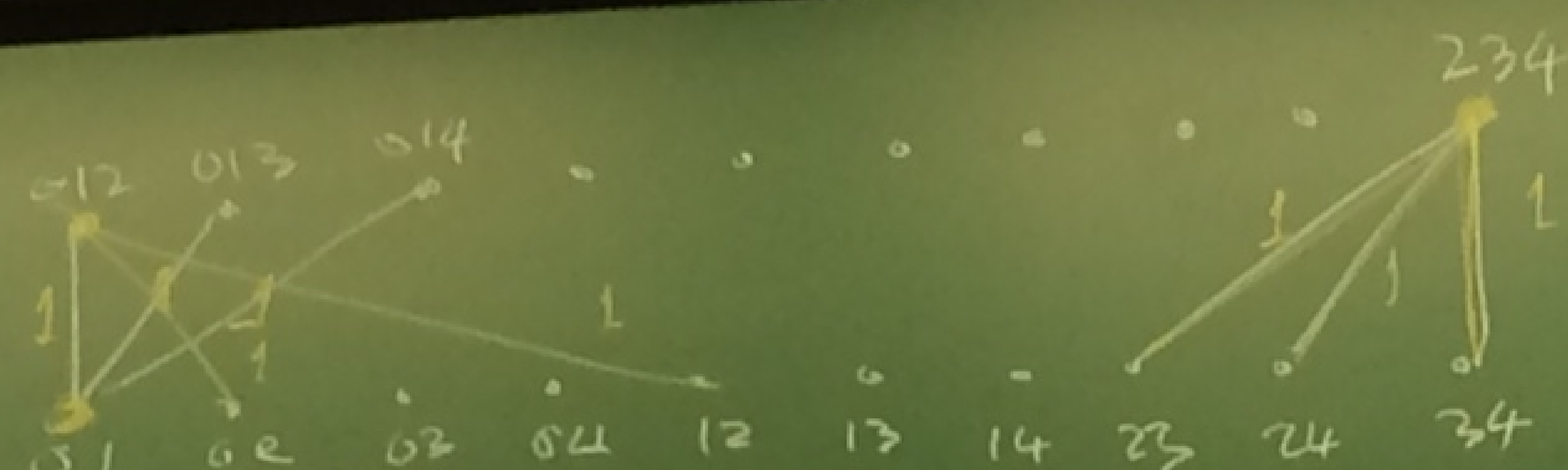


$\binom{n+1}{k+1}$

$\binom{n+1}{k}$

$\mathbb{R}^4 = \mathbb{R}^5 / \mathrm{Span}(1,1,1,1,1)$

For $S \subset E$, $e_S = \sum_{i \in S} e_i$

5 vectors in $\mathbb{R}^4$

$E = \{0, 1, 2, 3, 4\}$

Goal Show that L has exactly one negative eigenvalue

20 x 20 matrix

$$L_{ij} = \begin{cases} 2 & i=j \\ -1 & i\sim j \\ 0 & \text{otherwise} \end{cases}$$

$L = (\text{usual Laplacian}) - \mathrm{Id}.$

Balancing at 234

$\boxed{2}\, e_{234} = \boxed{1}\, e_{23} + \boxed{1}\, e_{24} + \boxed{1}\, e_{34}$.

Balancing at 01

$\boxed{2}\, e_{01} = \boxed{1}\, e_{012} + \boxed{1}\, e_{013} + \boxed{1}\, e_{014}$

Tropical Laplacian

$\underline{G}$: geometric graph in $\mathbb{R}^n$

$\underline{v}_1, \dots, \underline{v}_N$: vertices

w: "balanced" (positive) weight

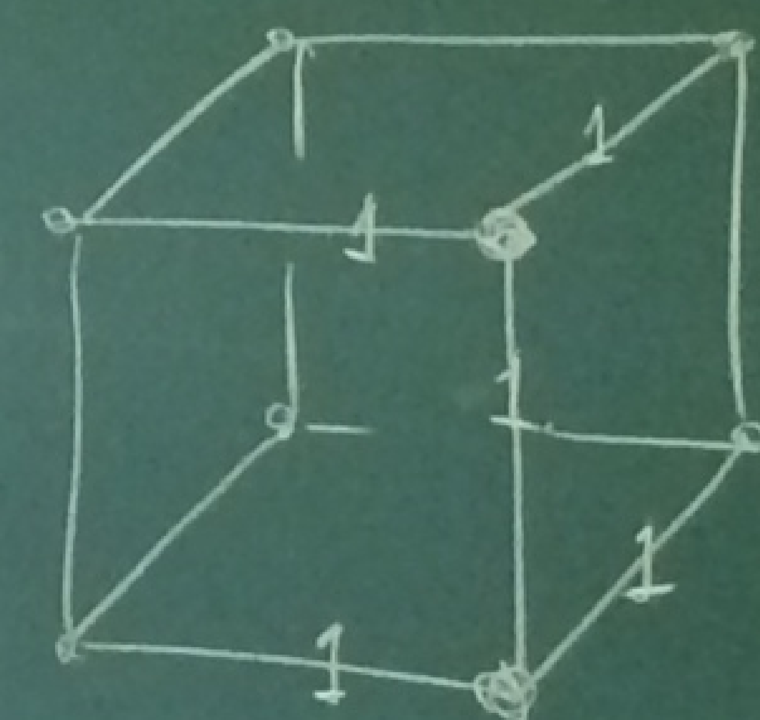
For each $\underline{v}_i$, there is a number d_i such that

$$d_i \underline{v}_i = \sum_{i \sim j} w_{ij} \underline{v}_j$$

$$L_{ij} = \begin{cases} d_i & i=j \\ -w_{ij} & i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Example

$\underline{G}$ graph of a simple polytope



Vertices

$(\pm 1, \pm 1, \pm 1)$

$$\boxed{1}(1,1,1) = \boxed{1}(-1,1,1) + \boxed{1}(1,-1,1) + \boxed{1}(1,1,-1)$$

L has exactly one negative eigenvalue.

Colin de Verdiere number of a graph

G: abstract graph

$\left\{ L : \right.$ ① entries corresp. to edge are negative
② " nonedge are zero
③ L has exactly one negative eigenvalue
④ Strong Arnold hypothesis.

μ = max. corank of L in {—}.

- $\mu \le$
- $\mu \le 2$
- G is planar iff $\mu = 3$ (V)
- G is linklessly embedable (in $\mathbb{R}^3$) iff $\mu = 4$ (V)

Conj

Any graph G can be colored using $\mu+1$ colors

Thm (Lovasz)

Let G be a 3-connected planar graph

There is a bijection

$$\left(\begin{array}{c} \text{Colin de Verdiere} \\ \text{matrices of } G \end{array} \right) \Big/ \sim \;\; \rightleftarrows \;\; \left(\begin{array}{c} \text{Steinitz representation} \\ \text{of } G \end{array} \right)$$

① If G is a graph of a 3-dim'l polytope, then there is a positive "balanced" weight for G.

② The "typical Laplacian" has exactly one negative eigenvalue.

Möbius function

P : poset

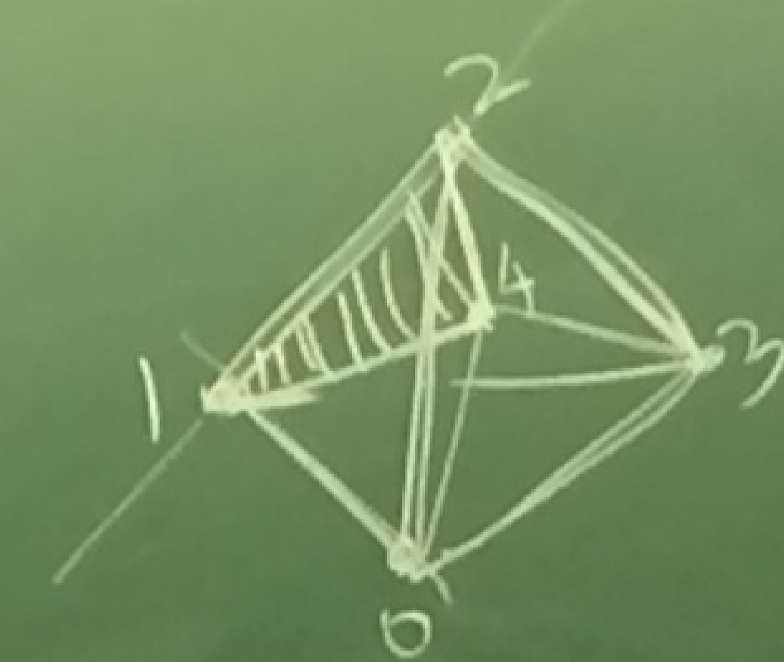
$\mu : P \times P \to \mathbb{Z}$

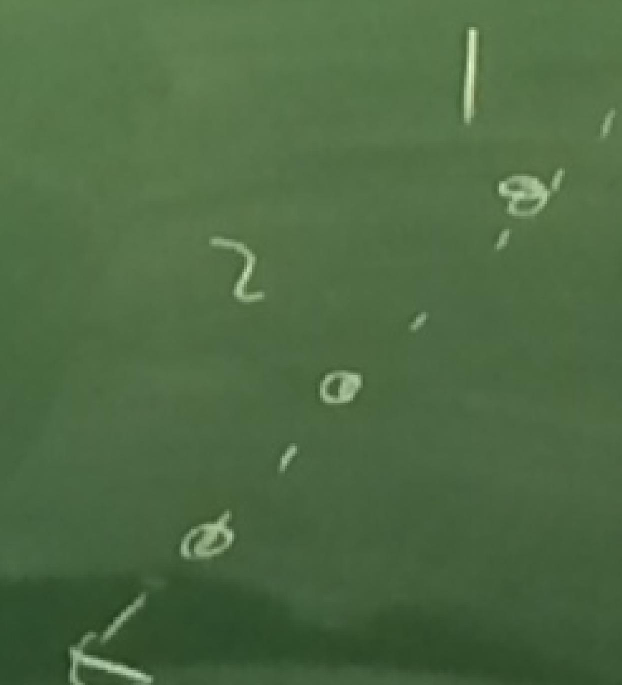
by ① $\mu(x, x) = 1$

② $\mu(x, y) = -\sum_{x \le z < y} \mu(x, z)$

M : matroid on $E = \{0, 1, 2, \cdots, n\}$ rank $r+1$

$$C_M = \left\{ \begin{array}{l} \underline{\text{flats}} \text{ of rank } r \quad \{0,1,2\} \to e_{012} \\ \text{flats of rank } r-1 \end{array} \right\}$$





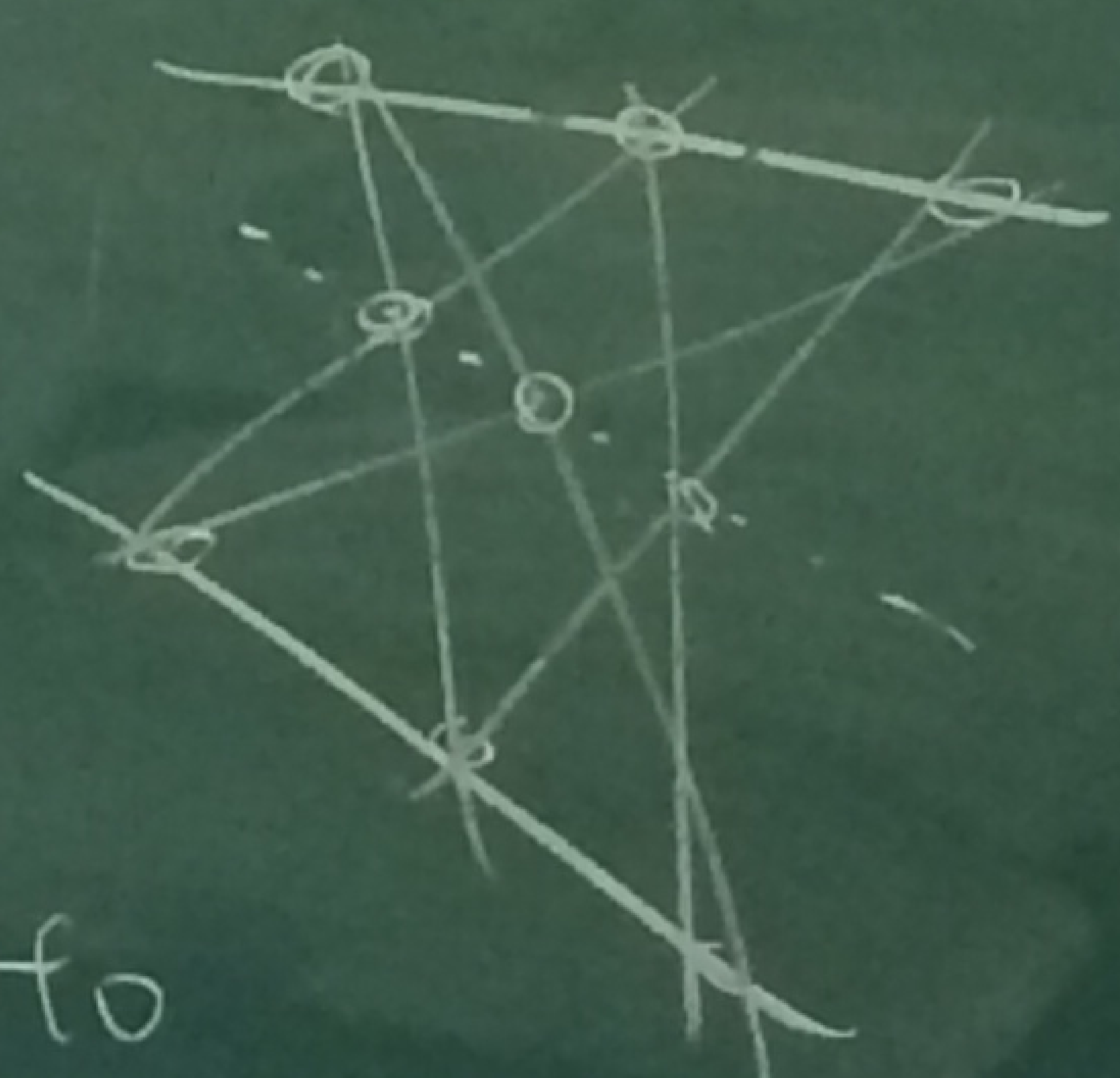
$\underline{G}(C_M) \subset \mathbb{R}^n$

<u>Thm</u> There is (a natural) choice of positive balanced weight for $\underline{G}(C_M)$.

<u>Conj</u> ☆ "The" tropical Laplacian of a matroid has exactly one negative eigenvalue

- ☆ holds for all matroids which are realizable over some field.

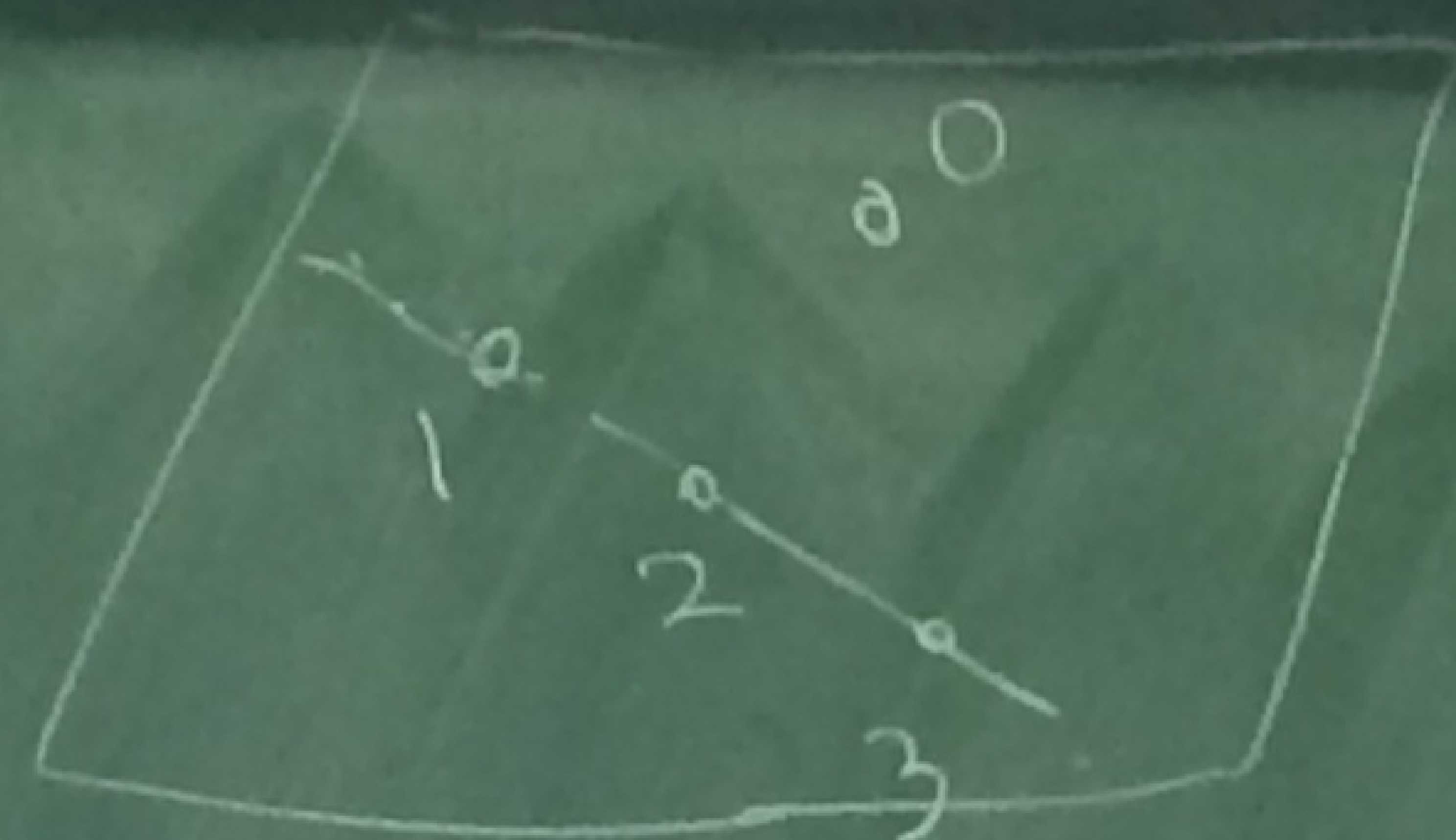
- ☆ holds for all matroids of rank 3

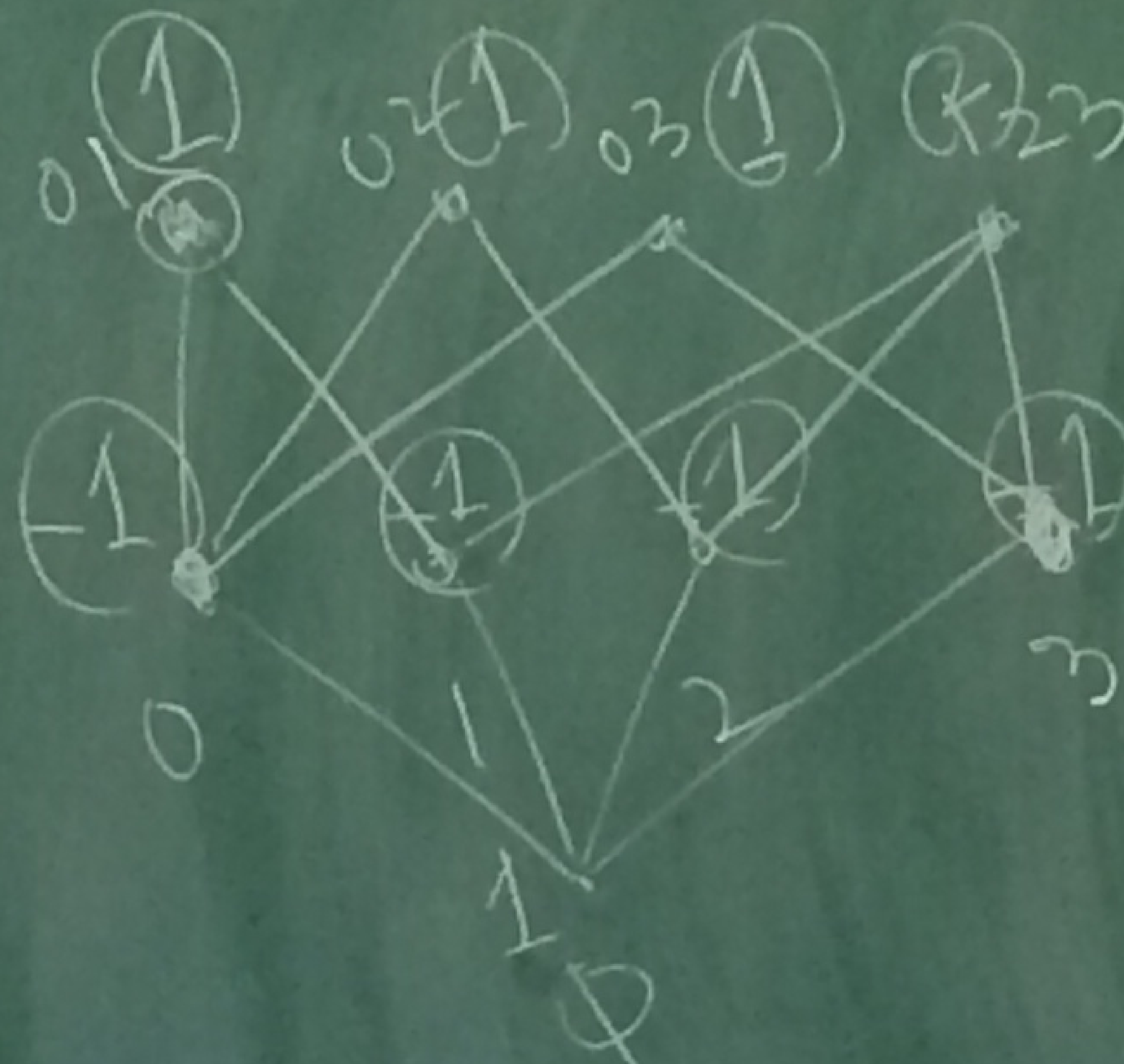


- ☆ holds for all matroids up to 9 elements.

nat

Matroid
rank 3
$\{0,1,2,3\}$





01

$\boxed{1}\, e_{01} = \boxed{1}\, e_0 + \boxed{1}\, e_1$

$\boxed{1}\, e_3 = \boxed{1}\, e_{03} + \boxed{1}\, e_{123}$

– G 15

G 15

Any

Using