The Loop O(n) Model

Joint with Hugo Duminil-Copin
Ron Peled
Wojciech Samotij

Hexagonal Lattice

H - suitable domain
(finite and connected)

Def: a loop a subgraph of H
isomorphic to a cycle

Def: a subset $\omega \subset E(H)$
is called a loop config
if $\deg_\omega(v) \in \{0, 2\}$ $\forall v$

if $\deg_\omega(v) \in \{0,2\}$ $\forall v$

Parameters

$x>0$ – edge-weight

$n>0$ – loop-weight

Probability distribution on

the set of loop-configs

$$\mathbb{P}(\omega) = \mathbb{P}_{H,n,x}(\omega) = \frac{1}{Z} x^{\#\text{edges}(\omega)} n^{\#\text{loops}(\omega)}$$

Taking $x \to \infty$ we led to

Def: a loop-config ω is called fully-packed

if $\deg_\omega(v) = 2$ $\forall v$

Prob distr on the set

of fully-packed loop-confs

$$\mathbb{P}(\omega) = \mathbb{P}_{H,n,\infty}(\omega) = \frac{1}{Z} n^{\#\text{loops}(\omega)}$$

$$= \sum_{\substack{w \\ \text{loop-config}}} \beta^{\#\text{edges}(w)} n^{\#\text{loops}(w)}$$

The Spin $O(n)$ Model

n – positive integer

$n=1 \longrightarrow$ Ising Model

$n=2 \longrightarrow$ XY model

$n=3 \longrightarrow$ Heisenberg

$\beta \geq 0$ – inverse temp

Def: a collection $S=(S_v)_v$ of spins $S_v \in \sqrt{n}\,\mathbb{S}^{n-1}$ is a spin-config

Prob. distr on the set of spin-config with desity

$$\frac{1}{Z} e^{\beta \sum_{u \sim v} \langle S_u, S_v \rangle} dS$$

$$dS = dS_1 \cdots dS_{|V|}$$

Cor: For $n \geq 3$ there is no phase transition

(exp decay of correlations)

the partition function

$$Z = \int e^{\beta \sum_{u \sim v} \langle S_u, S_v \rangle} dS = \int \prod_{u \sim v} e^{\beta \langle S_u, S_v \rangle} dS$$

$(e^t \approx 1 + t)$

$$\approx \int \prod_{u \sim v} (1 + \beta \langle S_u, S_v \rangle) dS$$

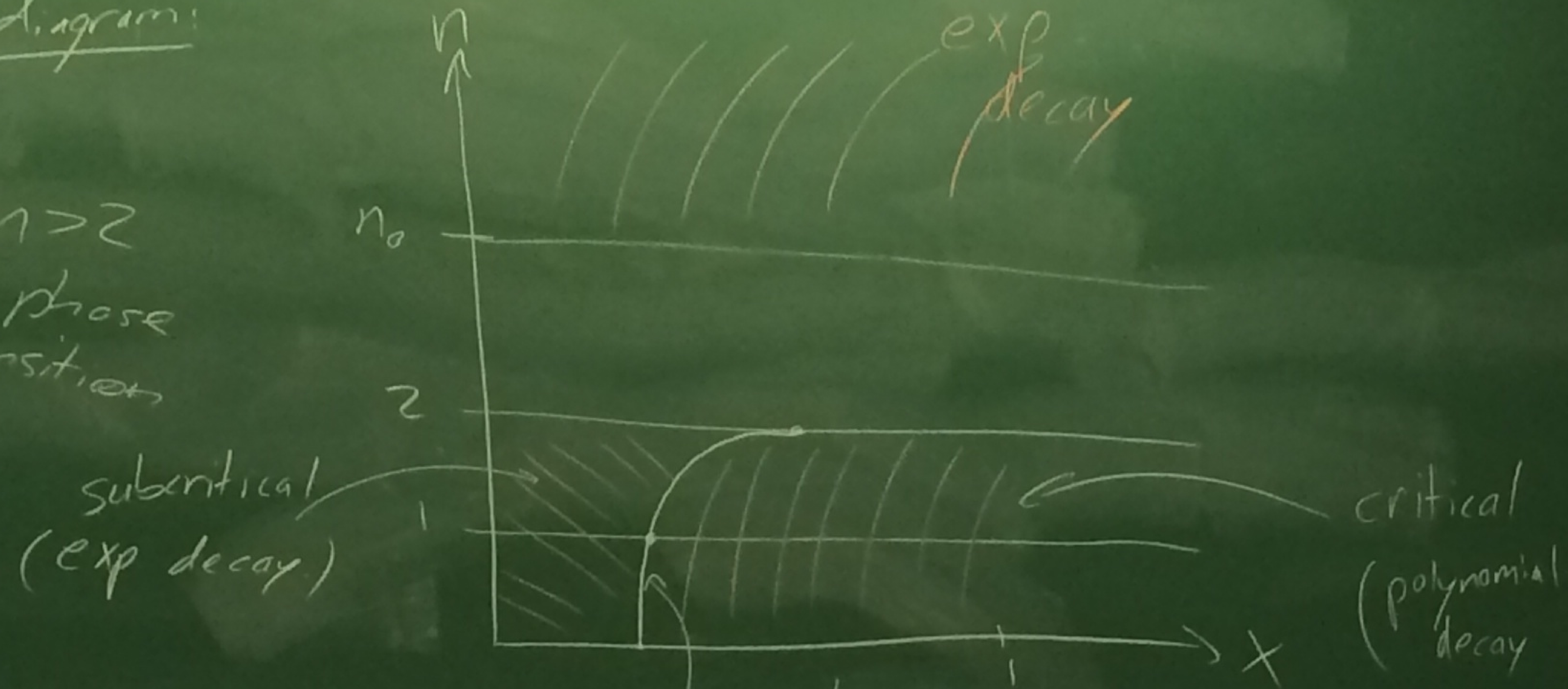
$$= \sum_{w \in E(H)} \beta^{|w|} \int \prod_{(u,v) \in w} \langle S_u, S_v \rangle dS = \begin{cases} 0 & \text{if } w \text{ is not a loop-config} \\ n^{\#loops(w)} & \text{otherwise} \end{cases}$$

$$= \sum_{w \text{ loop-config}} \beta^{\#edges(w)} n^{\#loops(w)}$$

(*) The size of the loop passing through the origin

<u>Phase diagram:</u>

<u>conj</u>: $n>2$
no phase transition



$$x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$$

$\sqrt{2+\sqrt{2-n}}$

Thm (no large loops)

$\exists\, C, n_0 > 0 \quad \forall n \geq n_0 \quad \forall x \in (0, \infty] \quad \exists\, \alpha = \alpha(n,x) > 0 \quad \forall H\text{-suitable}$

$$\mathbb{P}\left(\exists \text{ loop of length } k \text{ surrounding the origin}\right) \leq C e^{-\alpha k}$$

Joint with Hugo Duminil-Copin
Ron Peled
Wojciech Samotij

Suitable domain

$\mathbb{H}$ - hexagonal lattice

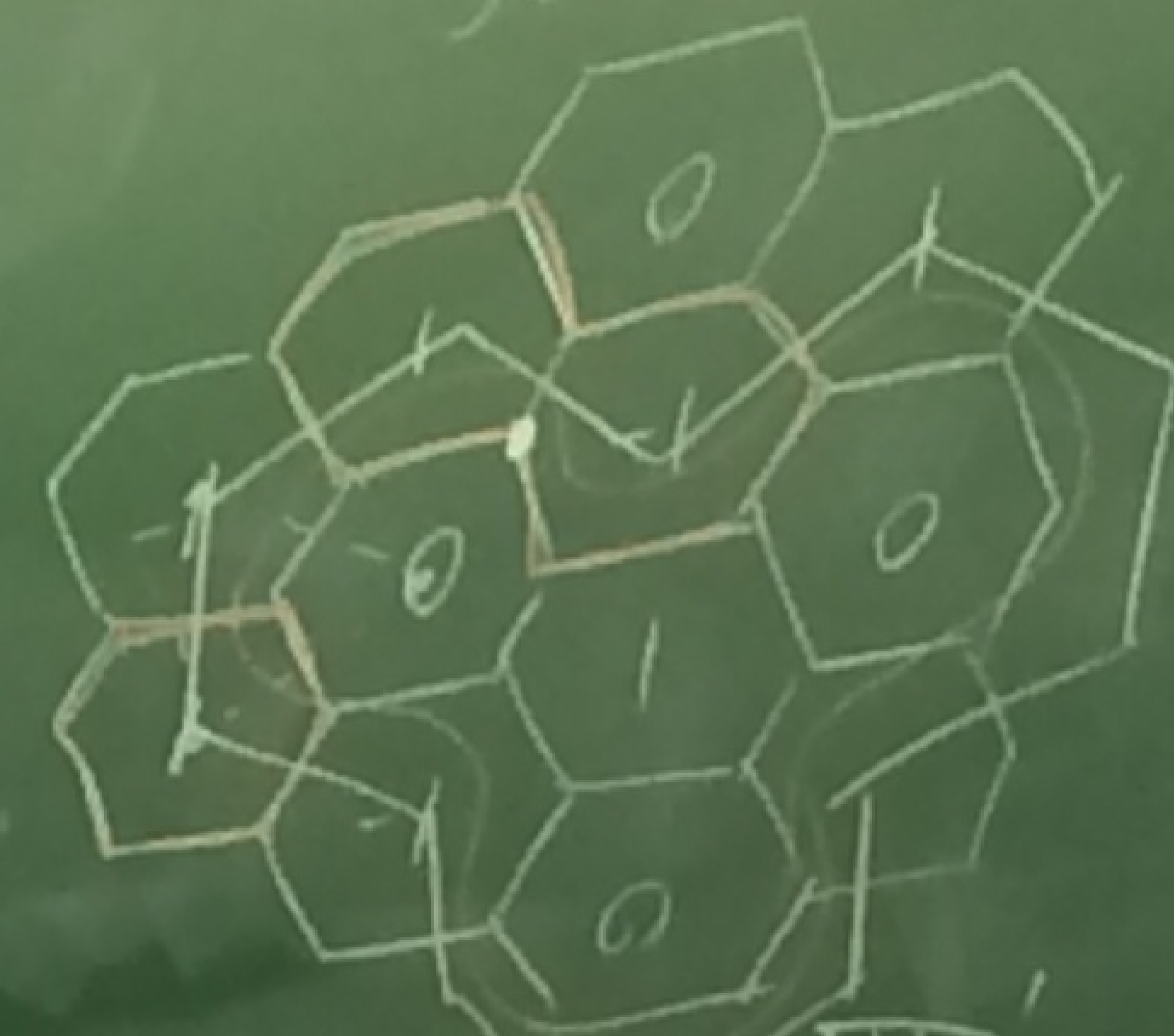
$\mathbb{T} = \mathbb{H}'$ - triangular lattice

$\gamma \subset \mathbb{T}$ simple closed path

$H =$ "inside of γ"

restriction: γ does not pass through $\mathbb{T}^0$

$\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$

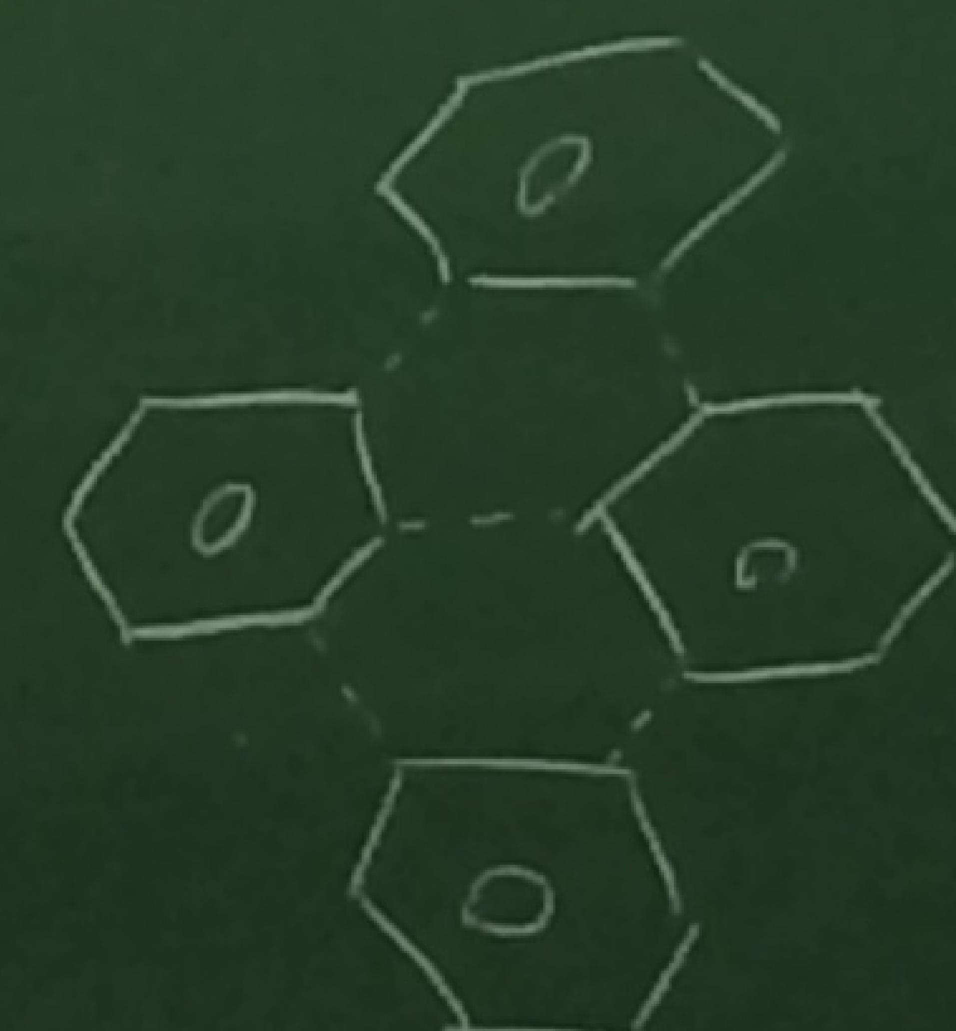


$\mathbb{T} = \mathbb{T}^0 \cup \mathbb{T}^1 \cup \mathbb{T}^{-1}$

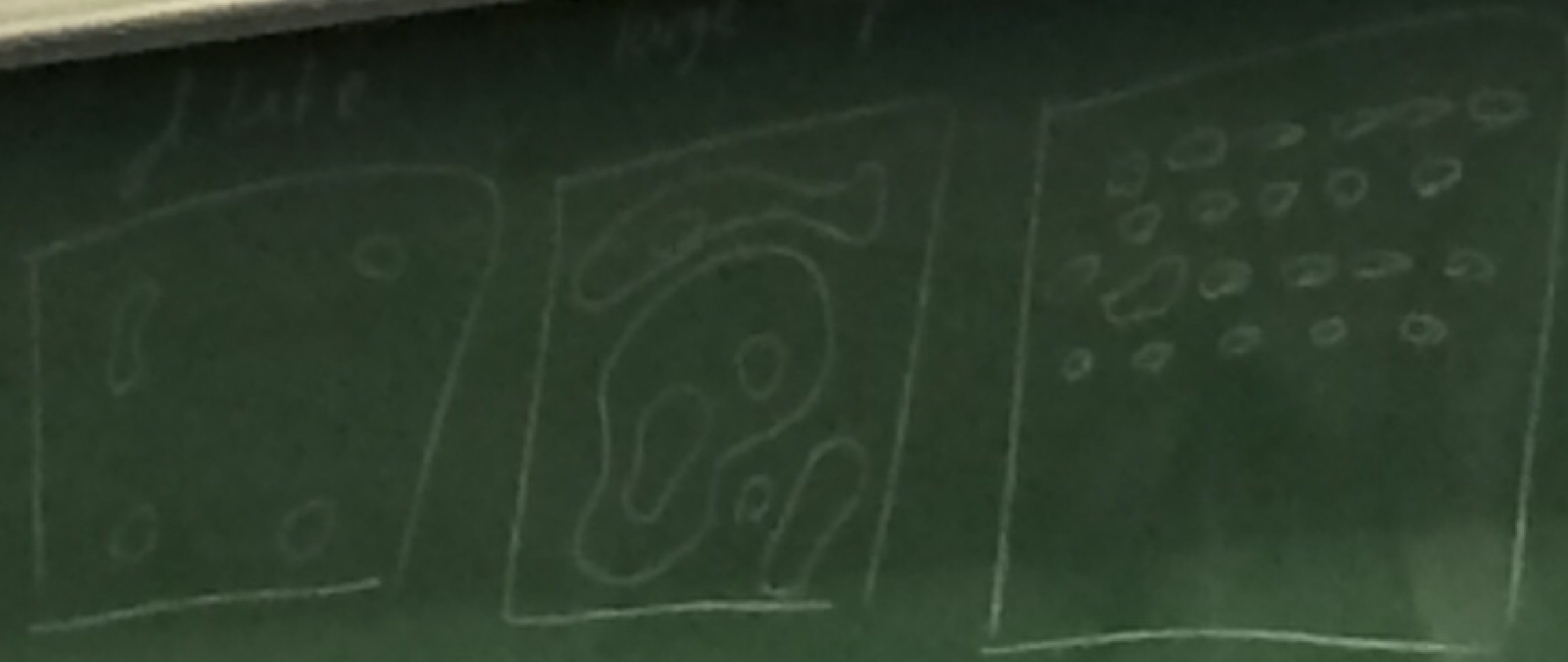
ground state:

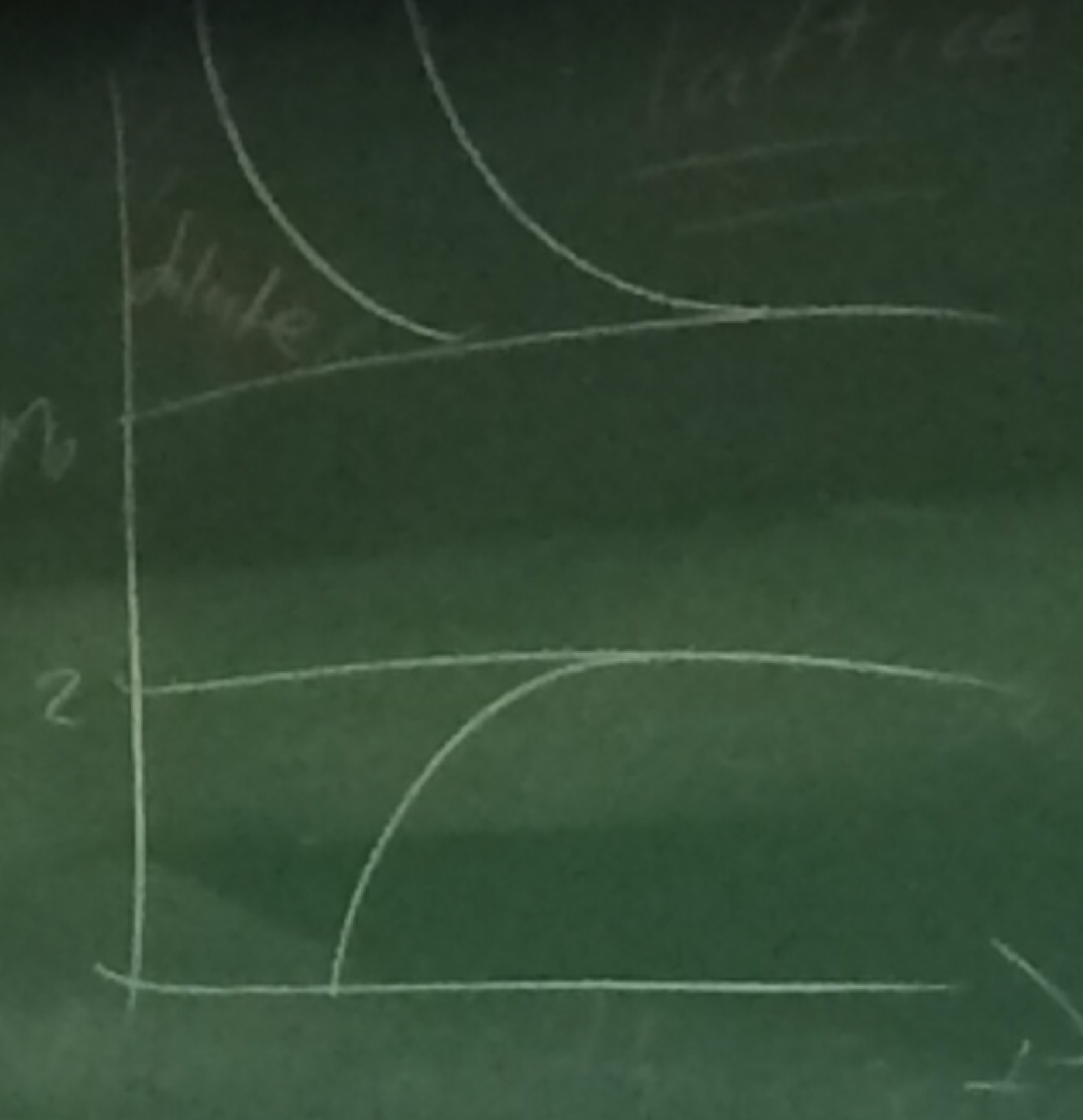
fully-packed loop config

maximizes $\nearrow$ #edges
$\searrow$ #loops



3 types of pictures (qualitative)





Thm (ordered/disordered)

$\exists C, c, n_0 > 0 \quad \forall n \geq n_0 \quad \forall x \in (0, \infty]$

(1) If $nx^6 \geq C$ then $\rightarrow$ ordered/lattice loops

(2) If $nx^6 \leq c$ then disordered/dilute

$$= \sum_{\substack{w \\ \text{loop-config}}} \beta^{\#\text{edges}(w)} n^{\#\text{loops}(w)}$$

<u>ordered</u> → $\mathbb{P}\left(\begin{array}{l}\text{origin is surrounded} \\ \text{by a trivial loop}\end{array}\right) \geq 0.99$

in particular, there exist at least $\exists$ Gibbs state.

disordered → unique Gibbs state arising from <u>vacant</u> boundary conditions

Open Questions

(1) Small n?

(2) Spin $O(n)$ model?

(3) Gap between $nx^6 = c, C$

unique trans point?

Number of Gibbs states?