Vector-valued version of the Mokobodzki theorem

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This talk is based on Section 4 of the paper

[KS] O.Kalenda and J.Spurný:
Baire classes of affine vector-valued functions,
http://arxiv.org/abs/1411.1874

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Remark

One can achieve $\|u_n\|_\infty \leq \|f\|_\infty$ for each n.

[Odell-Rosenthal 1975]

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Trivial part of the answer

YES, if $\dim E = d < \infty$.

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Trivial part of the answer

YES, if $\dim E = d < \infty$.

Moreover, one can achieve $\|u_n\|_\infty \leq d \cdot \|f\|_\infty$.

Negative results

Example [Mercourakis-Stamati 2002]

Let E be a separable reflexive Banach space, $X = (B_E, w)$ and $f : X \to E$ be the identity mapping. Then:

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- [Enflo 1973] There are separable reflexive spaces failing the c.a.p.
- [KS] If E fails the c.a.p., then f does not belong to any affine Baire class. I.e., f cannot be reached by iterated pointwise limits of sequences starting from affine continuous mappings.

Positive results I

Theorem [Mercourakis-Stamati 2002]

Let X be a compact convex set and E a Banach space with the bounded approximation property. Let $f : X \to E$ be an affine mapping of the first Baire class.

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Remarks

- The assumptions are slightly weaker
  ($X = (B_{F^*}, w^*)$ and E is separable).
- The proof contains a gap.

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Theorem [KS]

Let X be a compact convex set and E a Banach space with the b.a.p. Let $f : X \to E$ be an affine mapping of the first Baire class.

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Moreover, if E has the λ-b.a.p., then one can achieve

$$\|u_n\|_\infty \leq \lambda \cdot \|f\|_\infty.$$

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Question

Is the estimate optimal?
Can one replace b.a.p by a.p? Or by c.a.p?

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- Hence, $f = \lim_n \lim_m f_{n,m}$, in particular f is of the second affine Baire class.
- The problem is to show f is even of the first affine Baire class.

The tools used in the proof II

Lemma [Mercourakis-Stamati 2002]

Let K be a compact space, E a Banach space, $f_{n,m} : K \to E$ a double sequence of mappings.

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Then there are g_k, convex combinations of $f_{n,m}$, such that $g_k \to f$ pointwise.

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Back to the proof

- We have $f = \lim_n \lim_m f_{n,m}$, $f_{n,m}$ affine continuous, f of the first Baire class.
- So, it is enough to prove $\|f_{n,m}\|_\infty \leq \lambda \|f\|_\infty$.

Lemma (Mokobodzki)

Let X be a compact convex set, E a finite-dimensional Banach space. Let $f : X \to E$ be an affine mapping of the first Baire class.

Then there is a sequence of affine continuous mappings $u_n : X \to E$ such that $u_n \to f$ pointwise.

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Moreover, one can achieve

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- T is of the first Baire class in the weak* topology
- There are $T_n \in L(E^*, F) \approx L_{w^*}(F^*, E)$, $\|T_n\| \leq T$, $T_n \xrightarrow{w^*} T$ (by [Odell-Rosenthal 1975])

Further variants

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Let X be a compact convex set and E a Banach space. Let $f : X \to E$ be an affine mapping of the first Baire class.

Then there is a sequence of affine continuous mappings $u_n : X \to E$ such that $u_n \to f$ pointwise if one of the following assertions is satisfied:

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- $\mathfrak{A}(X)$ is isomorphic to a complemented subspace of an L^1-predual.

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Question

Is it enough to suppose that $\mathfrak{A}(X)$ has the a.p. (b.a.p., c.a.p.)?