DIFFERENTIAL PIETSCH MEASURES FOR DOMINATED POLYNOMIALS ON BANACH SPACES

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Relations Between Banach Space Theory and Geometric Measure Theory
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This is a (part of a) joint work with Daniel Pellegrino (João Pessoa) and Pilar Rueda (Valencia).

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The class of dominated polynomials, which goes back to Pietsch 1983, is one of the most studied nonlinear generalizations of the ideal of absolutely summing linear operators.

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Theorem. [Matos 1996] A continuous n-homogeneous polynomial $P\colon E \longrightarrow F$ is p-dominated if and only if there is a constant $C > 0$ and a regular Borel probability measure on $B_{E'}$ endowed with the weak* topology such that

$$\|P(x)\| \leq C\left(\int_{B_{E'}} |\varphi(x)|^p d\mu(\varphi)\right)^{\frac{n}{p}} \text{ for every } x \in E. \qquad (1)$$

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Any such measure μ is called a Pietsch measure for P.

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Holomorphy types (Nachbin 60's, Dineen 70's, ...), coherent sequences of polynomials (Carando-Dimant-Muro 2000's, Pellegrino-Ribeiro 2010's.)

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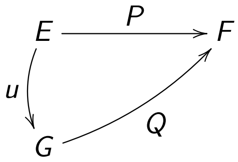
Question 1. (Rephrased) Does every p-dominated polynomial admit a differential Pietsch measure?

The factorization theorem

Theorem. An n-homogeneous polynomial $P\colon E \longrightarrow F$ is p-dominated if and only if there are a Banach space G, a p-summing linear operator $u\colon E \longrightarrow G$ and a continuous n-homogeneous polynomial $Q\colon G \longrightarrow F$ such that $P = Q \circ u$.



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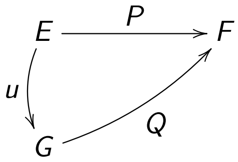

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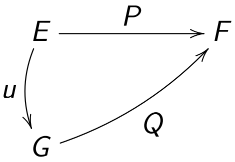

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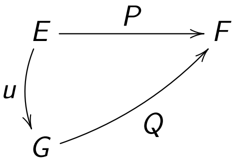

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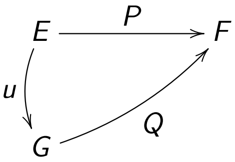

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proving that μ is a Pietsch measure for $\widehat{d}^k P(a)$.

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The problems with the proof of the factorization property are related to the (non)-injectivity of j_p^e.

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As to the existence of differential Pietsch measures for which j_p is injective we have:

Proposition. If $(B_{E'}, w^*)$ is separable, then any p-dominated n-homogeneous polynomial on E admits a differential Pietsch measure for which the canonical map j_q is injective for every $1 \leq q < \infty$.

THANK YOU VERY MUCH!