

3-Coloring the Discrete Torus

or

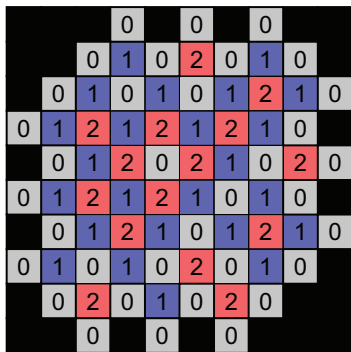
Rigidity of zero temperature 3-states anti-ferromagnetic Potts model

Ohad N. Feldheim
Joint work with Ron Peled

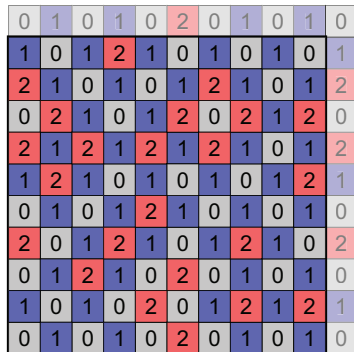
Department of Mathematics
Weizmann Institute of Science

Warwick University, April, 2014

3-Colorings of the Grid/Torus

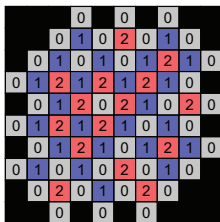


Zero boundary conditions

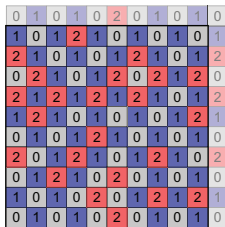


Periodic boundary conditions

Random 3-Colorings



Zero boundary conditions

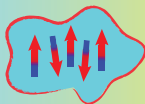


Periodic boundary conditions

- Uniformly chosen proper 3-coloring (Given boundary conditions)
- High dimension \mathbb{Z}^d , and \mathbb{T}_n^d .

Additional Motivation

Physics



Antiferromagnetism

Mathematical
Physics



*q-states antiferromagnetic
Potts model*

Combinatorics



*q-colorings of the
discrete torus*

- Generalizes the celebrated Ising model.
- Each point takes one of q values.
- Neighbors dislike getting the same color.
- 3-coloring is the “zero temperature” version.

Properties of Interest

In a typical coloring:

- What is the typical relative frequency of the colors?
Is it $(1/3, 1/3, 1/3)$?

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- Does it look roughly like this?

1	0	1	0	1	0	2	0	1	0
0	1	0	1	0	1	0	1	0	1
2	0	2	0	1	0	2	0	1	0
0	1	0	2	0	1	0	1	0	1
1	0	2	0	1	0	1	0	1	2
0	1	0	2	0	1	0	2	0	1
2	0	1	0	2	0	1	0	1	0
0	1	0	1	0	2	0	2	0	2
1	0	1	0	2	0	1	0	2	0
0	1	0	1	0	2	0	1	0	1

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Conjecture:

1	0	1	0	1	0	2	0	1	0
0	1	0	1	0	1	0	1	0	1
2	0	2	0	1	0	2	0	1	0
0	1	0	2	0	1	0	1	0	1
1	0	2	0	1	0	1	0	1	2
0	1	0	2	0	1	0	2	0	1
2	0	1	0	2	0	1	0	1	0
0	1	0	1	0	2	0	2	0	2
1	0	1	0	2	0	1	0	2	0
0	1	0	1	0	2	0	1	0	1

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Conjecture:

$d = 2$ **No.**

1	0	1	0	1	0	2	0	1	0
0	1	0	1	0	1	0	1	0	1
2	0	2	0	1	0	2	0	1	0
0	1	0	2	0	1	0	1	0	1
1	0	2	0	1	0	1	0	1	2
0	1	0	2	0	1	0	2	0	1
2	0	1	0	2	0	1	0	1	0
0	1	0	1	0	2	0	2	0	2
1	0	1	0	2	0	1	0	2	0
0	1	0	1	0	2	0	1	0	1

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Conjecture:

$d = 2$ **No.**

$d > 2$ **Yes.**

1	0	1	0	1	0	2	0	1	0
0	1	0	1	0	1	0	1	0	1
2	0	2	0	1	0	2	0	1	0
0	1	0	2	0	1	0	1	0	1
1	0	2	0	1	0	1	0	1	2
0	1	0	2	0	1	0	2	0	1
2	0	1	0	2	0	1	0	1	0
0	1	0	1	0	2	0	2	0	2
1	0	1	0	2	0	1	0	2	0
0	1	0	1	0	2	0	1	0	1

Previous Results - Rigidity for 0-boundary

The conjecture has been established for 0-boundary conditions in high dimension.

0-boundary rigidity (Peled 2010)

In a typical 3-coloring with 0-boundary conditions nearly all the even vertices take the color 0.

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Formally: Let d be large enough, a uniformly chosen 3-coloring with **0-BC**, has:

$$\frac{\mathbb{E} |\{v \in V^{\text{even}} : g(v) \neq 0\}|}{|V^{\text{even}}|} < \exp\left(-\frac{cd}{\log^2 d}\right).$$

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- Does not work for periodic BC.
- Open in low dimensions.

Previous Results - Rigidity for the hypercube

The conjecture has also been supported on bounded tori.

Periodic boundary on the even hypercube (Galvin & Engbers 2011)

For every fixed n , for high enough dimension (depending on n), a typical 3-coloring with periodic boundary conditions is nearly constant on either the even or the odd sublattice.

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- Works also for q -colorings (and even more general!)
- Fixed n is less important for physicists.

Our Results - Rigidity on \mathbb{T}_d^n

We establish a parallel phenomenon for periodic BC.

Theorem (F., Peled)

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- n must be even.

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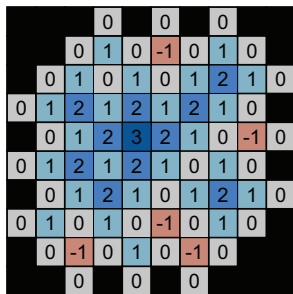
- n must be even.
- Introduces topological techniques to the problem.



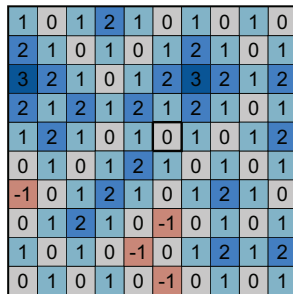
Proof Overview

Homomorphism Height Functions

$h : G \rightarrow \mathbb{Z}$ satisfying $|h(v) - h(u)| = 1$ if $v \sim u$.



Zero boundary conditions



Periodic boundary conditions

- Discretized “topographical map”.

Relation to 3-Colorings

On \mathbb{Z}^d there is a natural bijection.

1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1

Pointed 3-Colorings



mod 3



1	0	1	2	3	4	5	6	7	6
2	1	0	1	2	3	4	5	6	7
1	2	1	0	1	2	3	4	5	6
0	1	0	1	2	1	2	3	4	5
-1	0	1	2	1	0	1	2	3	4
0	1	0	1	2	1	2	3	4	5
-1	0	1	2	1	2	3	2	3	4
0	1	2	3	2	1	2	3	4	5
1	0	1	2	3	2	3	4	5	6
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Pointed HHFs

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1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1

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1	0	1	2	3	4	5	6	7	6
2	1	0	1	2	3	4	5	6	7
1	2	1	0	1	2	3	4	5	6
0	1	0	1	2	1	2	3	4	5
-1	0	1	2	1	0	1	2	3	4
0	1	0	1	2	1	2	3	4	5
-1	0	1	2	1	2	3	2	3	4
0	1	2	3	2	1	2	3	4	5
1	0	1	2	3	2	3	4	5	6
0	1	0	1	2	3	4	5	6	7

Pointed HHFs

This bijection **does not** extend to \mathbb{T}_n^d .

Rigidity of HHFs

More is known about HHFs than about 3-colorings:

Rigidity of HHFs on \mathbb{T}_n^d (follows from Peled 2010)

A typical pointed HHF on a high dimensional torus is nearly constant on either the even or the odd sublattice.

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Here Topology enters.

Introducing Quasi-Periodic HHFs

What are 3-colorings on \mathbb{T}_n^d in bijection with?

Introducing Quasi-Periodic HHFs

What are 3-colorings on \mathbb{T}_n^d in bijection with?

\mathbb{T}_n^d

1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1

Quasi periodic HHFs of \mathbb{Z}^d
 whose slopes are 0 mod 6

\mathbb{Z}^d

1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

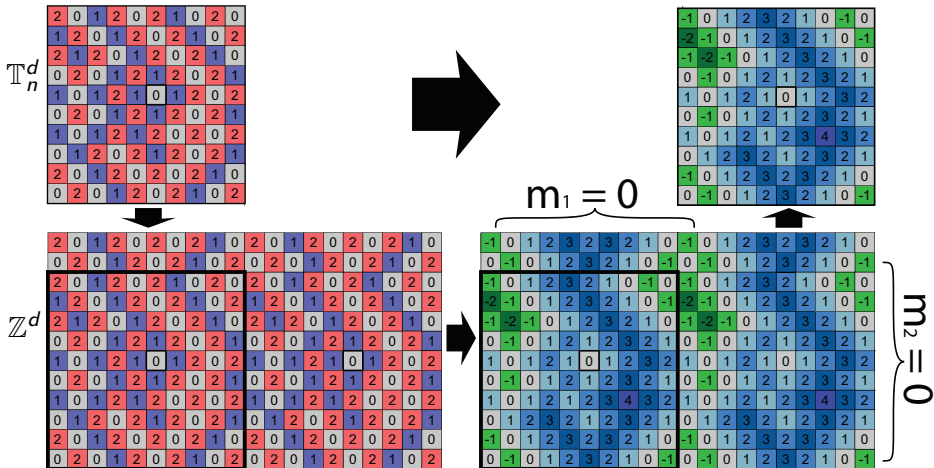
$m_1 = 6$

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

$m_2 = 0$

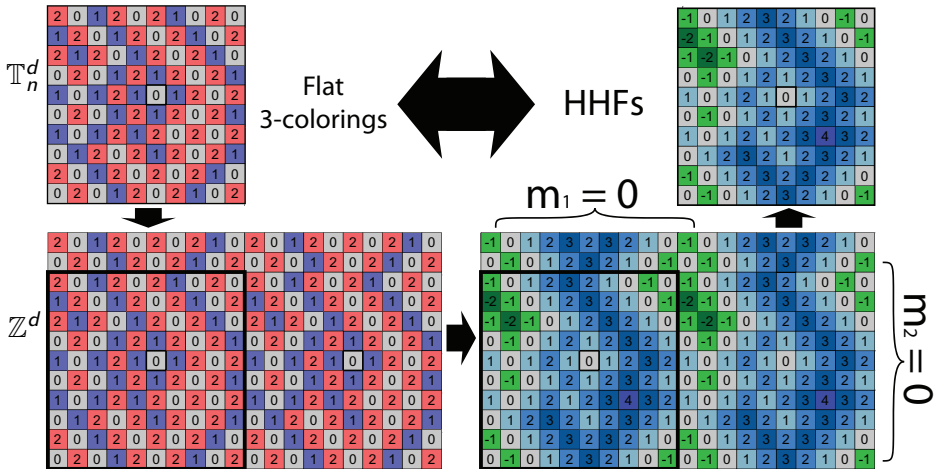
Flat Slope HHFs \leftrightarrow HHFs on \mathbb{T}_n^d

If all slopes are 0 (“flat” coloring) we get an HHF on \mathbb{T}_n^d .



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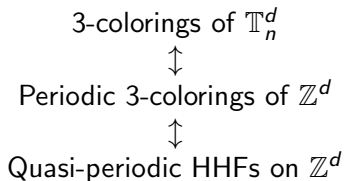


Pulling the HHFs result to 3-colorings

3-colorings of \mathbb{T}_n^d
 \updownarrow
 Periodic 3-colorings of \mathbb{Z}^d

1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

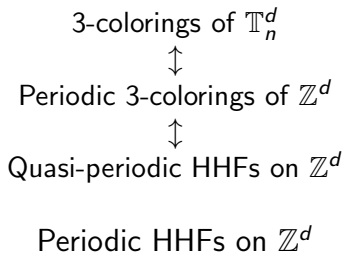
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1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
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0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

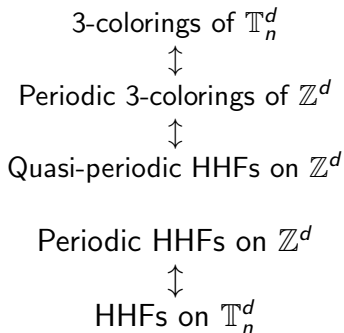
Pulling the HHFs result to 3-colorings



1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

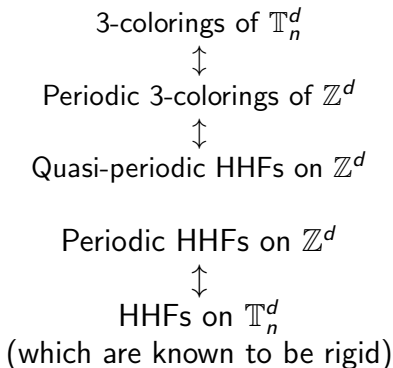
Pulling the HHFs result to 3-colorings



1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

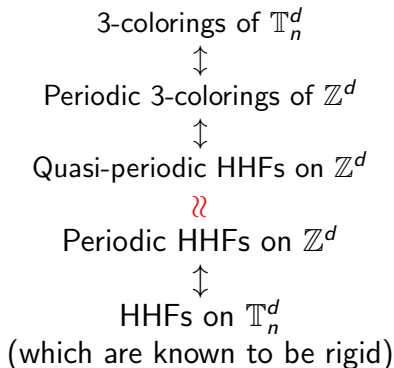
Pulling the HHFs result to 3-colorings



1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1	2
1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

Pulling the HHFs result to 3-colorings



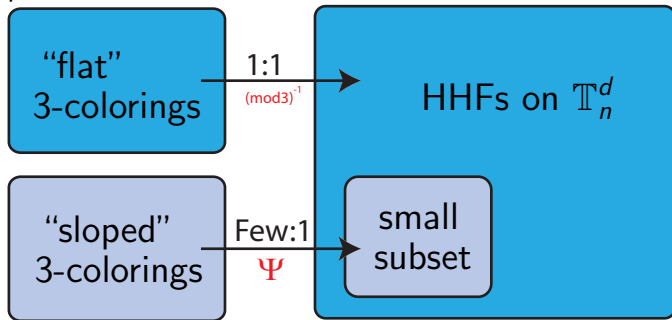
GOAL: Show that most quasi-periodic HHFs are periodic.

1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2	0
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1
1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1	0
2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0	1
1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1	2	0
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	0	1	2	0	1	2	0	1	2	1	0	1	2	0	1
0	1	0	1	2	1	2	0	1	2	0	1	0	1	2	1	2	0	1	2
2	0	1	2	1	2	0	2	0	1	2	0	1	2	1	2	0	2	0	1
1	0	1	2	0	2	1	2	0	1	2	0	1	2	0	2	1	2	0	1
0	1	0	1	2	0	2	0	1	2	0	1	0	1	2	0	2	0	1	2
0	1	0	1	2	0	1	2	0	1	0	1	0	1	2	0	1	2	0	1

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13	12
2	1	0	1	2	3	4	5	6	7	8	7	8	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13

Proving Most Quasi-periodic are Periodic

We construct a “flattening” map Ψ from quasi-periodic HHFs into periodic ones.



Flattening the slope

Introducing the reflection Ψ

- Denote $QP_m := \{h \in QP : m \text{ is the slope of } h\}$
- We construct $\Psi_m : QP_m \rightarrow QP_0$, a one-to-one mapping.

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12	13	12		
2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12	13
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12	13		

Ψ_m
 \rightarrow

-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1
-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1	0
-2	-1	0	1	2	3	2	1	0	-1	-2	-1	0	1	2	3	2	1	0	-1
-1	-2	-1	0	1	2	3	2	1	0	-1	-2	-1	0	1	2	3	2	1	0
0	-1	0	1	2	1	2	3	2	1	0	-1	0	1	2	1	2	3	2	1
1	0	1	2	1	0	1	2	3	2	1	0	1	2	1	0	1	2	3	2
0	-1	0	1	2	1	2	3	2	1	0	-1	0	1	2	1	2	3	2	1
1	0	1	2	1	2	3	4	3	2	1	0	1	2	1	2	3	4	3	2
0	1	2	3	2	1	2	3	2	1	0	1	2	3	2	1	2	3	2	1
-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1

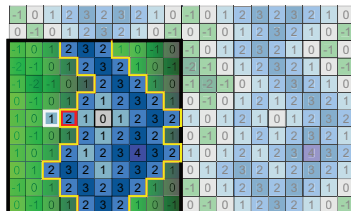
Flattening the slope

Introducing the reflection Ψ

- Denote $QP_m := \{h \in QP : m \text{ is the slope of } h\}$
- We construct $\Psi_m : QP_m \rightarrow QP_0$, a one-to-one mapping.



Ψ_m
 \rightarrow



- Observe that the image contains a long level set.

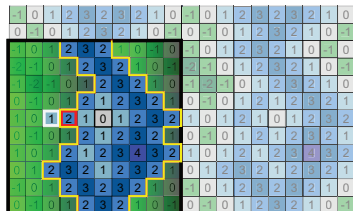
Flattening the slope

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- Denote $QP_m := \{h \in QP : m \text{ is the slope of } h\}$
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Ψ_m
 \rightarrow



- Observe that the image contains a long level set.
- Peled 2010: Long level sets are extremely uncommon.

Flattening the slope

Introducing the reflection Ψ

- Denote $QP_m := \{h \in QP : m \text{ is the slope of } h\}$
- We construct $\Psi_m : QP_m \rightarrow QP_0$, a one-to-one mapping.

1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	6	7	8	9	10	11	12	13
1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12	13	12	11	10
2	1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12	13	12	11
1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10	11	12
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	0	1	2	3	4	5	6	7	8	7	6	7	8	9	10
0	1	0	1	2	1	2	3	4	5	6	7	6	7	8	7	8	9	10	11
-1	0	1	2	1	2	3	2	3	4	5	6	7	8	7	8	9	8	9	10
0	1	2	3	2	1	2	3	4	5	6	7	8	9	8	7	8	9	10	11
1	0	1	2	3	2	3	4	5	6	7	6	7	8	9	8	9	10	11	12
0	1	0	1	2	3	4	5	6	7	6	7	8	9	10	11	12	13	12	11

Ψ_m
 \rightarrow

-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1
-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1	0
-1	-1	0	1	2	3	2	1	0	-1	-1	0	1	2	3	2	1	0	-1	0
0	-1	0	1	2	1	2	3	2	1	0	-1	0	1	2	1	2	3	2	1
-1	0	1	2	1	0	1	2	3	2	1	0	1	2	1	0	1	2	3	2
0	-1	0	1	2	1	2	3	2	1	0	-1	0	1	2	1	2	3	2	1
1	0	1	2	1	2	3	4	3	2	1	0	1	2	1	2	3	4	3	2
0	1	2	3	2	1	2	3	2	1	0	1	2	3	2	1	2	3	2	1
-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	1	2	3	2	1	0	-1	0	1

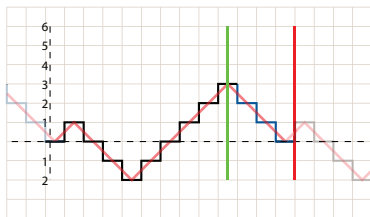
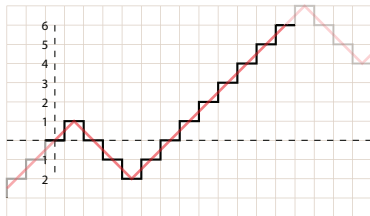
- Observe that the image contains a long level set.
- Peled 2010: Long level sets are extremely uncommon.
- We deduce the image of Ψ_m is small.



Ideas and Method

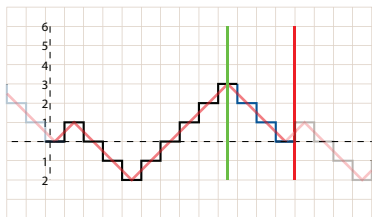
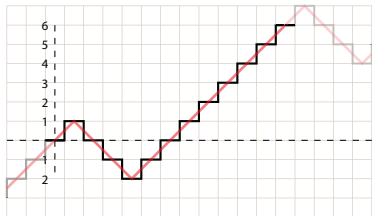
Flattening Intuition

- One-dimensional intuition: reflection.



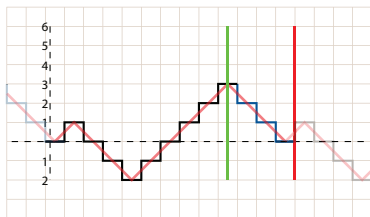
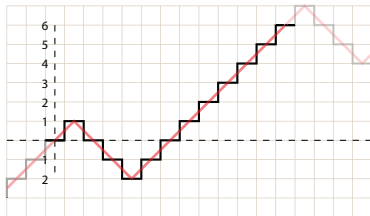
Flattening Intuition

- One-dimensional intuition: reflection.
- Where to reflect?



Flattening Intuition

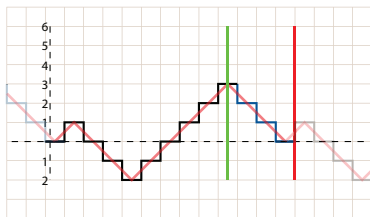
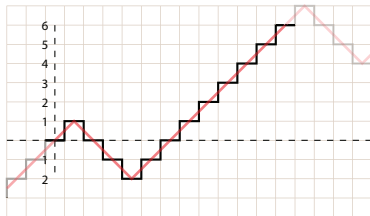
- One-dimensional intuition: reflection.
- Where to reflect?
 - immediately after height $\frac{m_1}{2}$



Flattening Intuition

- One-dimensional intuition: reflection.
- Where to reflect?
 - immediately after height $\frac{m_1}{2}$

Problem: several m_j -s. Can we fix them all at once?



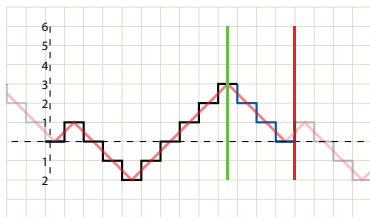
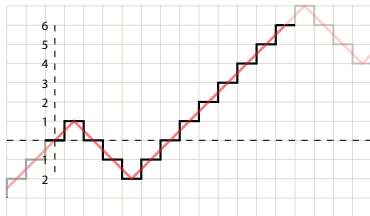
Flattening Intuition

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 - immediately after height $\frac{m_1}{2}$

Problem: several m_j -s. Can we fix them all at once?

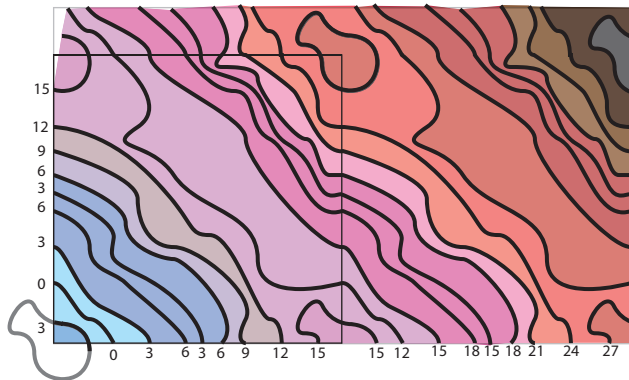
Answer:

Topology says - **Yes**.



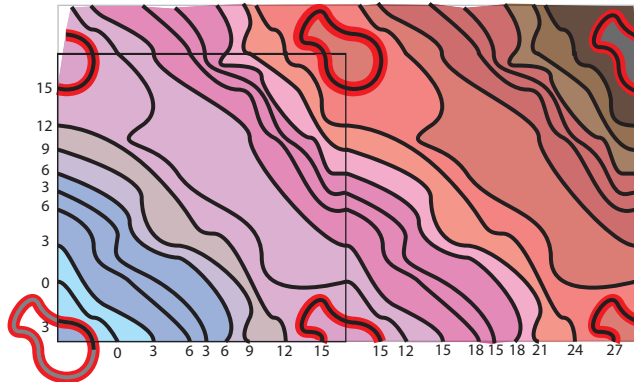
Flattening multi-dimensional functions

Quasi-periodic functions are homotopy equivalent to linear ones.



Flattening multi-dimensional functions

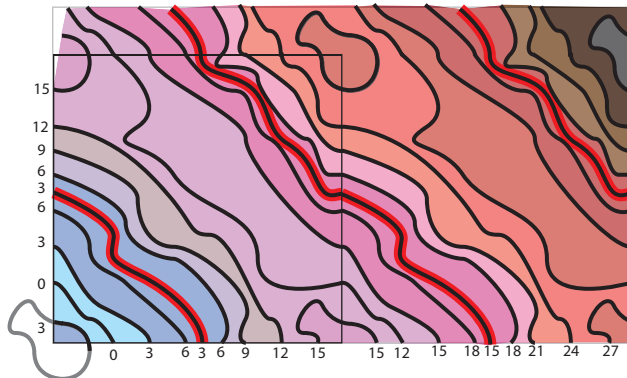
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- Two types of level contours:
Trivial level contours.

Flattening multi-dimensional functions

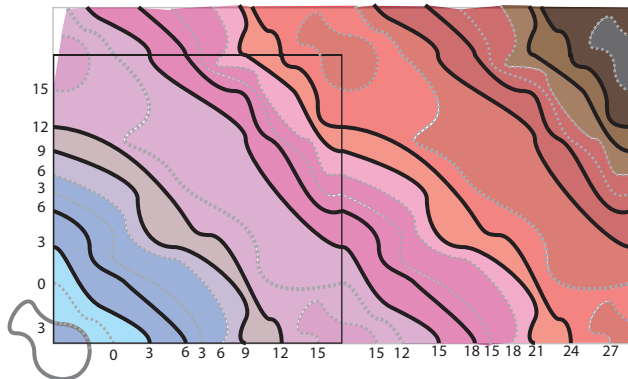
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- Two types of level contours:
 - Trivial level contours.
 - Non-Trivial level contours.

Flattening multi-dimensional functions

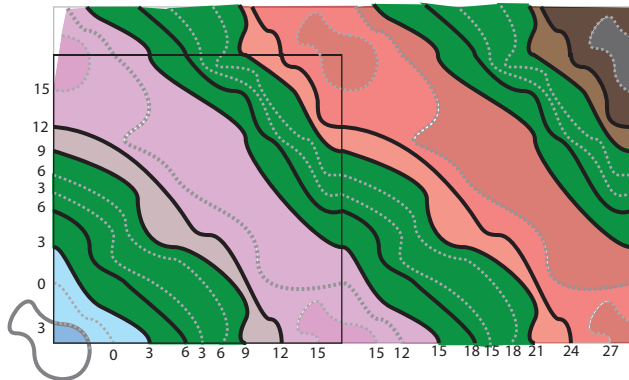
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Flattening multi-dimensional functions

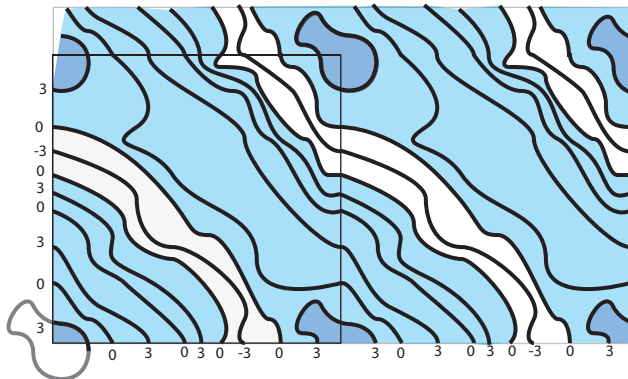
Quasi-periodic functions are homotopy equivalent to linear ones.



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- We find the proper reflection “domain” on the torus.

Flattening multi-dimensional functions

Quasi-periodic functions are homotopy equivalent to linear ones.



- We pick particular non-trivial level contours.
- We find the proper reflection “domain” on the torus.
- We make the reflection.

How to discretize?

Challenges of the discrete setting:

- Define level sets properly.
- Establish their structure.
- Identify trivial level sets.
- Prove the invertability of the reflection.

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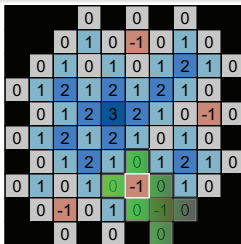
We will focus on **these** in this presentation.

Sublevel sets

Towards level sets

Sublevel set of v at height k

$LC_h^k(v)$ is the connected component of v in $G \setminus \{u \in G \mid h(u) = k\}$

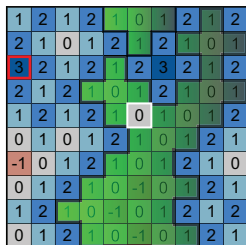


Sublevel Components

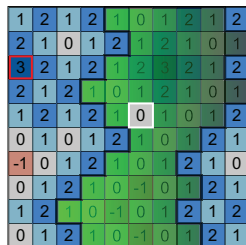
The fundament of level sets

Sublevel component from v to u at height k

$LC_h^k(v, u)$ is the complement of the connected component of u in $G \setminus LC_h^k(v)$



$LC_h^k(v)$



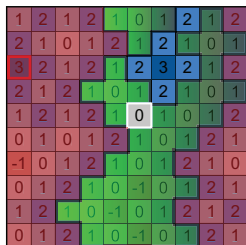
$LC_h^k(v, u)$

Sublevel Components

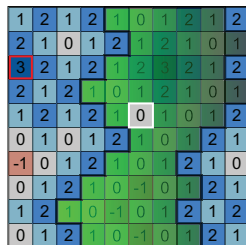
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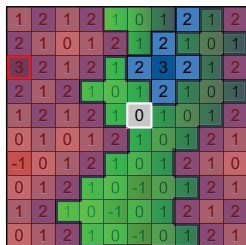
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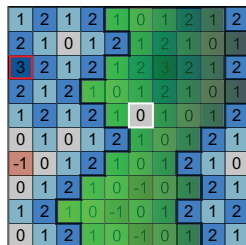
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$LC_h^k(v)$



$LC_h^k(v, u)$

The edge boundary of a sublevel component is called a level set.

3 Types of Level Components

A trichotomy

For $t \in n\mathbb{Z}^d$, and a set $U \subset \mathbb{Z}^d$ we call $U + t$ a translate of U .

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3 types of level components

Let $U = \text{LC}_h^k(u, v)$ be a sublevel component with non-empty boundary. One of the following holds:

- (Trivial) All of U 's translates are disjoint.
- (Trivial) All of U^c 's translates are disjoint.
- (Non-trivial) The translates of U are totally ordered by inclusion.

Trichotomy - example

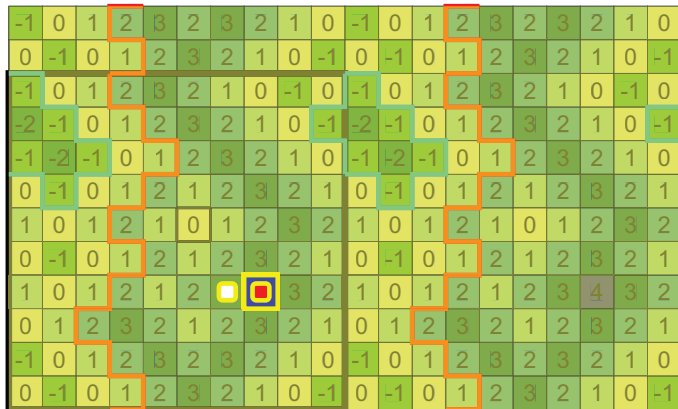
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-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1	0
-2	-1	0	1	2	3	2	1	0	-1	-2	-1	0	1	2	3	2	1	0	-1
-1	-2	-1	0	1	2	3	2	1	0	-1	-2	-1	0	1	2	3	2	1	0
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-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1

Trichotomy - example

-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1
-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1	0
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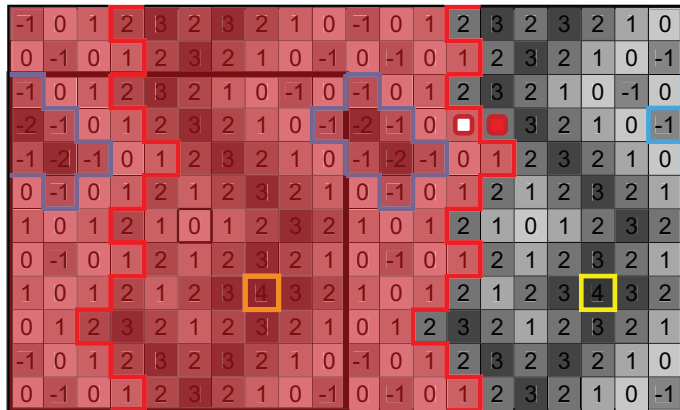
Trivial (Disjoint translates)

Trichotomy - example



Trivial (Disjoint complement translates)

Trichotomy - example



Non-Trivial (Ordered)

Trichotomy - example

-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1
-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1	0
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0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1

Trivial sublevel components do not create slope.

Formula for heights

Denote $\mathcal{L} = \{A : \exists u_1, u_2 \in \mathbb{Z}^d : A = LC_h^{h(u_2)}(u_1, u_2)\}$

Formula for $h(u) - h(v)$

$$h(u) - h(v) = |\{A \in \mathcal{L} : v \in A, u \notin A\}| - |\{A \in \mathcal{L} : v \notin A, u \in A\}|$$



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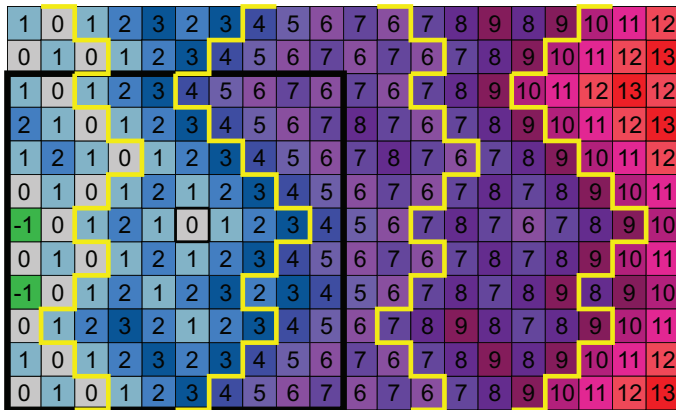
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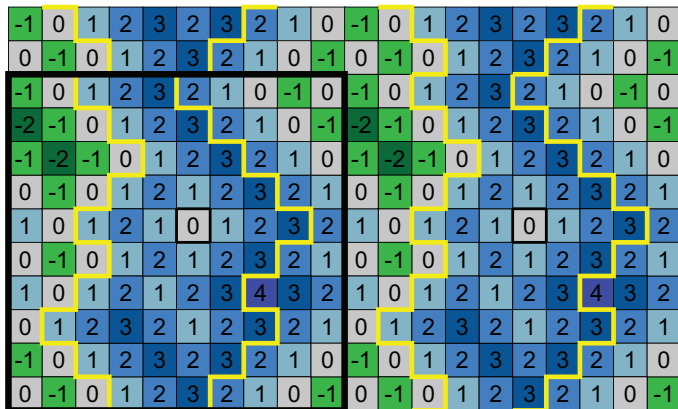
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- Used, for example, to show the existence of non-trivial sublevel components for sloped function.

The Discrete Picture



The Discrete Picture



The Discrete Picture

-1	0	1	2	3	2	3	2	1	0	-1	0	1	2	3	2	3	2	1	0
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0	-1	0	1	2	3	2	1	0	-1	0	-1	0	1	2	3	2	1	0	-1

Open Problems

Ordered by estimated difficulty

- Odd Tori.

Open Problems

Ordered by estimated difficulty

- Odd Tori.
- 4-colors and more.

Open Problems

Ordered by estimated difficulty

- Odd Tori.
- 4-colors and more.
- Non-zero temperature.

Open Problems

Ordered by estimated difficulty

- Odd Tori.
- 4-colors and more.
- Non-zero temperature.
- Low dimension.



Thank you