A new connection between irreversible random walks and electric networks

Work in progress, joint with
Áron Folly

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Combinatorics and Statistical Mechanics
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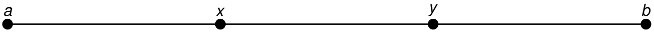
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



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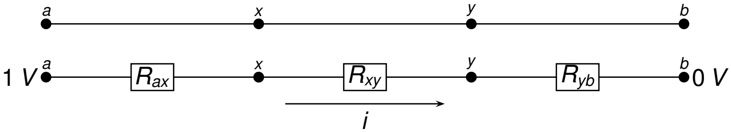

Electric resistor network: the voltage u is harmonic ($C = 1/R$):

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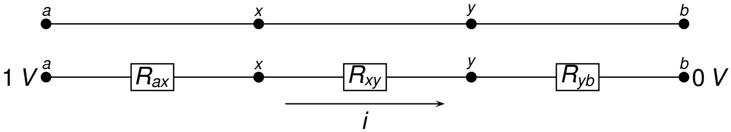

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Notice $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$, so the chain is reversible.

Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

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$\mathbf{E}_a$(signed current $x \to y$ before absorbed in b)

$= n_x P_{xy} - n_y P_{yx} = (u_x - u_y) C_{xy} = i_{xy}$. normalisation...

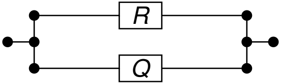
$P_{xy} = C_{xy}/C_x$ $\qquad$ $C_x = \mu_x$

Reducing a network

Series:

$$R_{\text{eff}} = R + Q$$

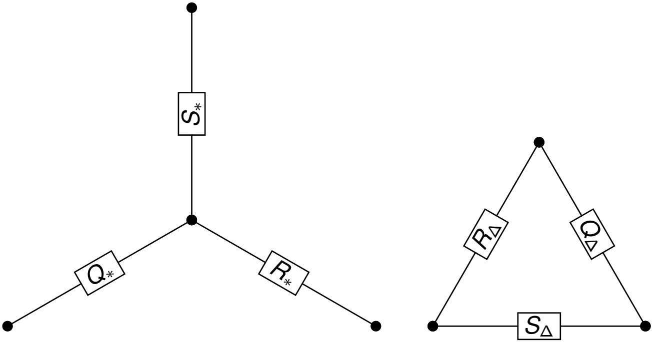
Parallel:



$$\frac{1}{R_{\text{eff}}} = \frac{1}{R} + \frac{1}{Q}$$

Reducing a network

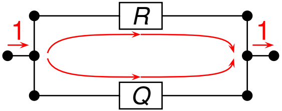
Star-Delta:



$$R_* = \frac{Q_\Delta S_\Delta}{R_\Delta + Q_\Delta + S_\Delta}, \qquad R_\Delta = \frac{R_* Q_* + R_* S_* + Q_* S_*}{R_*}.$$

Thomson, Dirichlet principles

Thomson principle:



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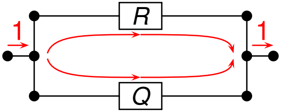

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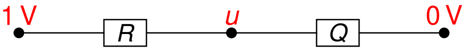

The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$.

Monotonicity, transience, recurrence

The monotonicity property:

Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

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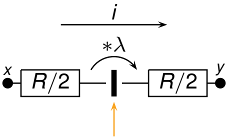
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

$\rightsquigarrow$ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

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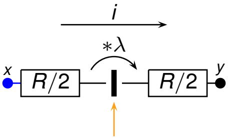

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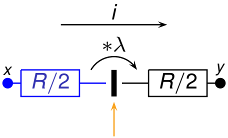

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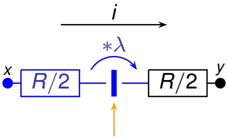

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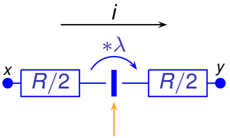

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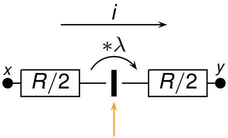

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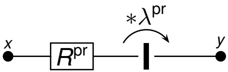
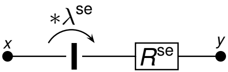


$$(u_x - i \cdot R^{pr}) \cdot \lambda^{pr} = u_y$$



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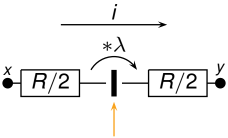

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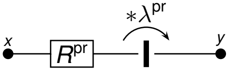

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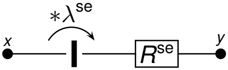

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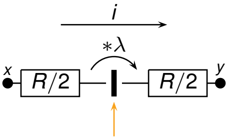

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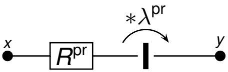

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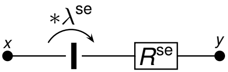

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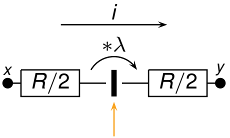

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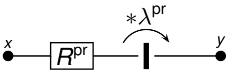

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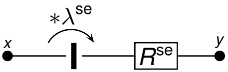

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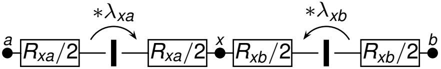
Harmonicity



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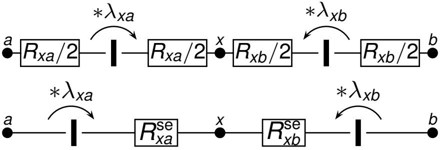

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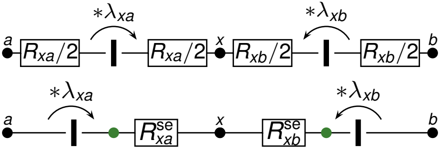

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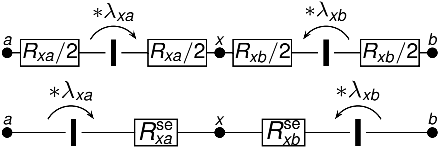

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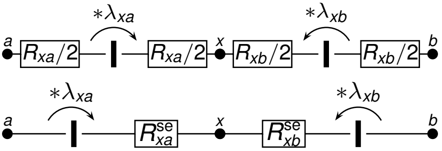

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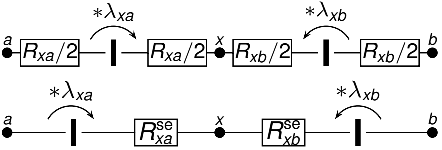

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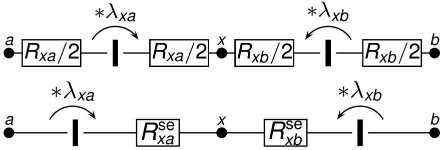

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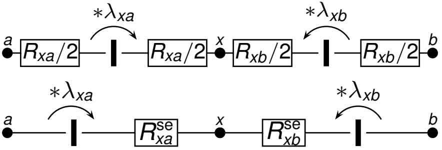

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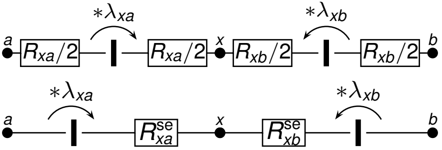

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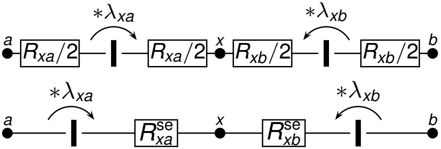
$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



$$u_x = \sum_y \frac{C^{se}_{xy}}{\sum_z C^{se}_{xz}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy} \frac{2}{\lambda_{xy}+1}}{\sum_z C_{xz} \frac{2}{\lambda_{xz}+1}} \cdot \lambda_{xy} u_y$$

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From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

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Stationary distribtuion:

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"Markovity" property

$$u_x = \sum_z P_{xz} u_z; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

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$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

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Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

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Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

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$$\begin{aligned}&\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b)\\ &= n_x P_{xy} - n_y P_{yx} = (\hat{u}_x \gamma_{xy} - \hat{u}_y \gamma_{yx}) D_{xy} = \hat{i}_{xy}.\end{aligned}$$

normalisation...

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Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

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$\rightsquigarrow$ In particular, i_a is proportional to $u_a - u_b$. We have effective resistance.

What works

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Modulo normalisation. . .

$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b) = \hat{i}_{xy}.$$

in the reversed network!

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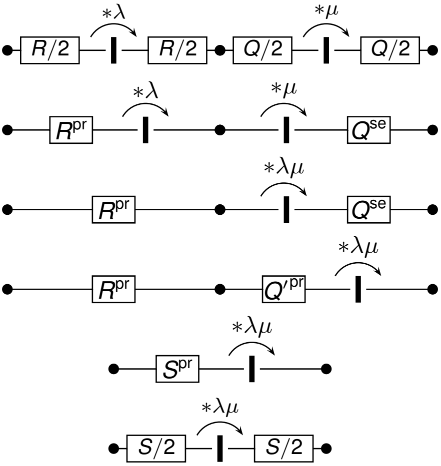
in the reversed network!

Theorem

Commute time $= R_{\mathit{eff}} \cdot$ *all conductances.*

The electric network

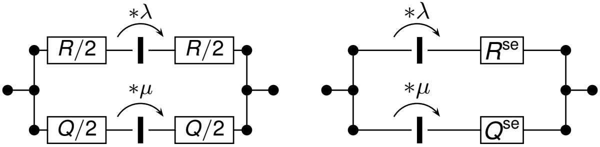
Series:



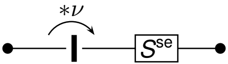
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

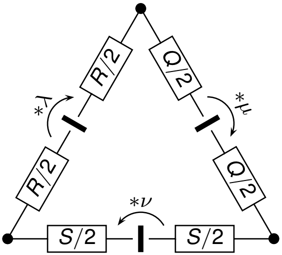
$$\nu = \frac{Q\lambda(\mu+1) + R\mu(\lambda+1)}{Q(\mu+1)+R(\lambda+1)}.$$

The electric network

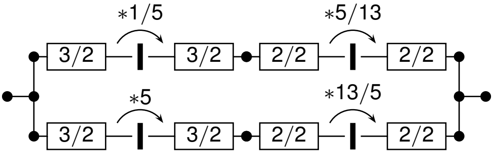
Star-Delta:

Star to Delta works,

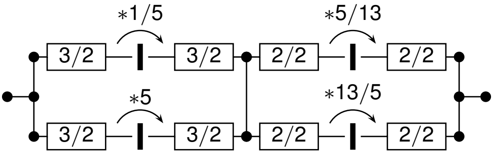
Delta to Star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).



Nonmonotonicity



$$R_{\text{eff}} = \frac{27}{14}.$$



$$R_{\text{eff}} = \frac{5}{2} = \frac{35}{14}.$$

Dirichlet principle

Classical case:

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$$(i_u)_{xy} = C_{xy} \cdot \big(u(x) - u(y)\big),$$

$$E_{\mathrm{Ohm}}(i_u) = \sum_{x \sim y} (i_u)_{xy}^2 \cdot R_{xy}.$$

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Irreversible case (A. Gaudillière, C. Landim / M. Slowik):

$$(i_u^*)_{xy} = \frac{2\lambda_{xy}}{\lambda_{xy}+1} C_{xy}\, u(x) - \frac{2}{\lambda_{xy}+1} C_{xy}\, u(y),$$

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Thank you.