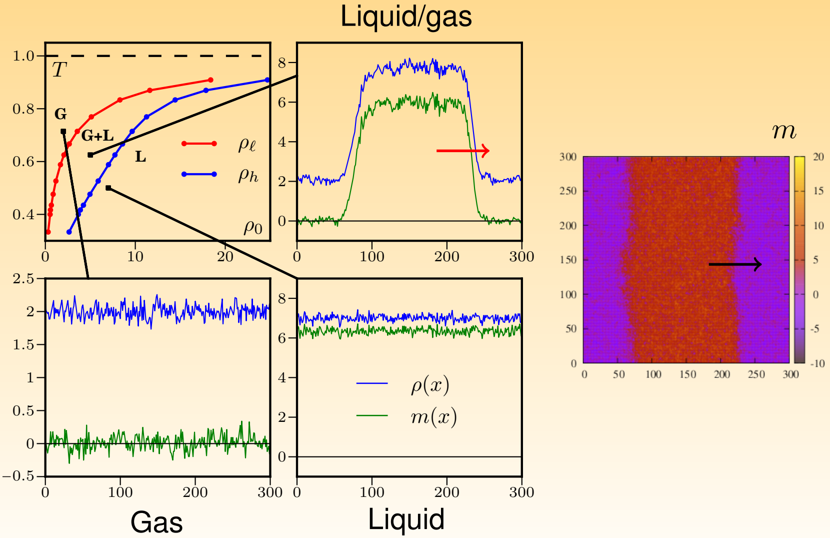
  $\Leftrightarrow$ Fully connected Ising models on each site
- Self-propulsion $\Leftrightarrow$ Diffusion biased by the spins for $\epsilon \neq 0$

Phase diagram in 2d •



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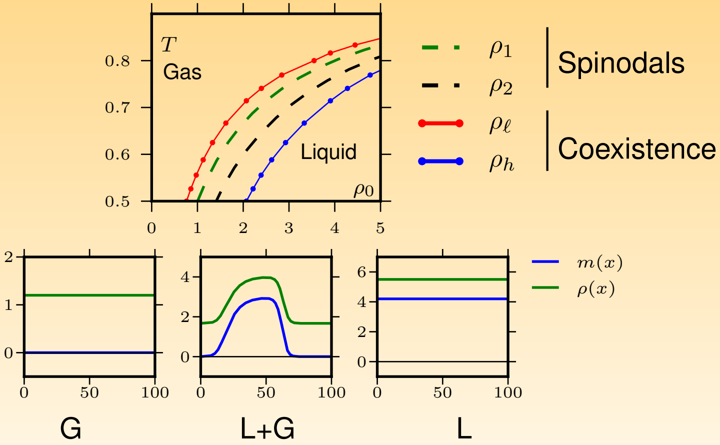
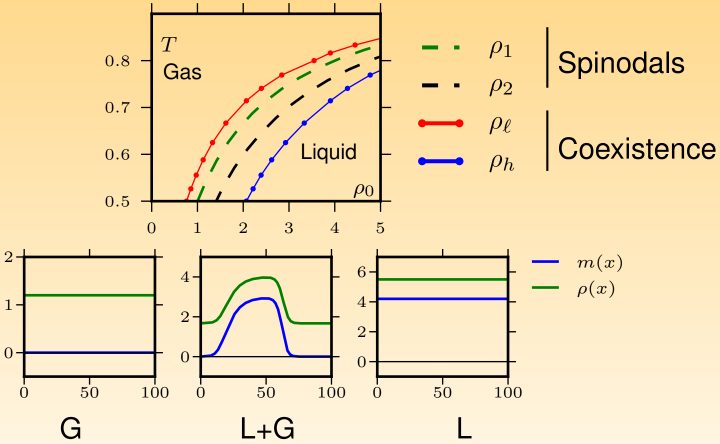
- MF only valid at $\rho = \infty \longrightarrow$ Refined-Mean-Field-Model (RMFM)

Simulations of the RMFM



Simulations of the RMFM



- Same phenomenology as microscopic model
- $\rho_1(\beta) < \rho_2(\beta) \longrightarrow$ Always observe phase-separated profiles

Phase-separated profiles

- Propagating shocks between $\rho_\ell, m_\ell = 0$ and $\rho_h, m_h \neq 0$
- Stationary solutions in comoving frame of velocity c

$$D\rho'' + c\rho' - vm' = 0 \tag{1}$$

$$Dm'' + cm' - v\rho' + 2(\beta - 1 - \frac{r}{\rho}) - \alpha\frac{m^3}{\rho^2} = 0 \tag{2}$$

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- Solvable at large densities $\rho_1 = \frac{r}{\beta-1} \gg r$
  1: Solve (1) to get $\rho = \rho_\ell + \frac{v}{c}\sum_{k=0}^{\infty}\left(-\frac{D}{c}\nabla\right)^k m$
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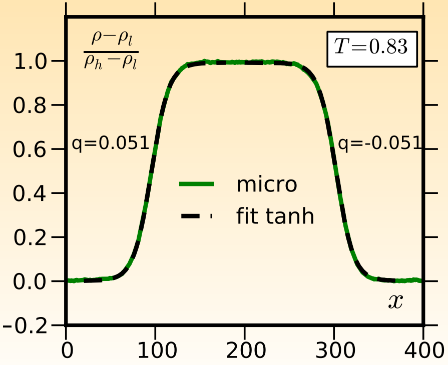
$$D(1+\frac{v^2}{c^2})m'' + [c - \frac{v^2}{c} - \frac{2Dvr}{c^2\rho_1^2}m]m' - \frac{2r(\rho_1-\rho_\ell)}{\rho_1^2}m + \frac{2rv}{c\rho_1^2}m^2 - \alpha\frac{m^3}{\rho_1^2} = 0$$

Symmetric front solutions $\beta \simeq 1$

$$m^{\pm}(x) = \frac{m_h}{2}(\tanh(q^{\pm}(x - ct)) + 1)$$

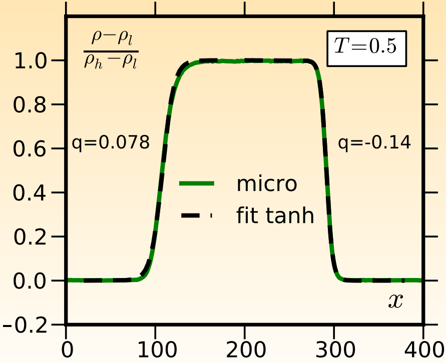
$$c = v \qquad q^{\pm} = \pm\frac{r}{3\rho_1\sqrt{\alpha D}} \qquad m_h = \frac{4r}{3\alpha}$$



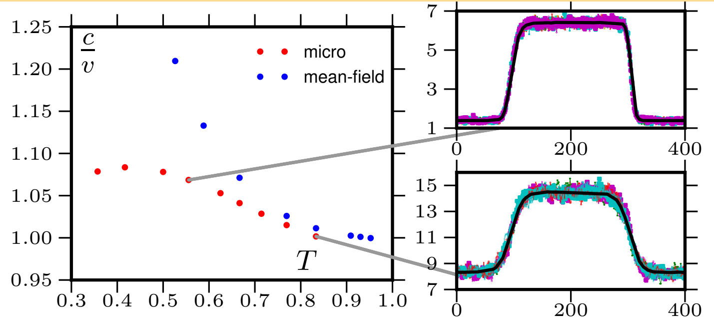
Asymmetric front solutions $\beta > 1$

$$m^{\pm}(x) = \frac{m_h}{2}(\tanh(q^{\pm}(x - ct)) + 1)$$

$$c = v + \frac{2Dr^2}{3v\alpha\rho_1^2} \quad q^{\pm} = \pm\frac{r}{3\rho_1\sqrt{\alpha D}} - \frac{r^2}{6\alpha v\rho_1^2} \quad m_h = \frac{4r}{3\alpha} - \frac{8Dr^3}{9v^2\alpha^2\rho_1^2}$$



The flock fly faster than the birds



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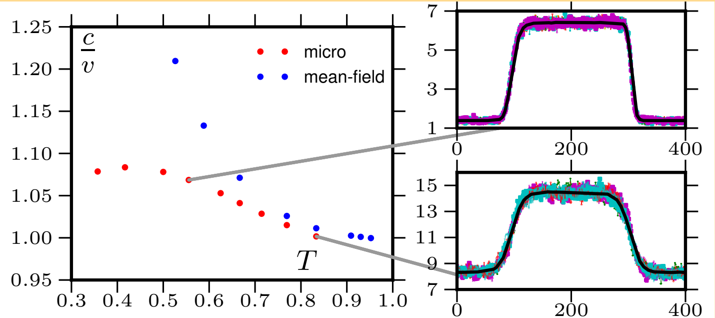
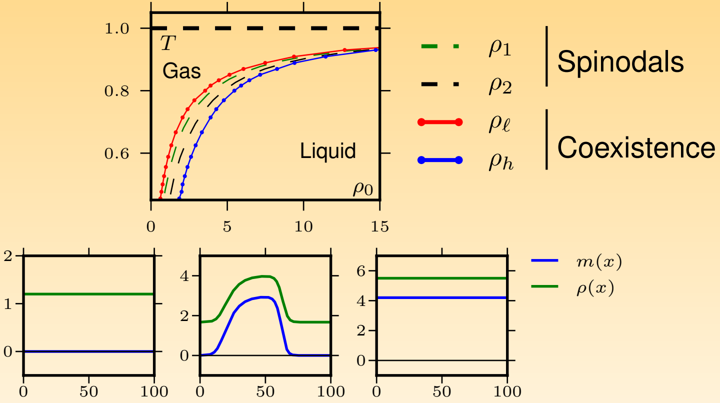


- $c = v + \frac{2Dr^2}{3v\alpha\rho_1^2}$
- $v \longrightarrow$ microscopic velocities
- $\frac{2Dr^2}{3v\alpha\rho_1^2} \longrightarrow$ FKPP-like contribution

A liquid-gas transition in the canonical ensemble



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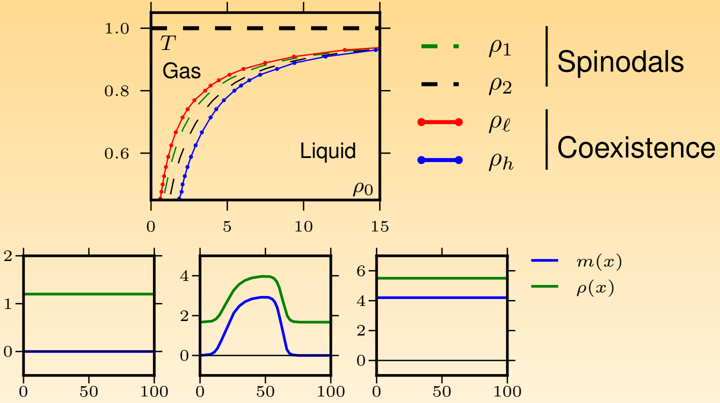
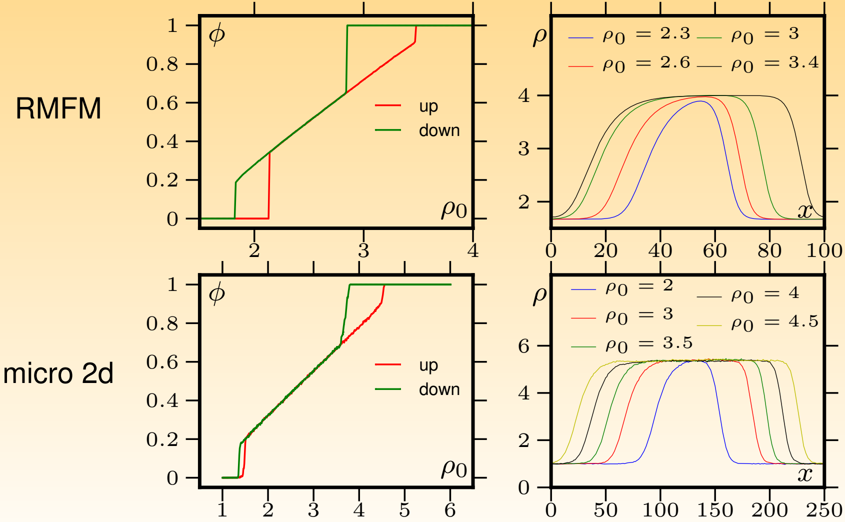


- No length selection mechanism → Phase-separation
- Gas and liquid have different symmetries
- First-order liquid-gas transition with $\rho_c = \infty$

Hysteresis loops



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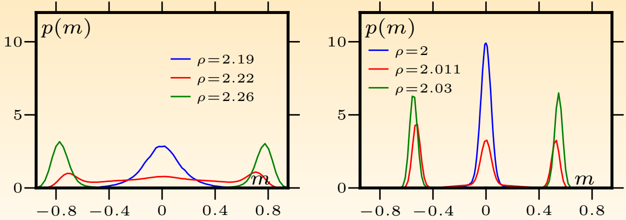
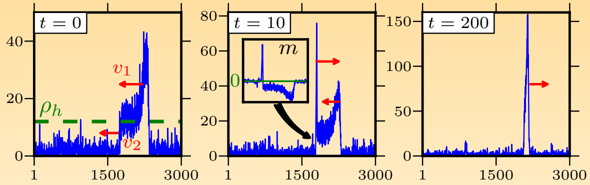
- Magnetization $\propto$ liquid fraction $\longrightarrow$ Continuous when $L = \infty$
- Finite size of interfaces $\longrightarrow$ Discontinuous jump of m
- Width and mean of the peaks go to 0 as $L \to \infty$



$L = 80$ $\qquad$ $L = 150$

Flocking in 1d: Role of fluctuations

- In 1D, single-site fluctuations can flip a domain ⟶ Reversals •



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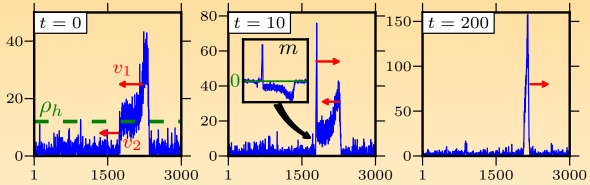

- In 2D: fluctuations of a complete line of length $\propto L$

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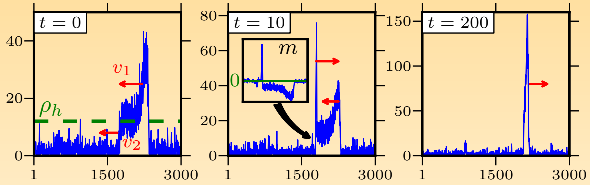

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  $\longrightarrow$ Vanishing probability as $L \to \infty$

- Similar to other 1D flocking models [Czirók *et al.*1999, O'loan& Evans 1999]

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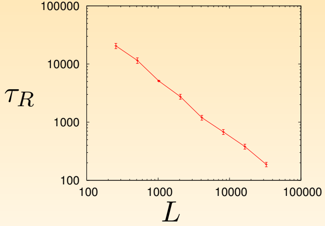

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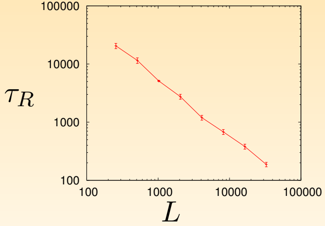

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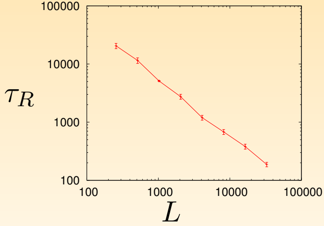

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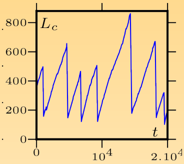
Only two phases in the thermodynamic limit

Dynamics in the low temperature phase •



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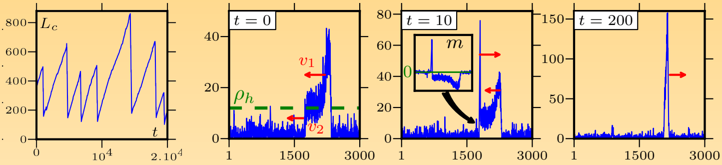

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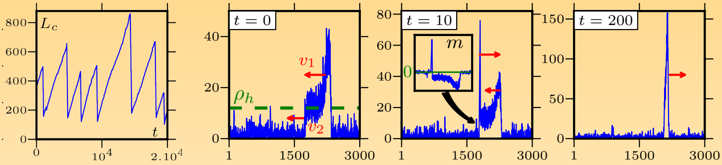

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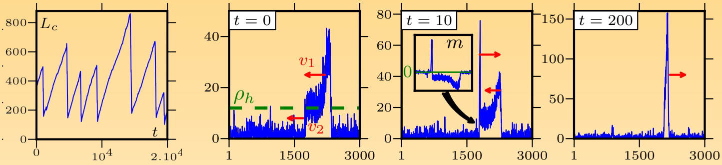

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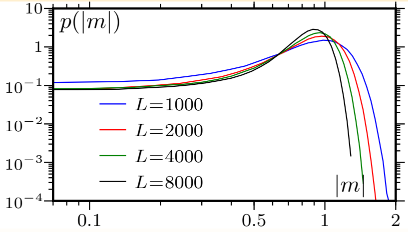

→ No ergodicity breaking

A Nightmarish FSS

- $\rho < \rho_\ell$: disordered gas
- $\rho > \rho_1$: alternating clusters
- $\rho_\ell < \rho < \rho_1$: phase coexistence
- Binder cumulant $G(\beta) = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}$

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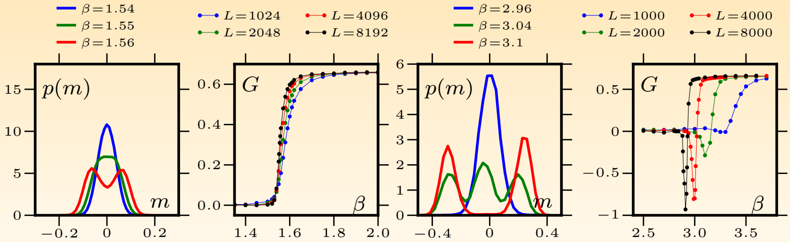

$D = 1; \rho = 3$ $D = 10; \rho = 0.5$

Very difficult to analyze!

$G(\beta) = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}$ across a transition

- Simplest case of phase coexistence:

$$P(m,\beta) \propto \alpha(\beta)\, e^{-\frac{(m+m_0)^2}{2\sigma^2}} + (1-2\alpha(\beta))\, e^{-\frac{m^2}{2\sigma^2}} + \alpha(\beta)\, e^{-\frac{(m-m_0)^2}{2\sigma^2}}$$

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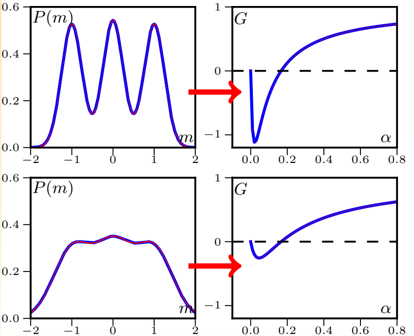

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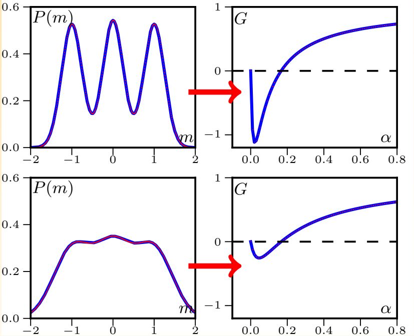
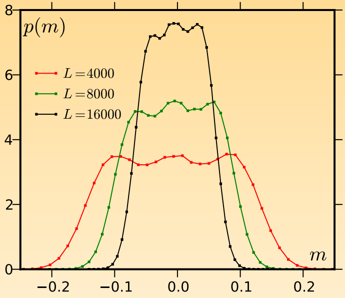
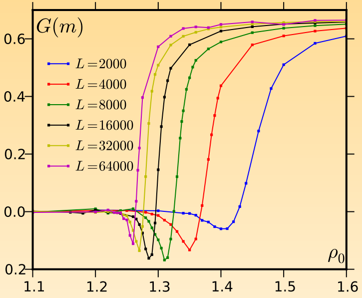
$m_0 = 1$
$\sigma = 0.45$



- No negative peak ⟶̸ 2nd order transition

Merging of the peaks as L increases (D=10, eps=0.9, beta=1.7)





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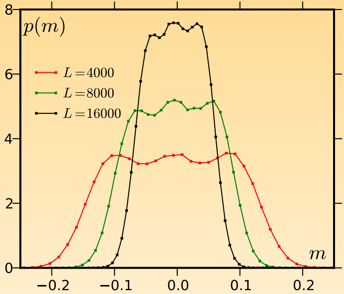

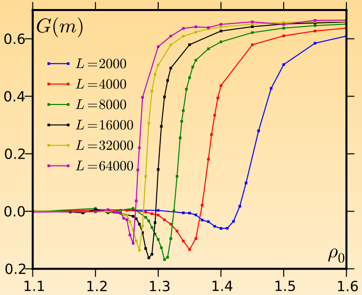

- $L < L_c$ critical nucleus $\longrightarrow$ continuous
- $L \gtrsim L_c$ $\longrightarrow$ discontinuous
- $L \gg L_c$ $\longrightarrow$ continuous

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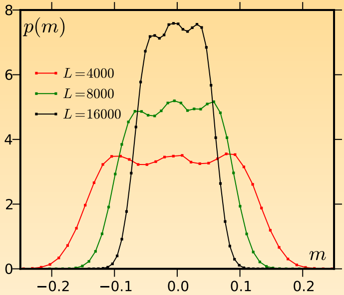

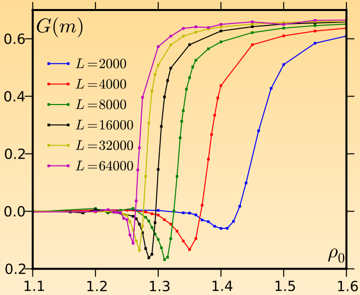

- $L < L_c$ critical nucleus $\longrightarrow$ continuous
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$\longrightarrow$ Grand canonical ensemble would be much simpler!

Active spins – summary

- New flocking model using active Ising spins
- Flocking trans. $\longrightarrow$ Liquid-gas transition in canonical ensemble
- Symmetry of the liquid phase $\longrightarrow$ $\rho_c = \infty$
- No spontaneous symmetry breaking in 1d
- FSS very difficult in the canonical ensemble

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Perspectives:

- Is all this generic?

More general flocking models

- Different symmetries for liquid and gas
  ⟶ Critical point at $\rho_c = \infty$

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  $\longrightarrow$ Critical point at $\rho_c = \infty$
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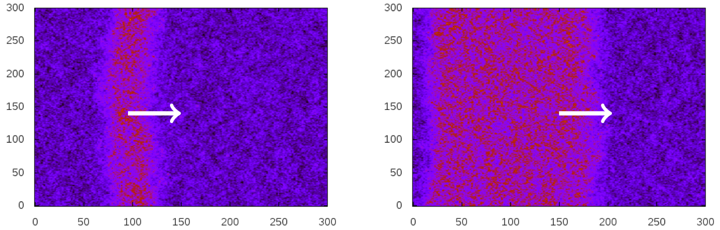
Ising interaction



$\rho_1 < \rho_2$

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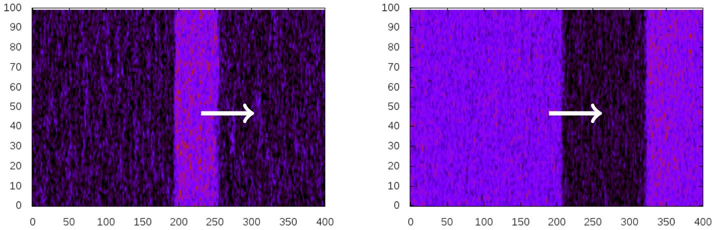
Off-lattice Ising interaction



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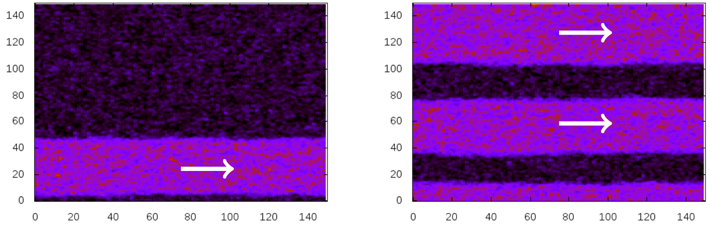
4-color Potts interaction



$\rho_1 \quad < \quad \rho_2$

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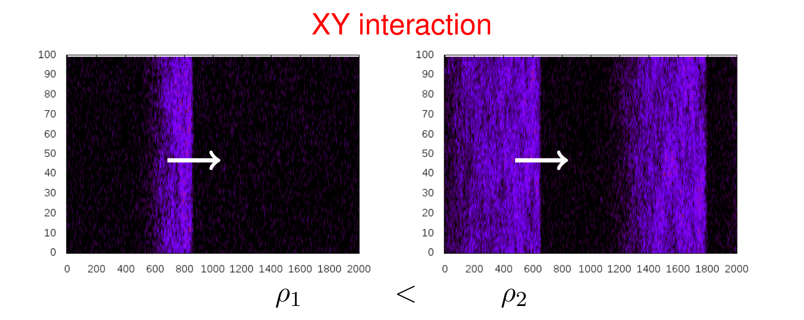

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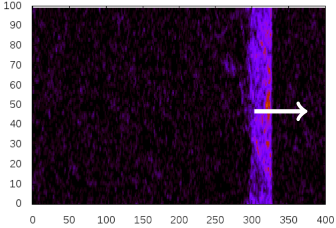
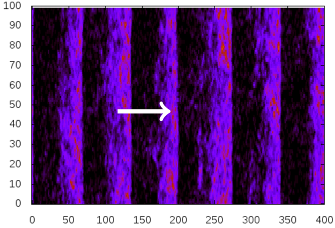
Vicsek model





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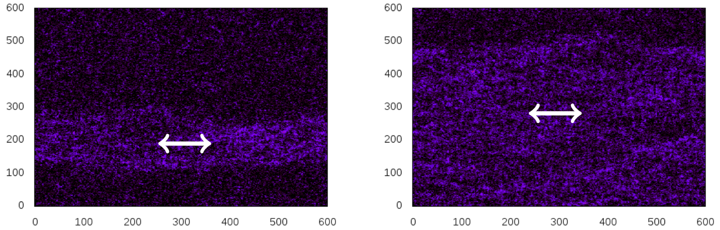
Nematic Vicsek model



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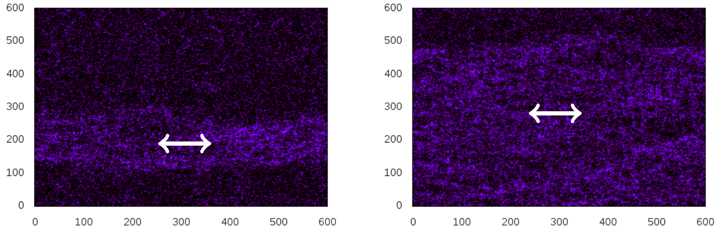

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A.Solon, J. Tailleur, Phys. Rev. Lett. 111, 078101 (2013)