S_n = permutation group

C = conjugacy class in S_n:

$\forall x \in C, \forall g \in S_n,$

$g x g^{-1} \in C$

Let $X_t^0 = \ldots \gamma_{N_t}$

where γ_i ... unif on C

N_t an ...

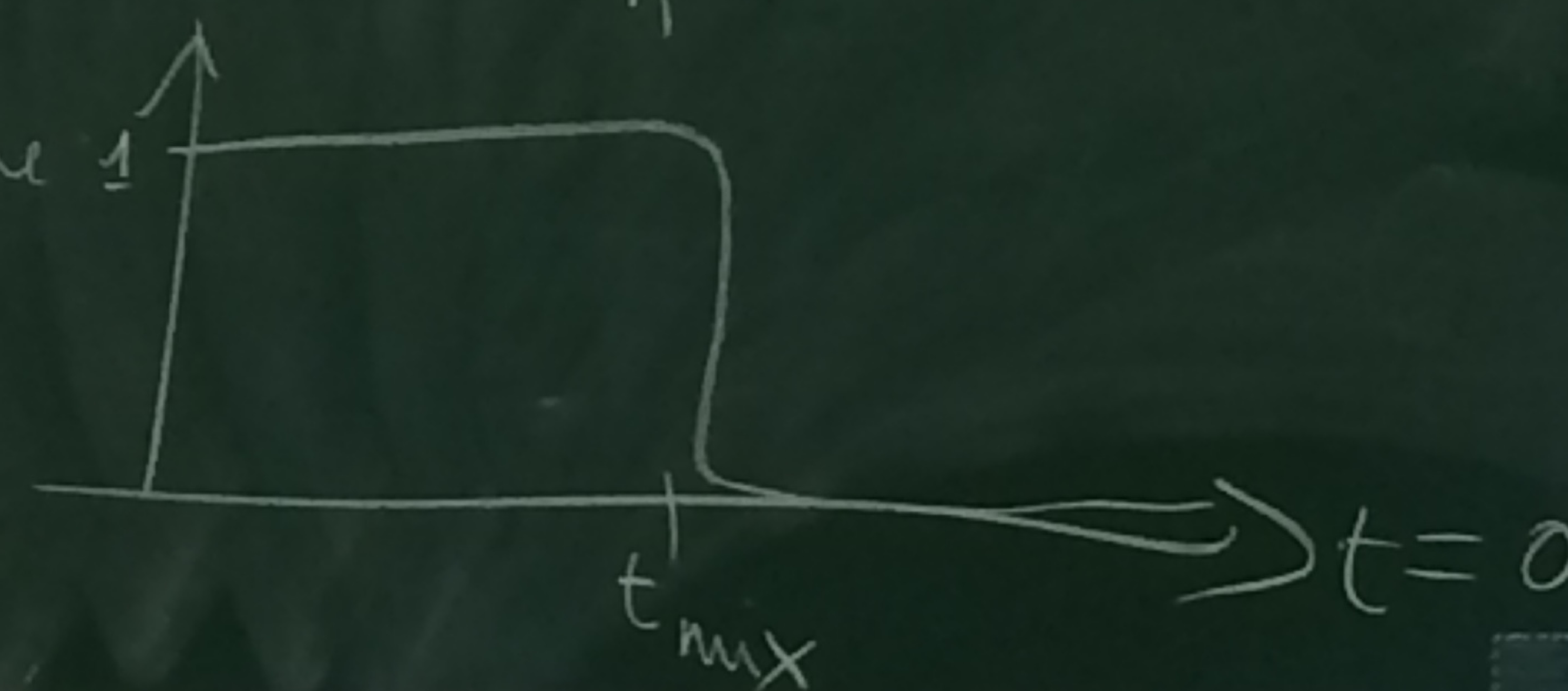
<u>Q.</u> What is the mixing time of this random walk?

Let π be invariant distr. of X

(π unifom on S_n or A_n)

$$d(t) = \| X_t^0 - \pi \|_{TV} = \sup_{A \subset S_n} | \mathbb{P}(X_t^0 \in A) - \pi(A) |$$

Picture 1



Old question

Diaconis - Shahshahani 1981:

$C = \{\text{transpositions}\}$

Theorem. Cutoff phenomenon occurs and $t_{mix} \sim \frac{1}{2} n \log n$. as $n \to \infty$

Why might this be true?

Before $\frac{1}{2} n \log n$, the # fixed points is not right

---

Here, let $|C|$ be the number of non-fixed points for any $g \in C$.

Guess if $\frac{|C|}{n} \longrightarrow 0$

then mixing cannot take place

before $t = \frac{n}{|C|} \log n$

Guess cutoff takes place at that time.

- Odlyzko, Flatto, Wales 1985
- Rachman 1996 made two conjecture.
- Roussel 2000 k-cycles

$k \leq 7$.

- Lulov & Pak 2002

$|C| \simeq O(n)$

- B. Schramm Zeitouni

2013

k-cycles, arbitrary k

Theorem (B. Bati Sengul
in preparation)

Conjecture true for arbitrary conjugacy class $C = S_n$ at least if $|C|$ is bounded and probably if $\frac{|C|}{n} \to 0$.

2 ingredients:

1. Coupling of B. Schramm Zeitouni (a variant of a coupling of O. Schramm)

2. geometric idea
discrete notion of Ricci curvature on S_n introduced by Y. Ollivier in 2010?

Curvature

Suppose X_n a Markov chain on (S, d) = metric space

Def (Y. Ollivier)

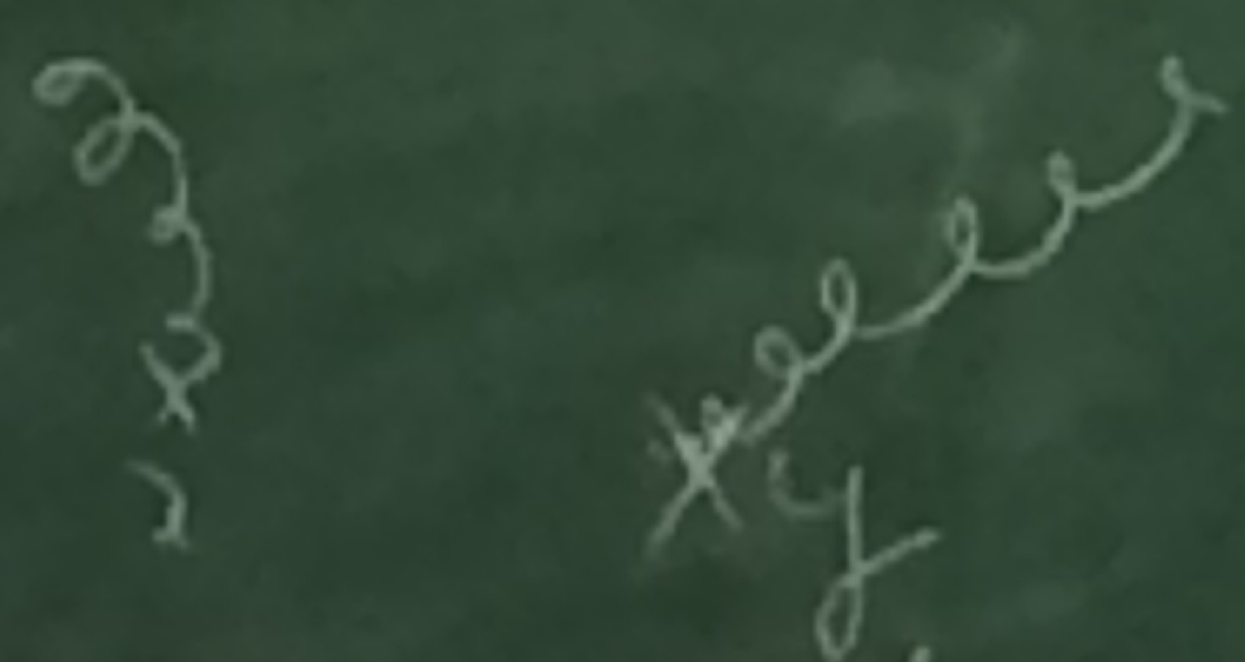
$$K(x,y) := 1 - \frac{W(X_1^x, X_1^y)}{d(x,y)}$$

where: X_n^x = the Markov chain started from x.

$W(\mu, \nu)$ = Kantorovich distance

$$= \inf_{\underbrace{X, Y}_{\text{all couplings}}} \mathbb{E}(d(X,Y)).$$

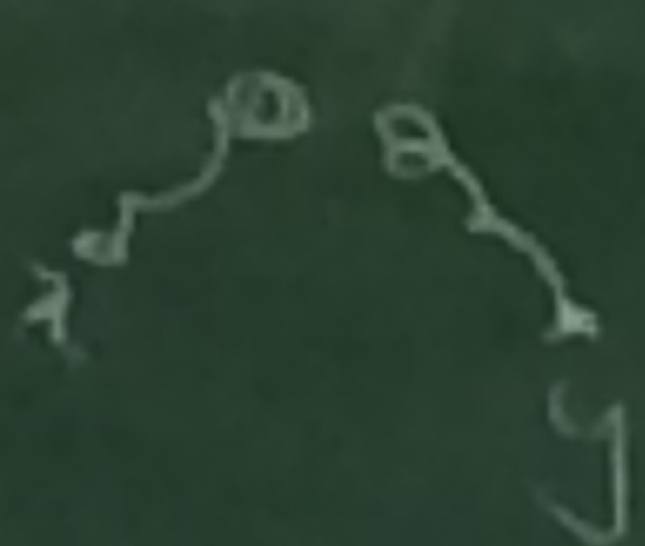
Intuition



$K<0$

$K \leq 0 \iff W(X^x_1, X^y_1) \geq d(x,y)$

$K>0 \iff$



Y Ollivier's question

what is K for random transpositions

$$K = \inf_{x \neq y} K(x,y)$$

Let $c>0$

$$K_c(x,y) = 1 - \frac{W(X^x_{cn/2}, X^y_{cn/2})}{d(x,y)}$$

$$K_c = \inf_{x\neq y} K_c(x,y)$$

Theorem 2. (B Sengul)

- if $c \leq 1$, then
$$\lim_{n\to\infty} K_c = 0$$
- if $c > 1$ then
$$\liminf_{n\to\infty} K_c \geq \Theta(c)^4$$
$$\limsup K_c \leq \Theta(c)^2$$

[for transpositions]

where $\Theta(c)$
$$= \mathbb{P}(\text{PGW}(c) \text{ survives forever});$$

Conjecture: $\lim_{n\to\infty} K_c = \Theta(c)^2$

Theorem 2 $\Rightarrow$ thm 1

in case of transpositions

write $t = \alpha \dfrac{cn}{2}$

then

$$W(X_t^x, X_t^y) \leq \underbrace{\Delta}_{\text{diam}(S)} \cdot (1-k_c)^{\alpha}$$

[contractivity]

$\Delta = n-1$

so need

$n(1-k_c)^{\alpha} \ll 1$

$\alpha(-\lg(1-k_c)) \simeq -\log n$

$\alpha \simeq \dfrac{\log n}{-\lg(1-k_c)}$

$\alpha t \simeq \dfrac{cn}{2} \cdot \dfrac{\log n}{-\lg(1-k_c)}$

$= \dfrac{1}{2} n \lg n \cdot \dfrac{c}{-\lg(1-k_c)}$

$K_c \geq \theta(c)^4$


Idea for (*).
Want an upper-bound on
$W(X_t^x, X_t^y)$ for $t = cn/2$
enough to consider the case
$x = 0$ (identity)
$y = \tau\tau'$ where τ, τ' are transpositions
For this, use Schramm's coupling
Theorem (Schramm) Assume $c > 1$
Consider $X^0_{cn/2}$
whp $X^0_{cn/2}$ has giant cycles, and their total mass $\simeq n\theta(c)$