Open problems from
Random walks on graphs and potential theory

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Abstract

The following open problems were posed by attendees (or non attendees by proxy) of the conference *Random walks on graphs and potential theory* held at the University of Warwick from 18th to 22nd of May 2015.

1 Stephan Wagner #1

Let G be a finite connected graph. For any two vertices u and v, the path between u and v in the uniform spanning tree on G is equivalent to a loop-erased random walk. The average length of this path is given by

$$\mathrm{ad}_G(u,v) = \frac{1}{\tau(G)} \sum_T d_T(u,v),$$

where $\tau(G)$ is the number of spanning trees of G, the sum is over all spanning trees T, and d_T is the distance in the particular tree. One might also consider the global quantity:

$$\mathrm{aad}_G = \frac{1}{\binom{|G|}{2}} \sum_{u,v} \mathrm{ad}_G(u,v).$$

Question 1. *Is there an efficient way to compute* ad_G *or* aad_G*, e.g. in the same way that hitting times of the ordinary random walk can be expressed in terms of effective resistances?*

Question 2. *How large can*

$$A_G(u,v) = \frac{\mathrm{ad}_G(u,v)}{d_G(u,v)}$$

be? (NB. d_G *is the distance in G.) Similarly, how large can*

$$B_G = \frac{\mathrm{aad}_G}{\frac{1}{\binom{|G|}{2}} \sum_{u,v} d_G(u,v)}$$

be?

It is trivial that both $A_G(u,v)$ and B_G are equal to 1 if G is a tree itself. Moreover, if G is the complete graph, one has that $B_G = \Theta(\sqrt{|G|})$. SW conjectured that $B_G = O(\sqrt{|G|})$ for any graph, but also gave an example demonstrating that $A_G(u,v) \neq O(\sqrt{|G|})$. (For the example given by SW, it was noted by Ori Gurel-Gurevich that $A_G = \Theta(\sqrt{|E|})$.) However, this was disproved by Endre Csóka during the workshop, who described a construction that gives $B_G \geq |G|^{3/4-\epsilon}$. Is this best possible?

2 Stephan Wagner #2

If $(\lambda_i)_{i=1}^n$ are the eigenvalues of the Laplacian on a finite weighted graph, then it is known that

$$\sum_{i=2}^{n} \frac{1}{\lambda_i} = \frac{1}{n} \sum_{\{u,v\} \subset V} r(u,v),$$

where $r(u,v)$ is the resistance between u and v. (The right-hand side here is sometimes referred to as the Kirchhoff index of the graph.) For example, if the graph is a tree, then

$$\sum_{i=2}^{n} \frac{1}{\lambda_i} = \frac{1}{n} \sum_{\{u,v\} \subset V} d_T(u,v),$$

where d_T is the weighted graph distance. In the continuous setting, consider the Sturm-Liouville problem on $[0,1]$,

$$(fu')' + \lambda u' = 0,$$

with boundary conditions $u'(0) = u'(1) = 0$. The corresponding eigenvalues satisfy

$$\sum_{\lambda \neq 0} \frac{1}{\lambda} = \int_0^1 \frac{x(1-x)}{f(x)} dx. \tag{1}$$

Question 3. *Is there a general theory that explains the latter result?*

Ben Hambly gave a proof of (1) making use of the heat kernel that seems to generalise to quantum graphs and possibly also domains in $\mathbb{R}^d$.

3 Omer Angel

First, let $M_1 := (\mathbb{Z}, E_1)$, $M_2 := (\mathbb{Z}, E_2)$ be independent, translation invariant matchings of $\mathbb{Z}$ (these might have the same or different distributions). Then take the graph which is the union of the two matchings,

$$G = (\mathbb{Z}, E_1 \cup E_2),$$

and define N to be the number of infinite paths in G.

Question 4. *Is it possible that*

$$\mathbb{P}(N = 1) = 1?$$

Example. *If we select M_1, M_2 uniformly from the set of non-crossing matchings, then $N \in \{0, 1\}$ almost-surely.*

Conjecture 1. *In the setting of the previous example, $N = 0$ almost-surely.*

Example. *If M_1, M_2 are selected uniformly from (the two possible) nearest neighbour matchings, then*

$$\mathbb{P}(N = 0) = \frac{1}{2}, \qquad \mathbb{P}(N = 1) = \frac{1}{2}.$$

Example. *Consider site percolation on the hexagonal lattice (i.e. cells coloured independently green and yellow with probabilities p and $1-p$ respectively). Moreover, assume the boundary condition along the horizontal axis is that the cells alternate green and yellow. The percolation interfaces in the upper and lower half plane then generate independent matchings of $\mathbb{Z}$. At criticality, $p = 1/2$, it holds that $N = 0$ almost-surely.*

4 Itai Benjamini (in absentia by proxy of Omer Angel)

Take some locally finite initial configuration of points on the real line and define some potential $F(x, y)$ to be the force between two points x, y, for example

$$F(x, y) := |x - y|^{-2}.$$

Letting $(a_n)_{n\in\mathbb{Z}}$ be the positions of the points, we say that the system is said to be in equilibrium if

$$\sum_{i \neq n} F(a_i, a_n)\text{sgn}(a_i - a_n) = 0, \qquad \forall n.$$

For a sequence $(a_n)_{n\in\mathbb{Z}}$ in arithmetic progression the total force is 0, this motivates the following question:

Question 5. *If $(a_n)_{n\in\mathbb{Z}}$ is in equilibrium, is*

$$a_n = \alpha n + \beta$$

for all n and some $\alpha, \beta \in \mathbb{R}$?

5 Antoine Gournay

Take Γ to be a transient Cayley graph of some group $\mathcal{G}$, and $\mathcal{P}$ to be the transition operator of a simple random walk on Γ. Define the green kernel of $\mathcal{P}$ to be

$$G(x, y) := \sum_{m \geq 0} \mathcal{P}_x^m(y).$$

where $\mathcal{P}_x^m(y)$ is the probability to get to y in m steps when starting from x. A metric (introduced by S. Brofferio) is obtained from the Green kernel by the following

$$d(x, y) := -\log\left(\frac{G(x, y)}{G(x, x)}\right).$$

The value inside the log can be interpreted as the probability to hit y when starting at x. For a subset F of vertices, let ∂F be the set of edges between F and its complement F^{c}. The following question was told to me by P. Mathieu.

Question 6. *If $B_r(x_0)$ is the ball of radius r centered at x_0 in the metric d above, then does*

$$\frac{|\partial B_r(x_0)|}{|B_r(x_0)|} \to 0$$

as $r \to \infty$?

Note that (for fixed x) the balls are level sets of the Green kernel. More precisely, let $f(y) = G(x, y)$ then $f^{-1}([G(x, x)e^{-r}, \infty[) = B_r(x)$.

In answering this, the following may be useful (this is a particular case of a result of Kaimanovich & Vershik):

Theorem 1. $(\Gamma, \mathcal{P})$ *is Liouville (*i.e. *there are no bounded harmonic functions) if and only if for the measure defined by*

$$\frac{1}{m} \sum_{i=0}^{m-1} \mathcal{P}_x^i = \mu_m$$

we have, $\forall g \in G$,

$$\lim_{m\to\infty} \sum_x |\mu_m(gx) - \mu_m(x)| = \lim_{m\to\infty} \|\delta_g * \mu_m - \mu_m\|_{\ell^1} = 0$$

Combine this with the "classical" result ($\leq$ 1960's)

Theorem 2. *If ν_n is a sequence of measures, S the set used to generate the Cayley graph and*

$$\epsilon_n := \sup_{s\in S} \sum_x |\nu_m(gx) - \nu_m(x)| \, .$$

Assume $\epsilon_n \to 0$. Then, there are level sets F_n of ν_n which satisfy $\frac{|\partial F_n|}{|F_n|} \leq \sqrt{\epsilon_n}$.

In fact, there are even "many" such level sets (in a weak sense).

Since the balls are the same as the level sets of $G(x, \cdot)$, that $G = \lim_{m\to\infty} m\mu_m$ and the level sets of μ_m are the same as those of $m\mu_m$, it seems reasonable to think that the answer to the question is positive when Γ is Liouville.

6 Wolfgang Woess

This problem is posed for the Baumslag–Solitar group $BS(2)$, which is the group

$$BS(2) := \left\{ \begin{pmatrix} 2^n & k2^{-l} \\ 0 & 1 \end{pmatrix} : k, l, n \in \mathbb{Z} \right\} = \langle a, b \mid ab = b^2 a \rangle.$$

In particular, consider a simple random walk on the corresponding Cayley graph. If $G(x, y)$ is the Green's function of the walk, then define the Martin kernel

$$K(x, y) := \frac{G(x, y)}{G(e, y)},$$

where e is the identity of the group. The possible limits of this function as $y \to \infty$ give the Martin boundary, and if such are understood, then one can describe the Martin compactification of the group.

Question 7. *Can we determine the asymptotics of the Martin kernel for some walk on $BS(2)$?*

Note that one might consider the simple random walk or a walk with drift where one of the generators (e.g. a) has larger probability than another (e.g. a^{-1}). It is likely that walks with a drift yield a richer Martin boundary.

7 Agelos Georgakopoulos

Let (X_n), (Y_n) be independent simple random walks on started simultaneously at the same vertex O of a transient infinite graph.

Question 8. *If we define $F(x, y) := \mathbf{P}_x(\tau_y < \infty)$, then does the function*

$$\frac{F(X_n, Y_n)}{F(X_n, O)F(O, Y_n)}$$

converge almost-surely?

Equivalently, the question is whether $\mathbf{P}_x (\tau_0 < \infty \mid \tau_y < \infty) |_{(x,y)=(X_n,Y_n)}$ converges a.s.

Note that the following is known: if we fix X_n then

$$\frac{F(X_n, Y_n)}{F(O, Y_n)}$$

converges almost-surely, and likewise fixing Y_n (because this coincides with the Martin kernel. AG believes that almost-sure convergence can proven by adapting a long proof from the paper L. Naïm. Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel. *Annales Inst. Fourier*, 7:183–281, 1957. However, this offers little insight. The main question is really why does the above converge; is there a probabilistic proof?

8 Perla Sousi

Let G be a finite graph, and X_n Y_n be independent (continuous time) simple random walks started from vertices sampled according to the stationary distribution π. Also, suppose Z is sampled according to π. Define

$$M^{X,Y} := \inf \{t \geq 0 : X_t = Y_t\}$$

and define $M^{X,Z}$, $M^{Y,Z}$ similarly.

Question 9. *Is there a uniform lower bound on the probability*

$$\mathbb{P}\left(M^{X,Y} \leq M^{X,Z} \wedge M^{Y,Z}\right)?$$

If we take three independent simple random walks X_n, Y_n, W_n, then

$$\begin{aligned}\mathbb{P}\left(M^{X,Y} \leq M^{X,W} \wedge M^{Y,W}\right) &= \mathbb{P}\left(M^{X,W} \leq M^{X,Y} \wedge M^{Y,W}\right)\\ &= \mathbb{P}\left(M^{Y,W} \leq M^{X,W} \wedge M^{Y,X}\right)\\ &= \frac{1}{3}\end{aligned}$$

by symmetry. The intuition is that if we instead replace W_n by the stationary point Z, then this should only increase the probability the walks $X_n,\ Y_n$ collide before either visits Z. In a paper soon to appear, PS and co-authors show that for transitive graphs, the relevant probability is bounded below by $\frac{1}{4}$.