Setup 00000 **Examples CE** 0000000 000

Classic

Inductive 0000 KS-equations

Series-revisited

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Convergence of cluster expansions: A review of the main strategies and their relations

Collaborators: R. Bissacot (São Paulo), A. Procacci (Minas Gérais), B. Scoppola (Roma "La Sapienza")

Contributors: R. Kotecký, S. Ramawadh, A.D. Sokal, C. Temmel, D. Ueltschi

Warwick, April 9, 2014

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Basic setu	р					

The basic setup

Goal: To study systems of objects constrained only by a "non-overlapping" condition

Countable family \mathcal{P} of objects: polymers, animals, ..., characterized by

• An *incompatibility* constraint:

$$\begin{array}{ll} \gamma & \sim \gamma' & \quad ext{if } \gamma, \gamma' \in \mathcal{P} & \quad ext{incompatible} \\ \gamma & \sim \gamma' & \quad ext{if } \gamma, \gamma' \in \mathcal{P} & \quad ext{compatible} \end{array}$$

For simplicity: each polymer incompatible with itself $(\gamma \nsim \gamma, \forall \gamma \in \mathcal{P})$

• A family of *activities* $\boldsymbol{z} = \{z_{\gamma}\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$.

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Basic set	1D					

The basic ("finite-volume") measures

Defined, for each *finite* family $\mathcal{P}_{\Lambda} \subset \mathcal{P}$, by weights

$$W_{\Lambda}(\{\gamma_1, \gamma_2, \dots, \gamma_n\}) = \frac{1}{\Xi_{\Lambda}(\boldsymbol{z})} z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j < k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

for $n \ge 1$ $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{P}_{\Lambda}$, and $W_{\Lambda}(\emptyset) = 1/\Xi_{\Lambda}$, where

$$\Xi_{\Lambda}(z) = 1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n_{\Lambda}} z_{\gamma_1} z_{\gamma_2} \dots z_{\gamma_n} \prod_{j < k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

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 $\Lambda =$ some label, often finite subset of a countable set

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Correlatio	on functions					

Polymer correlation functions

For $\gamma_1, \ldots, \gamma_k$ mutually compatible polymers in \mathcal{P}_{Λ}

$$\operatorname{Prob}_{\Lambda}(\{\gamma_1,\ldots,\gamma_k \text{ are present}\}) = z_{\gamma_1}\cdots z_{\gamma_k} \frac{\Xi_{\Lambda\setminus\{\gamma_1,\ldots,\gamma_k\}^*}}{\Xi_{\Lambda}}$$

where

 $\{\gamma_1, \ldots, \gamma_k\}^*$ = polymers incompatible with $\gamma_1, \ldots, \gamma_k$

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Setup ○○○●○	Examples 0000000	CE 000000	Classic 00000	Inductive 0000	$\mathbf{KS-equations}$	Series-revisited		
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The questions:								

- Existence of the limit $\mathcal{P}_{\Lambda} \to \mathcal{P}$ ("thermodynamic limit")
- ▶ Properties of the resulting measure (mixing properties, dependency on parameters,...)

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- Asymptotic behavior of Ξ_{Λ} (analyticity!)
- ▶ Control of correlation functions

Setup ○○○○●	$\frac{\mathbf{Examples}}{0000000}$	CE 000000	Classic 00000	Inductive 0000	$\mathbf{KS-equations}$	Series-revisited
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Immediate:

 Physics: Grand-canonical ensemble of polymer gas with activities z_γ and hard-core interaction

 Statistics: Invariant measure of point processes with not-overlapping grains and birth rates z_γ

Less immediate:

- Statistical mechanical models at high and low temperatures are mapped into such systems
- ▶ More generally: most perturbative arguments in physics involve maps of this type (choice of the "right" variables)

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► Zeros of the partition functions Ξ_{Λ} (phase transitions, sphere packing, chromatic polynomials, Lovász lemma)

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Hard-core examples							

Example zero: Hard-core lattice gases

Measures on configurations $\omega \in \mathbb{L}^E$ with

- \mathbb{L} =vertices of a graph (eg. \mathbb{Z}^d),
- $E = \{0, 1\}$ ("1"=occupied)
- ▶ No occupied neighbors are allowed
- ► Allowed configurations have weights $\sim \exp(\mu\beta |\Gamma|)$ (μ =Gibbs chemical potential, β =inverse temperature)

This is a polymer model with

- $\blacktriangleright \mathcal{P} = \{ \text{vertices of } \mathbb{L} \}$
- $x \not\sim y$ iff x and y are graph neighbors

 $\triangleright \ z_x = \mathrm{e}^{\beta u}$

(For Sokal-like people *all* polymer models are of this type)

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Hard-core examples						

Single-call loss networks

Defined through the following dynamics:

- $\mathcal{P} = \text{finite subsets of } \mathbb{Z}^d$ —the *calls*
- A call γ is attempted with Poissonian rates z_{γ}
- ▶ Call succeeds if it does not intercept existing calls
- Once established, calls have an $\exp(1)$ life span

Invariant measures correspond to the polymer expansion:

• $\mathcal{P} = \text{finite connected families of links of a graph —the$ *calls*

- $z_{\gamma} =$ Poissonian rate for the call γ
- Compatibility = use of disjoint links (disjoint calls)

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Expansions from stat-mech

Low-T expansions

Ising model at low T:

- ▶ Polymers = connected closed surfaces (contours)
- ► Compatibility = no intersection
- $\blacktriangleright z_{\gamma} = \exp\{-2\beta J |\gamma|\}$

LTE for Ising ferromagnets:

- $\blacktriangleright \mathcal{P} =$ connected families of (excited) bonds (contours)
- $\blacktriangleright z_{\gamma} = \exp\{-2\beta \sum_{B \in \gamma} J_B\}$
- $\blacktriangleright \ \gamma \sim \gamma' \text{ iff } \underline{\gamma} \cap \underline{\gamma}' = \emptyset \text{ (disjoint bases); } (\underline{\gamma} = \cup \{B : B \in \gamma\})$

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High-T expansions

General HTE:

 $\blacktriangleright \mathcal{P} = \{\text{connected finite subsets of bonds}\}$

 $z_{\boldsymbol{B}} = \int_{\underline{\boldsymbol{B}}} \prod_{A \in \boldsymbol{B}} (e^{-\beta \phi_A(\omega)} - 1) \bigotimes_{x \in \underline{\boldsymbol{B}}} \mu_E(d\omega_x)$

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- $\blacktriangleright \mathcal{P} = \left\{ \boldsymbol{B} \in \mathcal{B}_{\Lambda} : \underline{\boldsymbol{B}} \text{ connected }, \sum_{B \in \boldsymbol{B}} B = \emptyset \right\} \text{ (cycles)}$
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Random geometrical models

FK representation of Potts models:

 $\mathcal{P} = \{ \gamma \subset \subset \mathbb{L} \}$ $z_{\gamma} = q^{-(|\gamma|-1)} \sum_{\substack{B \subset B_{\gamma} \\ (\gamma, B) \text{ connected}}} \prod_{\{x, y\} \in B} v_{xy}$

with $v_{xy} = e^{\beta J_{xy}} - 1$

- ► Compatibility = non-intersection
- If $v\{x, y\} = -1 \rightarrow$ chromatic polynomial $(\beta \rightarrow \infty \text{ with } J_{xy} < 0, \text{ i.e. zero-temperature}$ antiferromagnetic Potts)
- General v_{xy} : multivariate version of Tutte polynomial.

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Geometrical polymer models

Previous examples: polymers formed by points of a set These are the original polymer models of Gruber and Kunz:

- A set \mathbb{V} (sites)
- A family \mathcal{P} of finite subsets of \mathbb{V} (grains, connected sets)

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- Activity values $(z_{\gamma})_{\gamma \in \mathcal{P}}$
- The relation $\gamma \sim \gamma' \iff \gamma \cap \gamma' = \emptyset$

More generally: $\gamma = (\gamma, \text{decoration}), \gamma \subset \mathbb{V}$

$$\gamma \sim \gamma' \iff \underline{\gamma} \cap \underline{\gamma}' = \emptyset$$
$$P_{\Lambda} = \{ \gamma \in \mathcal{P} : \gamma \subset \Lambda \}, \Lambda \subset \mathbb{C}$$



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Continuous systems							

Generalization : Continuous polymer systems More generally,

$$\frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}_{\Lambda}^n} \longrightarrow \frac{1}{n!} \int_{\mathcal{P}_{\Lambda}^n} d\gamma_1 \cdots d\gamma_n$$

where $d\gamma_1 \cdots d\gamma_n$ is an appropriate product measure Also, for book-keeping purposes: $z_{\gamma} = z \xi_{\gamma}$

That is, we consider measures on $\sum_n \mathcal{P}^n_{\Lambda}$ with projections on \mathcal{P}^n_{Λ}

$$\frac{1}{\Xi_{\Lambda}} \frac{z^n}{n!} \xi_{\gamma_1} \xi_{\gamma_2} \cdots \xi_{\gamma_n} \prod_{j < k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}} d\gamma_1 \cdots d\gamma_n$$

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The expansion							

Cluster expansions

Write $\Xi_{\Lambda}(\boldsymbol{z})$ as formal exponentials of a formal series

$$\Xi_{\Lambda}(\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}_{\Lambda}^n} \phi^T(\gamma_1, \dots, \gamma_n) \, z_{\gamma_1} \dots z_{\gamma_n}\right\}$$

or

$$\Xi_{\Lambda} \stackrel{\mathrm{F}}{=} \exp\left\{\sum_{n=1}^{\infty} \frac{z^{n}}{n!} \int_{\mathcal{P}_{\Lambda}^{n}} \phi^{T}(\gamma_{1}, \dots, \gamma_{n}) \xi_{\gamma_{1}} \dots \xi_{\gamma_{n}} \, d\gamma_{1} \cdots d\gamma_{n}\right\}$$

- ▶ The series between curly brackets is the *cluster expansion*
- $\phi^T(\gamma_1, \ldots, \gamma_n)$: Ursell or truncated functions (symmetric)

- Clusters: Families $\{\gamma_1, \ldots, \gamma_n\}$ s.t. $\phi^T(\gamma_1, \ldots, \gamma_n) \neq 0$
- ▶ Clusters are *connected* w.r.t. "~"

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or

$$\Xi_{\Lambda} \stackrel{\mathrm{F}}{=} \exp\left\{\sum_{n=1}^{\infty} \frac{z^{n}}{n!} \int_{\mathcal{P}_{\Lambda}^{n}} \phi^{T}(\gamma_{1}, \dots, \gamma_{n}) \xi_{\gamma_{1}} \dots \xi_{\gamma_{n}} \, d\gamma_{1} \cdots d\gamma_{n}\right\}$$

- ▶ The series between curly brackets is the *cluster expansion*
- $\phi^T(\gamma_1, \ldots, \gamma_n)$: Ursell or truncated functions (symmetric)

- Clusters: Families $\{\gamma_1, \ldots, \gamma_n\}$ s.t. $\phi^T(\gamma_1, \ldots, \gamma_n) \neq 0$
- ▶ Clusters are *connected* w.r.t. "~'

Setup	Examples	\mathbf{CE}	Classic	Inductive	KS-equations	Series-revisited	
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The expansion							

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Strategy						

Pinned expansions

Telescoping, it is enough to consider one-polymer ratios

$$\left[\log\frac{\Xi_{\Lambda}}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\right](\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \sum_{n=1}^{\infty}\frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n_{\Lambda}\\ \exists i:\ \gamma_i=\gamma_0}} \phi^T(\gamma_1,\ldots,\gamma_n) z_{\gamma_1}\ldots z_{\gamma_n}$$

Algebraically simpler alternative:

$$egin{aligned} & \left[rac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda
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For continuous systems $\sum_{\gamma \in \mathcal{P}_n} z_{\gamma} \to z^n \int_{\mathcal{P}_n} d\gamma$

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Strategy						

Classical cluster-expansion strategy Find a A-*independent* polydisc

$$\mathcal{R} = \left\{ oldsymbol{z} : |z_{\gamma}| \leq
ho_{\gamma} \,, \, \gamma \in \mathcal{P}
ight\}$$

where cluster expansions converge *absolutely*

Equivalently, find $\rho_{\gamma} > 0$ independent of Λ such that

$$\Theta_{\gamma_0}(\boldsymbol{\rho}) := \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n \\ \exists i: \ \gamma_i = \gamma_0}} \left| \phi^T(\gamma_1, \dots, \gamma_n) \right| \ \rho_{\gamma_1} \dots \rho_{\gamma_n}$$

 or

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n} \left| \phi^T(\gamma_0, \gamma_1, \dots, \gamma_n) \right| \rho_{\gamma_1} \dots \rho_{\gamma_n}$$

are finite

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Setup 00000	$\begin{array}{c} \mathbf{Examples} \\ \texttt{0000000} \end{array}$	CE ○00●00	Classic 00000	Inductive 0000	$\mathbf{KS-equations}$	Series-revisited		
Strategy								
Consequences								

- No Ξ_{Λ} has a zero (no phase transitions!)
- Explicit series expressions for free energy and correlations
- Explicit mixing

$$\frac{\operatorname{Prob}(\{\gamma_0, \gamma_x\})}{\operatorname{Prob}(\{\gamma_0\})\operatorname{Prob}(\{\gamma_x\})} - 1 \bigg| = \bigg| e^{F[d(\gamma_0, \gamma_x)]} - 1 \bigg|$$

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with $F(d) \to 0$ as $d \to \infty$

▶ Central limit theorem

Setup 00000	Examples	CE ○00●00	Classic 00000	Inductive 0000	$\mathbf{KS-equations}$	Series-revisited	
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Setup 00000	$\frac{\mathbf{Examples}}{0000000}$	CE ○000●○	Classic 00000	Inductive 0000	$\mathbf{KS-equations}$	Series-revisited
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Observations

Due to an alternating-sign property

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Hence,

- ▶ No loss in absolute convergence
- Only loss: insisting on convergence on polydiscs
- ▶ In particular, unphysical singularity

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Approaches							

Kirkwood-Salzburg equations (1971):

- ▶ System of linear coupled equations for the correlations
- ▶ Method used by Gruber and Kunz

Classical (1982–4):

- ▶ Based on *tree-graph bound*
- Seiler \rightarrow Cammarota \rightarrow Brydges

No-cluster-expansion (1986, 1994):

- ▶ Inductive argument (no mention of expansion)
- Kotecký-Preiss \rightarrow Dobrushin

Classical revisited (2007):

- ▶ Based on *tree-graph identities* due to Penrose
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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1st ingredient							

Tree-graph bound

Classical approach valid only for geometrical translation-invariant polymers Basic inequality

$$\left|\phi^{T}(\gamma_{0},\gamma_{1},\ldots,\gamma_{n})\right| \leq \left|\mathcal{T}_{\mathcal{G}(\gamma_{0},\gamma_{1},\ldots,\gamma_{n})}\right|$$

where $\mathcal{T}_{\mathcal{G}} = \{$ connected spanning trees of $\mathcal{G}\}$ Hence:

$$\Pi_{\gamma_0}(oldsymbol{
ho}) \ \le \ \sum_{n \ge 0} rac{1}{n!} \overline{T}_n(\gamma_0)$$

where $\overline{T}_0 = 1$ and

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited		
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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1st ingred	ient					

Interchanging sum over polymers with sum over trees:

$$\overline{T}_{n}(\gamma_{0}) = \sum_{\tau \in \mathcal{T}_{n+1}^{0}} \sum_{\substack{(\gamma_{1}, \dots, \gamma_{n}) \text{ s.t.} \\ \tau \subset \mathcal{G}_{(\gamma_{0}, \gamma_{1}, \dots, \gamma_{n})}}} \rho_{\gamma_{1}} \cdots \rho_{\gamma_{n}}$$
$$= \sum_{\tau \in \mathcal{T}_{n+1}^{0}} \overline{T}_{\tau}(\gamma_{0})$$

where

$$\mathcal{T}_{n+1}^0 = \{ \text{trees of vertices } 0, 1, \dots n, \text{rooted in } 0 \}$$

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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2nd ingredient							

Summing "from leaves down"

To compute \overline{T}_{τ} start summing over γ 's at leaves:

$$\prod_{j=1}^{s_i} \sum_{\gamma_{(i,j)} \nsim \gamma_i} \rho_{\gamma_{(i,j)}} = \left[\sum_{\gamma \nsim \gamma_i} \rho_{\gamma} \right]^{s_i}$$

For translation-invariant geometrical polymers,



Then, for each γ_i that is ancestor of leaves

$$\rho_{\gamma_i} \longrightarrow \rho_{\gamma_i} |\gamma_i|^{s_i} \left[\sum_{\gamma \ni 0} \rho_{\gamma}\right]^{s_i}$$

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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2nd ingree	dient					

Iterate! The sum over successive ancestors yields

$$\overline{T}_{\tau}(\gamma_0) \leq |\gamma_0| \prod_{i=0}^n \left[\sum_{\gamma \ni 0} \rho_{\gamma} |\gamma|^{s_i} \right]$$

This bound depends only on s₀, s₁,..., s_n
 The sum over trees τ brings a factor

trees with coord. nbers

$$s_0, s_1 + 1, \dots, s_n + 1 = \binom{n}{s_0 + 1 \ s_1 \ \dots \ s_n}$$

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(Cayley formula)

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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(Cayley formula)

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Criterion						

Classical criterion

In consequence

$$\overline{T}_n(\gamma_0) \leq |\gamma_0| \ n! \sum_{\substack{s_0, s_1, \dots, s_n \\ \sum s_i = n-1}} \prod_{i=0}^n \left[\sum_{\gamma \ni 0} \rho_\gamma \ \frac{|\gamma|^{s_i}}{s_i!} \right]$$

Hence

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \leq |\gamma_0| \sum_{n \geq 0} \left[\sum_{\gamma \neq 0} \rho_\gamma \, \mathrm{e}^{|\gamma|} \right]^n$$

which converges if

$$\sum_{\gamma \ni 0} \rho_{\gamma} \, \mathrm{e}^{|\gamma|} < 1$$

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[Cammarota (1982), Brydges (1984)]

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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[Cammarota~(1982),~Brydges~(1984)]

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited		
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Ingredients								

The two stages of the inductive approach

1) The addition identity:

$$\Xi_{\Gamma \cup \{\gamma\}} = \Xi_{\Gamma} + z_{\gamma} \Xi_{\Gamma \setminus \{\gamma\}^*}$$
(1)

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for $\gamma\not\in \Gamma$

2) The right inductive hypothesis: There exist $a(\gamma) > 0$ and $\rho_{\gamma} > 0$ such that

$$\rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\right] \leq e^{a(\gamma)} - 1$$

for all γ

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Theorem						

The main result

Theorem

If there exist $a(\gamma) > 0$ and $\rho_{\gamma} > 0$ such that

$$\rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\right] \leq e^{a(\gamma)} - 1 \tag{2}$$

for all γ , then

$$\Theta^{\Lambda}_{\gamma}(-\boldsymbol{\rho}) \leq a(\gamma) \tag{3}$$

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uniformly in Λ for all γ

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Proof of the main result

Proof

Must show

$$\frac{\Xi_{\Lambda \cup \{\gamma\}}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})} \geq e^{-a(\gamma)}$$
(4)

By the addition identity

$$\frac{\Xi_{\Lambda \cup \{\gamma\}}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})} = 1 - \rho_{\gamma} \frac{\Xi_{\Lambda \setminus \{\gamma\}}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})}$$

Induction plus telescoping implies

$$\frac{\Xi_{\Lambda \cup \{\gamma\}}(-\rho)}{\Xi_{\Lambda}(-\rho)} \geq 1 - \rho_{\gamma} \exp\left[\sum_{\substack{\widetilde{\gamma} \neq \gamma \\ \widetilde{\gamma} \neq \gamma}} a(\widetilde{\gamma})\right]$$
$$\geq 1 - \rho_{\gamma} e^{-a(\gamma)} \exp\left[\sum_{\substack{\widetilde{\gamma} \neq \gamma \\ \widetilde{\gamma} \neq \gamma}} a(\widetilde{\gamma})\right]$$

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Setup	Examples	CE	Classic

Inductive ○○○● KS-equations

Series-revisited

Kotecký-Preiss criterion

Most popular: If $\exists b(\gamma) > 0$ such that

$$\sum_{\widetilde{\gamma} \not\sim \gamma} \rho_{\gamma} \, \mathrm{e}^{b(\widetilde{\gamma})} \, \leq \, b(\gamma)$$

then convergence for $|z_{\gamma}| \leq \rho_{\gamma}$

Can be proven

• By "defoliation" of Π (Procacci-Scoppola)

By an inductive argument (Kotecký-Preiss)

• Particular case of Dobrushin: If $a(\gamma) = \rho_{\gamma} e^{b(\gamma)}$, then

$$\rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \neq \gamma} a(\widetilde{\gamma})\right] \le \rho_{\gamma} \exp\left[\log \frac{a(\gamma)}{\rho_{\gamma}}\right] = a(\gamma) \le e^{a(\gamma)} - 1$$

Setup	Examples	CE	Classic					
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Kotecký-Preiss								

Inductive ○○○● KS-equations

Series-revisited

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Setup	Examples	CE	Classic
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Kotecký-Preiss

Inductive ○○○● KS-equations

Series-revisited

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Kirkwood-Salzburg approach

Strategy: Set up systems of linear equations for the functions

$$\Phi_{\Lambda}(oldsymbol{z},X) \;=\; rac{\Xi_{\Lambda\setminus X}(oldsymbol{z})}{\Xi_{\Lambda}(oldsymbol{z})}$$

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involving a basically Λ -independent operator K.

- Search for solutions in a suitable Banach space
- Solutions = fixed points
- K contraction uniform in Λ yields
 - Convergence with Λ
 - Analyticity of ratios and its limits



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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Equations						

Derivation of the equations

- ► For each $X \subset \mathcal{P}_{\Lambda}$ choose some (first) $\gamma \in X$
- Write addition identity as deletion identity, with $\Lambda \to \Lambda \setminus X$

$$\Xi_{\Lambda \setminus X} = \Xi_{\Lambda \setminus (X \setminus \{\gamma\})} - z_{\gamma} \, \Xi_{\Lambda \setminus (X \cup \{\gamma\}^*)}$$

• Dividing by Ξ_{Λ}

$$\Phi_{\Lambda}(X) = \Phi_{\Lambda}(X \setminus \{\gamma\}) - z_{\gamma} \Phi_{\Lambda}(X \cup \{\gamma\}^*)$$

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The identity Φ_Λ(Ø) = 1 is considered as inhomogenity
 The condition X ⊂ P_Λ is introduced as a factor

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Equations						

Kirkwood-Salzburg equations

The equations are:

$$\Phi_{\Lambda} = \chi_{\Lambda} \alpha + \chi_{\Lambda} K_{\boldsymbol{z}} \Phi_{\Lambda}$$

with

$$\chi_{\Lambda}(X) = \begin{cases} 1 & \text{if } X \subset \Lambda \\ 0 & \text{otherwise} \end{cases}, \quad \alpha(X) = \begin{cases} 1 & \text{if } |X| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $K_{\boldsymbol{z}}$ the operator on $\mathbb{C}^{\{\text{non-empty fin parts of } \mathcal{P}\}}$

$$(K_{\boldsymbol{z}}f)(X) = \mathbb{1}_{\{|X|\geq 2\}} f(X \setminus \{\gamma\}) - z_{\gamma} f(X \cup \{\gamma\}^*_{\Lambda})$$

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Functional analytical set-up

Standard treatment

Aiming at factorized weights, introduce norms

$$\|f\|_{a} = \sup_{X \subset \mathbb{C}^{\mathcal{P}}} \frac{|f(X)|}{\exp\left[\sum_{\widetilde{\gamma} \in X} a(\widetilde{\gamma})\right]}$$

for $a(\widetilde{\gamma}) > 0$. Then

$$|(K_{z}f)(X)| \leq ||f||_{a} \exp\left[\sum_{\substack{\widetilde{\gamma}\in X\\\widetilde{\gamma}\neq\gamma}} a(\widetilde{\gamma})\right] + |z_{\gamma}| ||f||_{a} \exp\left[\sum_{\substack{\widetilde{\gamma}\in (X\setminus\{\gamma\})\cup\{\gamma\}^{*}\\\widetilde{\gamma}\in\Lambda}} a(\widetilde{\gamma})\right]$$

and

$$\|K_{\mathbf{z}}\|_{a} \leq \frac{1}{\mathrm{e}^{a(\gamma)}} \Big[1 + |z_{\gamma}| \exp\Big(\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\Big) \Big]$$

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Series-revisited

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Convergence criterion

Gruber-Kunz condition

If for some $\rho_{\gamma} > 0$

$$\frac{1}{\mathrm{e}^{a(\gamma)}} \Big[1 + \rho_{\gamma} \exp\Big(\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma}) \Big) \Big] < 1$$
(5)

then, for $|z_{\gamma}| \leq \rho_{\gamma}$, the operators $1 - \xi_{\Lambda} K_{z}$ are invertible and

$$\Phi_{\Lambda} = \left[1 - \xi_{\Lambda} K_{z}\right]^{-1} \chi_{\Lambda} \alpha \tag{6}$$

is the only solution of the Λ -KS-equation.

Furthermore, as (5) is Λ -independent,

▶ The ratios converge

$$\Phi_{\Lambda}(X) \xrightarrow[\Lambda \to \mathbb{V}]{} \left([1 - K_{\boldsymbol{z}}]^{-1} \alpha \right)(X)$$

▶ The ratios, and their limits have analytic dependence on z

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Convergence criterion

Comparison with the inductive result

GK-condition (5) is identical to Dobrushin's

$$\rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\right] < e^{a(\gamma)} - 1$$

except that the inequality is *strict*

To fix it: alternative treatment of the KS equations

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Alternative strategy							

Alternative strategy

Find another way of making sense of the formula

$$\Phi_{\Lambda} = \left[1 - \chi_{\Lambda} K_{\boldsymbol{z}}\right]^{-1} \chi_{\Lambda} \alpha = \chi_{\Lambda} \sum_{n \ge 0} K_{\boldsymbol{z}}^{n} \chi_{\Lambda} \alpha \qquad (7)$$

This series is term-by-term dominated by

$$\Phi_{\rho} = \sum_{n \ge 0} K_{-\rho}^n \alpha$$

as long as $|z_{\gamma}| \leq \rho_{\gamma}$. (As in cluster expansions: singularities at negative fugacities)

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Alternativ	e strategy					

Inductive-like bound

Find functions $\xi(X)$ such that

$$\left(\alpha + K_{-\boldsymbol{\rho}} \,\xi\right)(X) \leq \xi(X) \tag{8}$$

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Recursively this implies that

$$\sum_{n=0}^{N} K_{-\boldsymbol{\rho}}^{n} \alpha \leq \xi$$

and hence Φ_{ρ} converges.

Reciprocally, if Φ_{ρ} is finite, (8) holds with $\xi = \Phi_{\rho}$ Conclusion:

(8) is necessary and sufficient for the convergence of Φ_{ρ}

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Alternativ	e strategy					

Why factorization

If X_1 and X_2 are disjoint,

$$\Phi_{\Lambda}(X_1 \cup X_2) = \frac{\Xi_{\Lambda \setminus (X_1 \cup X_2)}}{\Xi_{\Lambda \setminus X_2}} \frac{\Xi_{\Lambda \setminus X_2}}{\Xi_{\Lambda}} = \Phi_{\Lambda \setminus X_2}(X_1) \Phi_{\Lambda}(X_2) .$$

In the limit $\Lambda \to \mathbb{V}$ we should obtain

$$\Phi(X_1\cup X_2) = \Phi(X_1)\Phi(X_2) .$$

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It is natural, to look for factorized majorizing functions.

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited		
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Alternative strategy								

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited			
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Alternative strategy									

GK alla Dobrushin recovered

Postulating

$$\xi(X) = \prod_{\gamma \in X} \xi(\gamma)$$

(8) holds for all X iff it holds for a single-site:

 $(\alpha + K_{-\rho} \xi)(\{\gamma\}) \leq \xi(\{\gamma\})$

Writing $\xi(\{\gamma\}) = e^{a(\gamma)}$, this condition is

$$1 + \rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\right] \leq e^{a(\gamma)}$$

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as obtained via the inductive argument

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited			
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Alternative strategy									

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$$(\alpha + K_{-\rho} \xi)(\{\gamma\}) \leq \xi(\{\gamma\})$$

Writing $\xi(\{\gamma\}) = e^{a(\gamma)}$, this condition is

$$1 + \rho_{\gamma} \exp\left[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\right] \leq e^{a(\gamma)}$$

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as obtained via the inductive argument

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Comparison							

Comparison so far

Classical < KP < Inductive = KS (GK)

However, alternative KS leads to a sequence of bounds:

$$\left|\frac{\Xi_{\Lambda\setminus X}(\boldsymbol{z})}{\Xi_{\Lambda}(\boldsymbol{z})}\right| \ \le \ \frac{\Xi_{\Lambda\setminus X}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})} \ \le \ \left(\mathbb{T}_{\boldsymbol{\rho}}\right)^m \boldsymbol{\xi}^X \ \le \ \left(\mathbb{T}_{\boldsymbol{\rho}}\right)^n \boldsymbol{\xi}^X \ \le \ \boldsymbol{\xi}^X$$

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for all $m \leq n$



Comparison so far

Classical
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$$\left|rac{\Xi_{\Lambda\setminus X}(oldsymbol{z})}{\Xi_{\Lambda}(oldsymbol{z})}
ight|\ \le\ rac{\Xi_{\Lambda\setminus X}(-oldsymbol{
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ho})}\ \le\ ig(\mathbb{T}_{oldsymbol{
ho}}ig)^moldsymbol{\xi}^X\ \le\ oldsymbol{\xi}^X$$

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for all $m \leq n$

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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"Standard form" of the criteria

If we substitute

$$\mu_{\gamma} = \rho_{\gamma} e^{a_{\gamma}}$$
 (Kotecký-Preiss)
$$\mu_{\gamma} = e^{a_{\gamma}} - 1$$
 (Dobrushin)

We obtain convergence if there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\begin{split} \rho_{\gamma_0} \; \exp \Bigl[\sum_{\gamma \approx \gamma_0} \mu_{\gamma} \Bigr] \; &\leq \; \mu_{\gamma_0} \quad \text{(Kotecký-Preiss)} \\ \rho_{\gamma_0} \; \prod_{\gamma \approx \gamma_0} \Bigl(1 + \mu_{\gamma} \Bigr) \; &\leq \; \mu_{\gamma_0} \quad \text{(Dobrushin)} \end{split}$$



$\textbf{Comparison } \mathbf{D} \leftrightarrow \mathbf{KP}$

D improves KP because

$$\prod_{\gamma \not \sim \gamma_0} (1 + \mu_{\gamma}) \leq \exp \left[\sum_{\gamma \not \sim \gamma_0} \mu_{\gamma} \right]$$

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Differences:

- D lacks powers μ_{γ}^{ℓ}
- ▶ D exact for polymers with only self-exclusion



$\textbf{Comparison } \mathbf{D} \leftrightarrow \mathbf{KP}$

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			Obser	rvations		

- ▶ It looks as a hierarchy of approximations
- ▶ Why induction better than control of explicit series?
- ▶ Dobrushin extracts extra information Which one?
- ▶ Why the form

 $\rho_{\gamma_0} \varphi_{\gamma_0}(\boldsymbol{\mu}) \leq \mu_{\gamma_0}?$

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As for KS, is there some fix point of positive series?

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As for KS, is there some fix point of positive series?

Setup	Examples	CE	Classic	Inductive	KS-equations
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Series-revisited

1st ingredient

Classical revisited: 1st ingredient Penrose identity

$$\left|\phi^{T}(\gamma_{0},\gamma_{1},\ldots,\gamma_{n})\right| = \left|\mathcal{T}^{\mathrm{Pen}}_{\mathcal{G}(\gamma_{0},\gamma_{1},\ldots,\gamma_{n})}\right|$$

A Penrose tree for G_(γ0,...,γn) is a spanning tree s.t.
(P1) Brothers are compatible
(P2) Cousins are compatible
(P3) Nephews compatible with uncles with smaller in Hence, now

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) = \sum_{n \ge 0} \frac{1}{n!} \overline{T}_n(\gamma_0)$$

$$T_n(\gamma_0) = \sum_{(\gamma_1, \dots, \gamma_n)} \sum_{\tau \in \mathcal{T}_{\mathcal{G}}(\gamma_0, \gamma_1, \dots, \gamma_n)} \rho_{\gamma_1} \cdots \rho_{\gamma_n}$$

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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1st ingred	ent					

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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1st ingredient							

Approximation

Retain only (P1): Brothers may not be linked in \mathcal{G}

If $\{i, i_1\}$ and $\{i, i_2\}$ are edges of τ , then $\gamma_{i_1} \sim \gamma_{i_2}$

In this way $\boldsymbol{\rho} \boldsymbol{\Pi}(\boldsymbol{\rho}) \leq \boldsymbol{\rho}^*$, with

$$\rho_{\gamma_0}^* := \rho_{\gamma_0} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n} \sum_{\tau \in \mathcal{T}_n^0} \prod_{i=0}^n c_{s_i}(\gamma_i, \gamma_{i_1}, \dots, \gamma_{i_{s_i}}) \rho_{\gamma_{i_1}} \dots \rho_{\gamma_{i_{s_i}}} \right]$$

where i_1, \ldots, i_{s_i} = descendants of i and

$$c_n(\gamma_0, \gamma_1, \dots, \gamma_n) = \prod_{i=1}^n \mathbb{1}_{\{\gamma_0 \sim \gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i \sim \gamma_j\}}$$

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited	
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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2nd ingr	edient					

2nd ingredient: Iterative generation of trees Consider the function $T_{\rho}: [0,\infty)^{\mathcal{P}} \to [0,\infty]^{\mathcal{P}}$ defined by

$$\left(\boldsymbol{T}_{\boldsymbol{
ho}}(\boldsymbol{\mu})\right)_{\gamma_0} =
ho_{\gamma_0} \left[1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n} c_n(\gamma_0, \gamma_1, \dots, \gamma_n) \, \mu_{\gamma_1} \dots \mu_{\gamma_n}\right]$$

or

$$T_{
ho}(\mu) =
ho \, arphi(\mu)$$

Diagrammatically:



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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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2nd ingred	ient					

The diagrams of the series

$$T_{oldsymbol{
ho}}(T_{oldsymbol{
ho}}(oldsymbol{\mu})) \;=\; T_{oldsymbol{
ho}}^2(oldsymbol{\mu})$$

have black dots replaced by each of the preceding diagrams.

 $T^2_{
ho}(\mu) = ext{sums over trees of up to two generations}$ with • in 2nd generation

 $T^n_{
ho}(\mu) = ext{sums over trees of up to } n ext{ generations}$ with \bullet in n-th generation

and

$$T^n_{oldsymbol{
ho}}(oldsymbol{
ho}) \stackrel{ extsf{/}}{_{n o \infty}} oldsymbol{
ho}^*$$

Alternatively, ρ^* generated by replacing $\bullet \rightarrow \rho^*$:

$$ho^* \;=\;
ho \, arphi(
ho^*) \qquad ext{or} \qquad
ho^* \;=\; T_
ho(
ho^*)$$

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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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2nd ingred	liont					

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- $T^2_{
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and

Alternatively, ρ^* generated by replacing $\bullet \to \rho^*$: $\rho^* = \rho \, \varphi(\rho^*) \quad \text{or} \quad \rho^* = T_{\rho}(\rho^*)$

Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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$$T^n_{oldsymbol{
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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Setup	Examples	CE	Classic	Inductive	KS-equations	Series-revisited
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ho}^* \;=\; \boldsymbol{T}_{\!\boldsymbol{
ho}}(\boldsymbol{
ho}^*)$$

Setup	Examples	CE	Classic
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3rd ingredient

Inductive 0000 KS-equations

Series-revisited

Convergence of a positive series

 $T^n_{
ho}(
ho)$ converges *if, and only if,* exists μ s.t.

$$T_{\rho}(\mu) \leq \mu \tag{9}$$

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Indeed, (9) + positiveness $\Rightarrow T_{\rho}^{n}(\mu)$ decreasing and bdd below

 $0 \leq (\boldsymbol{\rho}^* \leq) \boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\mu}) \leq \cdots \leq \boldsymbol{T}_{\boldsymbol{\rho}}^2(\boldsymbol{\mu}) \leq \boldsymbol{\mu}$

Reciprocally, if there is convergence (9) holds for $\mu = \rho^*$



Convergence of a positive series

 $T^n_{\rho}(\rho)$ converges *if*, and only *if*, exists μ s.t.

$$T_{\rho}(\boldsymbol{\mu}) \leq \boldsymbol{\mu} \tag{9}$$

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Indeed, (9) + positiveness $\Rightarrow T_{\rho}^{n}(\mu)$ decreasing and bdd below $0 \leq (\rho^{*} \leq) T_{\rho}^{n}(\mu) \leq \cdots \leq T_{\rho}^{2}(\mu) \leq \mu$

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Setup	Examples	CE	Classic	Inductive	KS-equati
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Series-revisited 0000000000

Revisited classical criterion

New criterion

For

$$c_n(\gamma_0, \gamma_1, \dots, \gamma_n) = \prod_{i=1}^n \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i \sim \gamma_j\}}$$
$$\left(T_{\rho}(\boldsymbol{\mu})\right)_{\gamma_0} = \rho_{\gamma_0} \left[1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{\substack{(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n \\ \gamma_0 \nsim \gamma_i, \gamma_i \sim \gamma_j, 1 \le i, j \le n}} \mu_{\gamma_1} \dots \mu_{\gamma_n}\right]$$
$$= \rho_{\gamma_0} \Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})$$

$$ho_{\gamma_0} \, \Xi_{\{\gamma_0\}^*}(oldsymbol{\mu}) \, \leq \, \mu_{\gamma_0}$$

$$ho \, \Pi \ \leq \
ho^* \ \leq \ T^{n+1}_
ho(\mu) \ \leq \ T^n_
ho(\mu) \ \leq \ \mu$$

Same for continuous polymers with $\sum_{\gamma \in \mathcal{P}_n} \mathbb{Z}_{\gamma} \xrightarrow{z_{\gamma}} \mathbb{Z}_{\gamma} \xrightarrow{z_{\gamma}} \mathbb{Z}_{\gamma}$

Setup	Examples	CE	Classic	Inductive	KS-equations
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Series-revisited

Revisited classical criterion

New criterion

For

$$c_n(\gamma_0, \gamma_1, \dots, \gamma_n) = \prod_{i=1}^n \mathbb{1}_{\{\gamma_0 \approx \gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i \approx \gamma_j\}}$$
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$$= \rho_{\gamma_0} \Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})$$

Hence, convergence if

$$\rho_{\gamma_0} \, \Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu}) \, \leq \, \mu_{\gamma_0}$$

Furthermore,

$$ho \, oldsymbol{\Pi} \ \le \ oldsymbol{
ho}^* \ \le \ oldsymbol{T}_{oldsymbol{
ho}}^{n+1}(oldsymbol{\mu}) \ \le \ oldsymbol{T}_{oldsymbol{
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Same for continuous polymers with $\sum_{\gamma \in \mathcal{P}_n} z_{\gamma} \xrightarrow{z_{\gamma}} z_{\gamma}^n \int_{\mathcal{P}_n} d\gamma$

Setup	Examples	CE	Classic	Inductive	KS-equations
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Series-revisited

Revisited classical criterion

New criterion

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$$= \quad \rho_{\gamma_0} \Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})$$

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ho}^* \ \le \ oldsymbol{T}_{oldsymbol{
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Setup	Examples	CE	Classic	Inductive	KS-equations
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Series-revisited

Revisited classical criterion

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$$= \rho_{\gamma_0} \Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})$$

Hence, convergence if

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Furthermore,

$$ho \Pi \leq
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ho}(m\mu) \leq T^n_{
ho}(m\mu) \leq m\mu$$

Same for continuous polymers with $\sum_{\gamma \in \mathcal{P}_n} z_{\gamma} \xrightarrow{} z_{\gamma} \xrightarrow{} \int_{\mathcal{P}_n} d\gamma$

SetupExamplesCEClassicInductiveKS-ec0000000000000000000000000000000

KS-equations

Series-revisited

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Explanation and comparison

Explanation of the different criteria

If we replace $\gamma_i \sim \gamma_j$ by the weaker requirement $\gamma_i \neq \gamma_j$:

$$\begin{aligned} c_n^{\text{Dob}}(\gamma_0, \gamma_1, \dots, \gamma_n) \ &= \ \prod_{i=1}^n \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i \neq \gamma_j\}} \\ & \left(\boldsymbol{T}_{\boldsymbol{\rho}}^{\text{Dob}}(\boldsymbol{\mu}) \right)_{\gamma_0} \ &= \ \rho_{\gamma_0} \ \prod_{\gamma \nsim \gamma_0} (1 + \mu_{\gamma}) \end{aligned}$$

If requirement $\gamma_i \approx \gamma_j$ is ignored altogether,

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ight)_{\gamma_0} \,=\,
ho_{\gamma_0} \, \expiggl[\sum_{\gamma pprox \gamma_0} \mu_\gammaiggr] \end{aligned}$$

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KS-equations

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Explanation and comparison

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$$c_n^{\mathrm{KP}}(\gamma_0, \gamma_1, \dots, \gamma_n) = \prod_{i=1}^n \mathbb{1}_{\{\gamma_0 \approx \gamma_i\}}$$
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Comparison classical revisited \leftrightarrow inductive

The improvement is expressed by the inequality

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \leq \prod_{\gamma \nsim \gamma_0} (1 + \mu_{\gamma})$$

LHS contains only monomials of *mutually compatible* polymers Sources of improvement:

- (I1) $\Xi_{\{\gamma_0\}^*}$ has no triangle diagram (i.e. pairs of neighbors of γ_0 that are themselves neighbors)
- (12) In $\Xi_{\{\gamma_0\}^*}$, the only monomial containing μ_{γ_0} is μ_{γ_0} itself, $(\gamma_0 \text{ is incompatible with all other polymers in } \{\gamma_0\}^*)$

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Explanation and comparison

Directions for further research

- ▶ Incorporation of additional constraints in Penrose trees
- ▶ Use of other partition schemes
- ▶ Inductive proof?
- Extension to polymers with soft interactions (in progress)
- ▶ Uncountably many polymers (eg. quantum contours)
- ▶ Revisit "classical" results based on cluster expansions

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Part II

Alternative probabilistic scheme

The alternative treatment has the following features:

- ▶ It is probabilistic, hence only positive activities
- ▶ Basic measures = invariant measures for point processes
- ▶ Larger region of validity, but no analyticity
- ▶ Yields a "universal" perfect simulation scheme

Process

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Probabilistic approach (with P. Ferrari and N. Garcia)

Basic measures are invariant for the following dynamics:

- Attach to each polymer γ a poissonian clock with rate z_{γ}
- When the clock rings, γ tries to be born
- It succeeds if no other γ' present with $\gamma \nsim \gamma'$
- Once born, the polymer has an $\exp(1)$ lifespan

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Alternative scheme

1st step: free process

- ▶ Generate first a *free process* where *all* birth are succesful
- \blacktriangleright Associate to each born polymer γ a space-time cylinder

$$C^{\gamma} = \left(\gamma, [\operatorname{Birth}_{C^{\gamma}}, \operatorname{Death}_{C^{\gamma}}]\right)$$

2nd step: cleaning

To decide whether a given cylinder C^{γ} remains a live, determine its *clan of ancestors*

$$A_{1}(C^{\gamma}) = \left\{ C' : \operatorname{Base}_{C'} \approx \gamma, \operatorname{Birth}_{C^{\gamma}} \in [\operatorname{Birth}_{C'}, \operatorname{Death}_{C'}] \right\}$$
$$A_{n+1}(C^{\gamma}) = A_{1}(A_{n}(C^{\gamma}))$$
$$A(C^{\gamma}) = \bigcup_{n} A_{n}(C^{\gamma})$$

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Forward-forward scheme

If $\boldsymbol{A}(C^{\gamma})$ is finite. do the cleaning starting from the "mother cylinder"

- ▶ Keep mother
- ▶ Erase first children
- ▶ Keep new mothers

This is a *forward-forward* scheme

Backward-forward scheme

Ancestors clan can be constructed backwards (Poisson and exponential distributions are reversible)

To construct the clan of ancestors of a finite window Λ :

- Generate, backwards, marks at rate $z_{\gamma} e^{-s}$ for each $\gamma \nsim \Lambda$
- These are cylinders born at -s and surviving up to 0
- Take the first mark; ignore the rest. If its basis is γ_1
- ▶ Repeat with

$$\begin{split} \Lambda &\to & \Lambda \cup \{\gamma_1\} \\ s &\to & s - \begin{cases} \text{Birth}_{\gamma_1} & \text{if } \gamma \nsim \gamma' \\ 0 & \text{if } \gamma \nsim \Lambda, \gamma \sim \gamma_1 \end{cases} \end{split}$$

 $\blacktriangleright \cdots \longrightarrow \mathbf{A}^{\Lambda}$

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Perfect simulation

If

$$\mathbb{P}(\{\mathbf{A}^{\Lambda} \text{ finite}\}) = 1 \tag{10}$$

cleaning leads $\mathit{exactly}$ to a sample of the basic measure

Sufficient conditions for (10)?

- ▶ Clan of ancestors defines an *oriented percolation model*
- Lack of percolation $\implies (10)$
- Can dominate by a branching process:
 - \blacktriangleright branches = ancestors
 - ▶ branching rate = mean surface-area of cylinders:

$$\frac{1}{|\gamma|} \sum_{\theta \nsim \gamma} |\theta| \, z_{\theta} \, \times \, 1$$

(geometrical case)

Extinction condition

Extinction of the branching process implies (10)

Hence, perfect simulation if

$$\frac{1}{|\gamma|} \sum_{\theta \nsim \gamma} |\theta| \, z_{\theta} \, \leq \, 1$$

Under this condition

- $\operatorname{Prob} = \lim_{\Lambda} \operatorname{Prob}_{\Lambda}$ exists
- Mixing properties

 $\left|\operatorname{Prob}(\{\gamma_0,\gamma_1\}) - \operatorname{Prob}(\{\gamma_0\})\operatorname{Prob}(\{\gamma_1\})\right| \leq e^{-M\operatorname{dist}(\gamma_0,\gamma_1)}$

• CLT: If A depends on a finite # of polymers

$$\frac{1}{\sqrt{\Lambda}} \sum_{x \in \Lambda} \mathbb{1}_{\{A+x\}} \xrightarrow{\Lambda} \mathcal{N}(0, D)$$

with $D = \sum_{x} \operatorname{Prob}(A \cup A + x)$

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