Rich families in Asplund spaces and separable reduction of Fréchet (sub)differentiability

Marek Cúth, Marián Fabian

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Examples

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Clearly, $\mathcal{S}_\square(X \times X^*)$ is a rich subfamily of $\mathcal{S}(X \times X^*)$.

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Easy fact. If $\partial f(x) \neq \emptyset$ and also $\partial(-f)(x) \neq \emptyset$, then f is Fréchet differentiable at x.

Theorem 6

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Summarizing: $\partial(f|_V)(x) = \big(\partial f(x) \cap Y\big)|_V = \big(\partial f(x)\big)|_V$ *for every* $x \in V$.

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Proof. Let $\mathcal{R}_1,\ \mathcal{R}_2$ be the rich families separably reducing the Fréchet subdifferentiability of f and $-f$ respectively.

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(Fabian89, Mordukhovič-Fabian02) Fuzzy calculus for Fréchet subdifferentials on an `Asplund` space is separably reducible via a suitable cofinal, even rich, family.

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