Rota's conjecture and the tropical Laplacian

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April 8, 2014

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A *matroid* on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

1. Every subset of an independent set is an independent set.
2. If an independent set A has more elements than independent set B, then there is an element in A which, when added to B, gives a larger independent set.

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  Call a subset of E independent if it is linearly independent.

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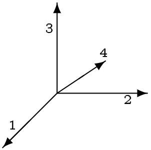
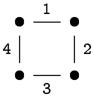
- Let G be a finite graph, and E the set of edges.

  Call a subset of E independent if it does not contain a circuit.

  This defines a *graphic matroid* M.





- A sequnce $a_0, \ldots, a_r$ is *log-concave* if for all i

$$a_{i-1}\, a_{i+1} \leq a_i^2.$$

- If there are *no internal zeroes*, log-concavity implies *unimodality*:

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_r \quad \text{for some } i.$$

- The chromatic polynomial of G is the function

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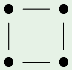
Example



$$\chi_G(q) \quad = \quad 1q^4 - 4q^3 + 6q^2 - 3q$$

Conjecture (Read '68, Hoggar '74)

The coefficients of the chromatic polynomial $\chi_G(q)$ form a unimodal (log-concave) sequence for any graph G.

- Let E be a finite subset of a vector space.
- Define the f-vector of E by

$$f_i = (\text{number of independent subsets of } E \text{ with size } i).$$

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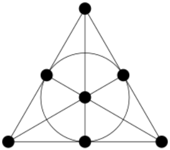

Example (Fano plane)

For $E = \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, we have

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

Conjecture (Welsh '69, Mason '72)

The sequence $f_0, f_1, \ldots, f_r$ is unimodal (log-concave) for any finite subset of a vector space.

Theorem (H.)

The conjecture on the chromatic polynomial is true for all graphs.

Theorem (Lenz)

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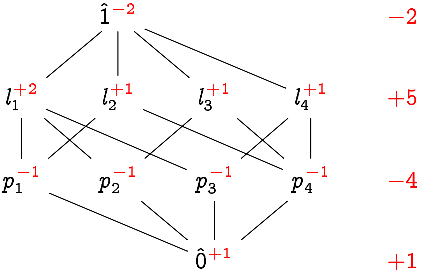
I will show how to prove these statements; but these are just a small part of a more general conjecture of Rota that is still *open* and *more interesting*.

Rota noted the significance of the *characteristic polynomial* $\chi_M(q)$ defined from the lattice of flats of a matroid M.

On the lattice of flats, the *Möbius function* is defined recursively by

$$\mu(\hat{0},\hat{0}) = 1 \quad \text{and} \quad \mu(\hat{0},F) = -\sum_{G<F} \mu(\hat{0},G).$$



The sum of the values of Möbius function determine the characteristic polynomial:

$$\chi_M(q) = 1q^3 - 4q^2 + 5q - 2.$$

The following conjecture specializes to the previous conjectures in the realizable case:

Conjecture (Rota, Heron, Welsh)

The coefficients of the characteristic polynomial $\chi_M(q)$ *form a log-concave sequence for any matroid* M.

Theorem (H.-Katz)

The coefficients of the characteristic polynomial $\chi_M(q)$ form a log-concave sequence if M is realizable over some field.

Does this gives a strong evidence for the general case?

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1. A subvariety defines a homology class.

   The homology class of a subvariety is called *prime*.

2. An *algebraic* class is an integral linear combination of prime classes.

   The Hodge conjecture characterizes algebraic classes (up to a multiple).

The homology group contains

(i) the set of *prime* classes, the classes of subvarieties,

(ii) the set of *effective* classes, the nonnegative linear combinations of prime classes.

Definition

A homology class $\xi \in H_{2d}(X;\mathbb{R})$ is said to be *prime* if some positive multiple of ξ is the class of a subvariety. Define

$$P_d(X) = \Big(\text{the closure of the set of prime classes in } H_{2d}(X;\mathbb{R})\Big).$$

This closed subset shows the asymptotic distribution of primes in the homology of X.

Example

If $X = \mathbb{P}^m \times \mathbb{P}^n$, then

$$H_{2d}(X;\mathbb{R}) = \Big\{\xi = \sum_i x_i[\mathbb{P}^{d-i} \times \mathbb{P}^i] \mid x_i \in \mathbb{R}\Big\}.$$

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Then $P_d(X)$ is the set

$\Big(\{x_i\}$ is a *log-concave* sequence of *nonnegative* integers with *no internal zeros*$\Big)$.

The orderly structure of primes is *not* visible if we work with integral homology classes.

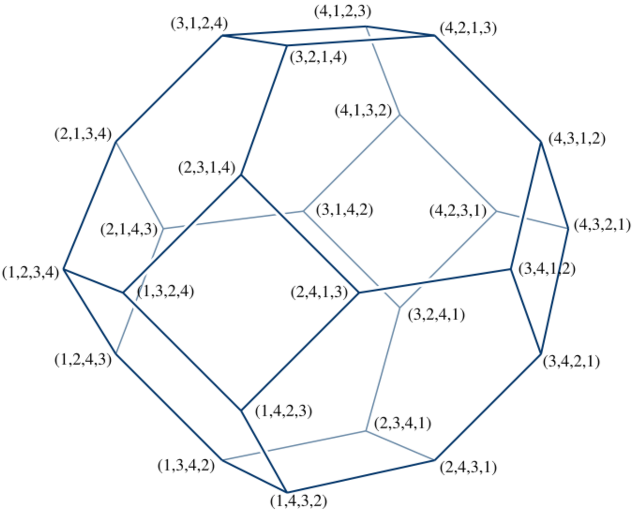
For example, there is no subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ with the homology class

$$1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros.

(Recall Hodge conjecture, Steenrod's problem, etc.)

I claim that matroid theory is a study of the toric variety X_{A_n} of the n-dimensional *permutohedron*:



- Let M be a (loopless) matroid on $E = \{0, 1, \ldots, n\}$ of rank $(r+1)$.
- Let $\mathscr{R}$ be a set of (nonzero) vectors $\{f_0, f_1, \ldots, f_n\}$ in an $(r+1)$-dimensional vector space over k.

If the matroid of $\mathscr{R}$ is M, then we say that $\mathscr{R}$ is a *realization* of M.

There is a ‘natural’ construction of

- an r-dimensional homology class of X_{A_n} from M, denoted Δ_M, and
- an r-dimensional subvariety of X_{A_n} from $\mathscr{R}$, denoted $Y_{\mathscr{R}}$.

The homology class Δ_M determines M, and the subvariety $Y_{\mathscr{R}}$ determines $\mathscr{R}$.

Theorem

Let k be a field, and M be a loopless matroid on E.

(i) *The homology class Δ_M is effective.*

(ii) *The homology class Δ_M is prime if and only if M is realizable over k.*

Theorem

More precisely,

(iii) *If* $\mathscr{R}$ *is a realization of* M, *then* $[Y_{\mathscr{R}}] = \Delta_M$.

(iv) *If* Y *is a subvariety with* $[Y] = \Delta_M$, *then* $Y = Y_{\mathscr{R}}$ *for some realization* $\mathscr{R}$ *of* M *over* k.

Theorem

Under the "anticanonical" map

$$\pi : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

the matroid homology class pushforwards

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Since prime classes map to prime classes, Rota's conjecture is true for all matroids which are realizable over some field.

In our language,

Conjecture (log-concavity conjecture)

For any matroid M and any field k,

$$\pi_*(\Delta_M) \in P_r(\mathbb{P}^n \times \mathbb{P}^n).$$

If Rota's conjecture is true, I believe it is because the same is true in X_{A_n}.

Conjecture (AG)

For any matroid M and any field k,

$$\Delta_M \in P_r(X_{A_n}).$$

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Conjecture (CO)

The tropical Laplacian of a matroid has exactly one negative eigenvalue.

(i) Conjecture (CO) holds for all matroids which are realizable over some field.

(ii) Conjecture (CO) implies the log-concavity conjecture.