# Robustness of spatial preferential attachment networks

**Peter Mörters**

UNIVERSITY OF BATH

joint work with

Emmanuel Jacob (ENS Lyon)

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Bollobás and Riordan defined a growing family of random graphs based on the preferential attachment paradigm and rigorously verified several of the conjectured emerging features.

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A plausible reason for the clustering in networks is the presence of individual features of the nodes such that similarity of features is an additional incentive to form links. We therefore propose a model in which preferential attachment is combined with spatial structure to address this. Similar models have been set up by Flaxman, Frieze and Vera (2006) and Aiello, Bonato, Cooper, Janssen and Pralat (2009).

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- A node born at time $t$ is connected by an edge to each existing node independently with probability

$$\varphi\Big(\frac{t\varrho}{f(k)}\Big),$$

where

- $k$ is the indegree of the older node at time $t$,
- $\varrho$ is the distance of the nodes,
- $\varphi\colon [0,\infty) \to [0,1]$ is a decreasing profile function,
- $f\colon \mathbb{N}_0 \to (0,\infty)$ is an increasing attachment rule.

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Conventions:

- We normalise $\varphi$ so that $\int \varphi(|x|)\, dx = 1$.
- We assume $\gamma := \lim_{k\to\infty} \frac{f(k)}{k}$ exists and $0 < \gamma < 1$.

Suppose the graph $G_{t-}$ is given, and a vertex is born at time $t$. Then, for each vertex in $G_{t-}$ with indegree $k$, the probability that it is linked to the newborn vertex is equal to

$$\int \varphi\left(\frac{td(0,y)}{f(k)}\right)\,dy = \frac{f(k)}{t}\,2\int_0^{\frac{t}{2f(k)}} \varphi(x)\,dx \sim \frac{f(k)}{t}.$$

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The indegree process of a vertex is a time-inhomogeneous pure birth process, starting from 0 and jumping at time $t$ from state $k$ to state $k+1$ with intensity $f(k)/t$.

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Indegree processes of different vertices are dependent.

# Empirical degree distribution

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**Theorem 1** Jacob and M (2013)

The empirical indegree distributions $\mu_t$ converge in probability to a deterministic probability measure $\mu$ given by

$$\mu(k) = \frac{1}{1+f(k)} \prod_{\ell=0}^{k-1} \frac{f(\ell)}{1+f(\ell)} = k^{-(1+\frac{1}{\gamma})+o(1)},$$

i.e. the asymptotic indegree distribution is a power law with exponent

$$\tau = 1 + \tfrac{1}{\gamma}.$$

The empirical outdegree distribution converges to a light-tailed distribution, which does not influence the power law exponent.

For a finite graph $G$ we define the local clustering coefficient

$$c^{\text{loc}}(v) := \frac{\#\text{triangles containing } v}{\#\text{adjacent edge pairs meeting at } v}$$

and the average clustering coefficient as

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**Theorem 2 Jacob and M (2013)**

The network $(G_t)_{t>0}$ is clustering in the sense that

$$c^{\mathrm{av}}(G_t) \to c^{\mathrm{av}}_{\infty} > 0 \qquad \text{in probability.}$$

Suppose $\varphi$ is regularly varying at infinity with index $-\delta$, for $\delta > 1$. The parameter $\delta$ controls the probability of long edges and quantifies the clustering; the bigger $\delta$ the stronger the clustering.

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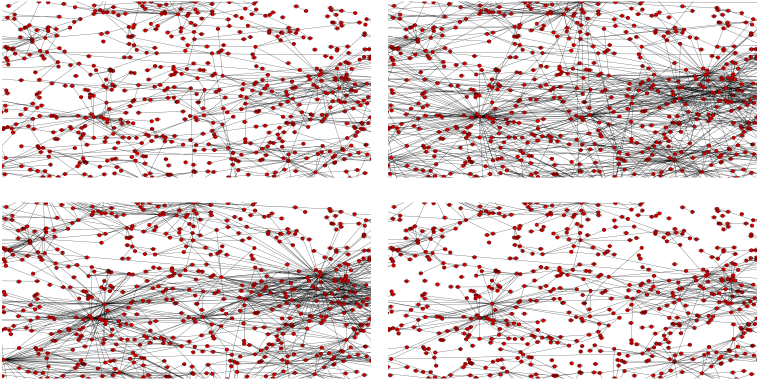

Simulations with parameters (clockwise from top left) (a) $\gamma = 0.5$, $\delta = 2.5$, (b) $\gamma = 0.75$, $\delta = 2.5$, (c) $\gamma = 0.5$, $\delta = 5$, (d) $\gamma = 0.75$, $\delta = 5$.

We now address the problem of robustness of the network $(G_t)_{t>0}$ under percolation. Let $C_t \subset G_t$ be the largest connected component in $G_t$ and denote by $|C_t|$ its size. We say that the network has a giant component if $C_t$ is of linear size or, more precisely, if

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \mathbb{P}\left( \frac{|C_t|}{t} \leq \varepsilon \right) = 0.$$

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We write ${}^p G_t$ for the random subgraph of $G_t$ obtained by Bernoulli percolation with retention parameter $p > 0$ on the vertices of $G_t$.
The network $(G_t)_{t>0}$ is said to be robust if, for any $p > 0$, the network $({}^p G_t)_{t>0}$ has a giant component.

**Theorem 3 (Jacob and M. 2015)**

The network $(G_t)_{t>0}$ is robust if $\gamma > \frac{\delta}{1+\delta}$ or equivalently $\tau < 2 + \frac{1}{\delta}$.

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- Robustness requires strong preferential attachment and weak clustering and can fail for any power-law exponent if the clustering is too strong. For example, the spatial preferential attachment model of Aiello et al.(2009) has too strong clustering and is never robust.
- Robustness has also been shown by Deijfen, van der Hofstad and Hooghiemstra (2013) for a scale-free long range percolation model.

Let $C_t \subset G_t$ be the largest connected component in $G_t$. We say that the network has no giant component if $C_t$ has sublinear size, i.e.

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(a) if $\gamma < \frac{1}{2}$ or equivalently $\tau > 3$;

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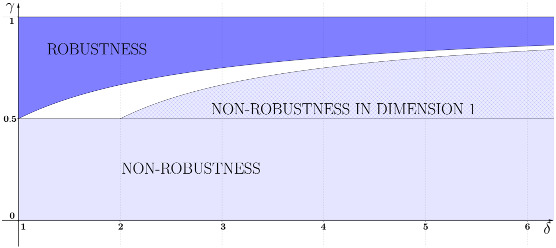

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- Robustness does not occur if the profile function $\varphi$ is not heavy-tailed, i.e. as $\delta \to \infty$.
- We conjecture non-robustness for $\gamma < \frac{\delta}{\delta+1}$ or equivalently $\tau > 2 + \frac{1}{\delta}$.

# Proof strategy for non-robustness

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- The network is non-robust if, for some $p > 0$ the percolated limit model ${}^pG^\infty$ has no infinite component.
- The main problem is that, if $n$ is large, there is no easy upper bound for the probability that $n$ distinct vertices $x_1, \ldots, x_n$ in $G^\infty$ form a path $\{x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_n\}$.

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- Two increasing events occur disjointly if the Poisson points can be divided into two subsets such that the first event holds if the points falling in the first subset are present, and the second event occurs if the points falling in the second subset are present.
- The BK-inequality of van den Berg and Kesten (1985) states that the probability of increasing events occurring disjointly is bounded by the probability of their product.

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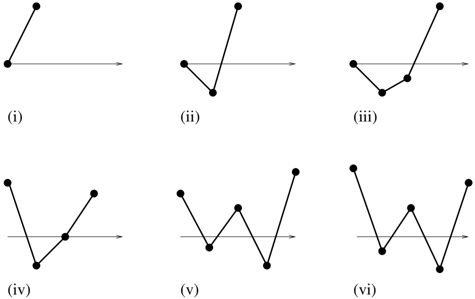
- With this trick the path can be split into pieces which up to symmetry are one of the following six types.



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Now any vertex $z_1' \in G^\infty$ younger than $x_0$ and $x_2$ but older than $z_1$ can only influence the indegree of either $z_0$ or $z_2$ at the birth time of $z_1$, but never both.

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The construction of quick paths is not using spatial information and gives non-robustness only in the case $\tau > 3$. To show non-robustness in the case $\tau > 2 + \frac{1}{\delta-1}$ a refinement is needed.

## Proof strategy for non-robustness

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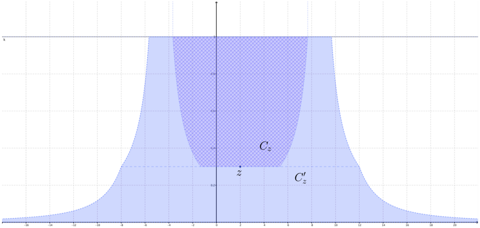
For a point $z$ define a region around $z$ and show that the typical number of vertices outside this region that are connected to $z$, or any other vertex in $C_z$, is small. Including only edges that straddle the boundary of the region in a reduced quick path improves the bound if $\delta > 2$.



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- The phase transition between the robust and non-robust phase does not occur at $\tau = 3$, but at a smaller value depending on the clustering strength. This is a new phenomenon.
- The proof of non-robustness is based on an upper bound for the probability of a special class of paths in the network, called the quick paths. We use disjoint occurence of events and the BK inequality to break up paths into short bits whose probabilities can be estimated.

Talks based on


- Spatial preferential attachment networks:
  Power laws and clustering coefficients
  Emmanuel Jacob and Peter Mörters.
  *Annals of Applied Probability*. 25 (2015) 632-662.
- Robustness of scale-free spatial networks
  Emmanuel Jacob and Peter Mörters.
  Submitted to *Annals of Probability*.
  arXiv:1504.00618


Thank you very much for your attention!