Subspaces of L_p defined on trees

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Lindenstrauss and Pelczynski (1968) introduced $\mathcal{L}_p$ spaces having the property that each finite dimensional subspace is contained in another finite dimensional subspace which is not farther than a fixed constant from ℓ_p^n space.

Definition

Let $1 \leq p \leq \infty$ and $1 \leq \lambda \leq \infty$. A Banach space X is a $\mathcal{L}_{p,\lambda}$ space if for each finite dimensional subspace F of X there is a finite dimensional subspace E of X containing F such that $d(E, \ell_p^n) \leq \lambda$, where $n = dim(E)$ and $d(E, \ell_p^n)$ is the Banach-Mazur distance between E and ℓ_p^n.

A Banach space X is a $\mathcal{L}_p$ space if it is $\mathcal{L}_{p,\lambda}$ space for some $1 \leq \lambda < \infty$.

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Theorem

Let $1 < p < \infty$, $p \neq 2$ and X be a Banach space such that X is not isomorphic to ℓ_2. Then X is a $\mathcal{L}_p$ space if and only if X is isomorphic to a complemented subspace of $L_p(\mu)$ for some μ.

Classical Examples: Let $1 < p < \infty$, $p \neq 2$. ℓ_p, $\ell_p \oplus \ell_2$, $\ell_p(\ell_2)$ and L_p are mutually non isomorphic $\mathcal{L}_p$ spaces.

Rosenthal's space X_p (1970):

Definition

Let $2 < p < \infty$ and $w = \{w_n\}$ be a sequence of positive scalars. Define $X_{p,w} := \{\{x_n\} : x_n \in \mathbb{R}, \sum |x_n|^p < \infty \text{ and } \sum |w_n x_n|^2 < \infty\}$.

For $x = \{x_n\} \in X_{p,w}$ define the norm $||x||_{p,w} := max\{(\sum |x_n|^p)^{\frac{1}{p}}, (\sum |w_n x_n|^2)^{\frac{1}{2}}\}$.

If w satisfies the condition: for each $\epsilon > 0, \sum_{w_n<\epsilon} w_n^{\frac{2p}{p-2}} = \infty$, then $X_{p,w}$ is not isomorphic to any of the classical $\mathcal{L}_p$ space.

For any sequence of scalars w, w' satisfying the above condition, $X_{p,w}$ and $X_{p,w'}$ are isomorphic.

More Examples: Let $2 < p < \infty$.

$B_p = \ell_p(X_n)$ where each $X_n \sim \ell_2$ with *sup* $d(X_n, \ell_2) = \infty$. (Rosenthal-1970)

$D_p \sim (\ell_p^{2^n} \otimes \ell_2)_{p,2,(1)}$. (Alspach-1974)

We consider $\cup_{n\geq 1}\mathbb{N}^n$ with a natural order $(x_1, \cdots, x_n) \prec (y_1, \cdots, y_m)$ provided $m \geq n$ and $x_i = y_i$ for $1 \leq i \leq n$.

Definition

(i) A tree on $\mathbb{N}$ is a subset of $\cup_{n\geq 1}\mathbb{N}^n$ with the property that $(x_1, \cdots, x_n) \in T$ whenever $(x_1, \cdots, x_n, x_{n+1}) \in T$.

(ii) A subtree S of a tree T is a subset of T such that S itself is a tree.

(iii) A branch F in T is a subset of mutually comparable elements of T.

(iv) A tree T is well founded provided there is no sequence $\{n_k\}$ in $\mathbb{N}$ such that $(n_1, \cdots, n_m) \in T$ for all $m \in \mathbb{N}$.

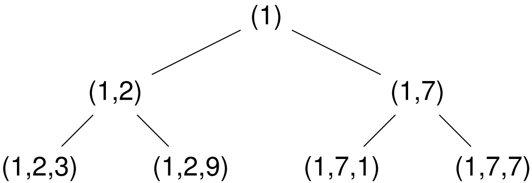


Figure: T

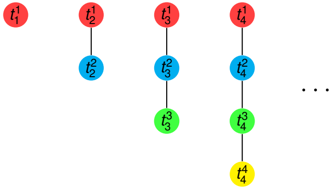


Figure: T_ω

Definition

For a tree T, we define the derived tree

$$D(T) = \cup_{n \geq 1} \{(n_1, \cdots, n_m) : (n_1, \cdots, n_m, n) \in T \text{ for some } n \in \mathbb{N}\}.$$

Proceeding by induction, we can construct a transfinite system of trees as follows.

(i) Take $T^0 = T$.

(ii) If T^α is obtained, let $T^{\alpha+1} = D(T^\alpha)$.

(iii) For a limit ordinal γ, define $T^\gamma = \cap_{\alpha<\gamma} T^\alpha$. If T is well-founded, then T^α's are strictly decreasing as α increases. Hence T^α will be empty for some ordinal α. For a well founded tree T, we will denote order of T, namely $o(T)$, to be the smallest ordinal for which $T^{o(T)} = \emptyset$.

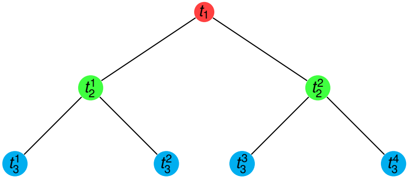
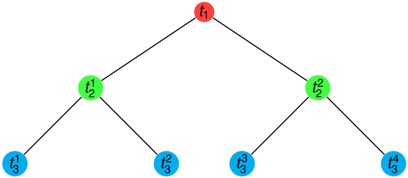


Figure: T^0



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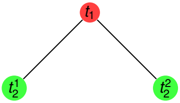
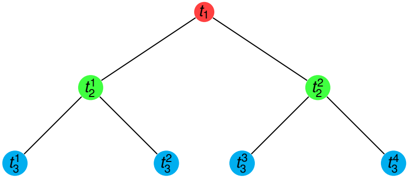


Figure: T^1



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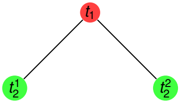
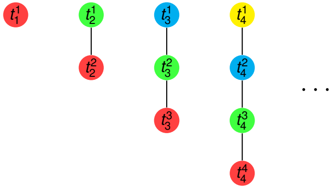


Figure: T^1

Figure: T^2



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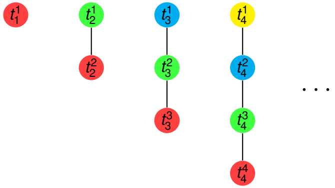

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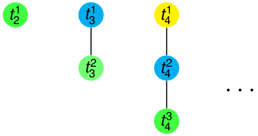

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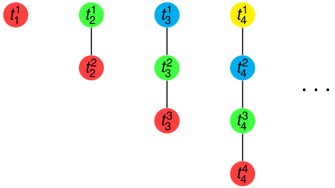
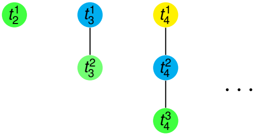


Figure: T^0



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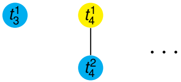

Figure: T^2

Definition

For countable ordinal $\alpha < \omega_1$ we will define the well founded canonical trees T_α inductively:

(i) Take T_1 to be the tree with a single element.

(ii) If T_α is obtained, define $T_{\alpha+1} = \dot{T}_\alpha$.

(iii) For a limit ordinal β define $T_\beta = \cup_{\alpha<\beta} T_\alpha$ with the relation $\prec_\beta = \cup_{\alpha<\beta} \prec_\alpha$ that is, if $u, v \in T_\beta$, $u \prec_\beta v$ if and only if $u, v \in T_\alpha$ for some $\alpha < \beta$ and $u \prec_\alpha v$. T_β may be visualized as simply setting the trees T_α side by side in such a way that if $\alpha_1 < \alpha_2 < \beta$ then T_{α_1} is kept on the left to T_{α_2}.

Figure: T_1

Figure: T_1

Figure: T_2

Figure: T_1

Figure: T_2

Figure: T_1

Figure: T_2

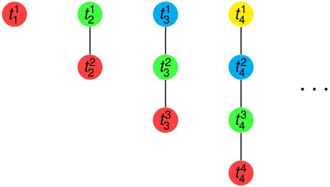

Figure: T_ω

Figure: T_1

Figure: T_2

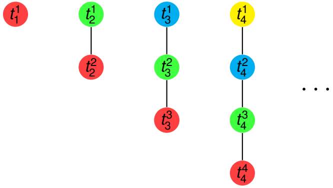

Figure: T_ω

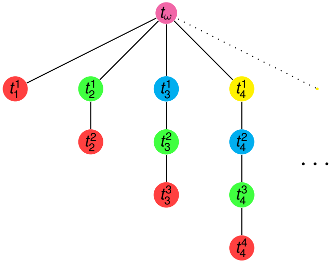

Figure: $T_{\omega+1}$

If we consider the full tree $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{N}^n$ then $G = \{-1, 1\}^{\mathcal{C}}$ is group with pointwise multiplication.

A measurable function f on G is said to depend only on the coordinates of a branch Γ of $\mathcal{C}$ provided $f(x) = f(y)$ whenever $x, y \in G$ with $x(c) = y(c)$ for all $c \in \Gamma$.

Definition

Let $1 \leq p < \infty$. For a tree T in $\mathcal{C}$ we denote X_T^p the closed linear span in $L_p(G)$ of the functions depending on finite branches in T.

Consider the tree T_{D_p}:

The space $D_p (2 < p < \infty)$ is isomorphic to $X^p_{T_{D_p}}$.

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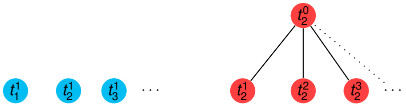

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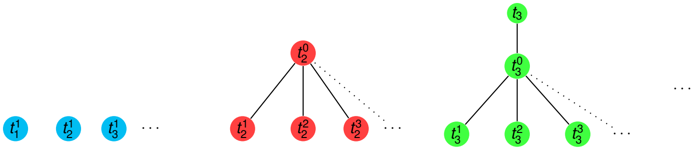

The space $D_p (2 < p < \infty)$ is isomorphic to $X^p_{T_{D_p}}$.

Theorem (Bourgain, Rosenthal, Schechtman 1981)

(*i*) *For* $1 < p < \infty$*, the space* $X^p_{\mathcal{C}}$ *is isomorphic to* $L_p(G)$*.*

(*ii*) *Let* $1 \leq p < \infty$*. If* T_1, T_2 *are trees such that* $T_1 \subseteq T_2$ *then* $X^p_{T_1}$ *is a complemented subspace of* $X^p_{T_2}$*.*

Thus $\{X^p_{T_\alpha}\}_{\alpha<\omega_1}$ are $\mathcal{L}_p$ spaces for $1 < p < \infty$, $p \neq 2$.

Let D_n be the set of all n-strings of 0 and 1.

For a Banach space$X$ define X^{D_n} to be the set of all functions from D_n to X, which can be identified with the set of all 2^n tuples $(u_1, ..., u_{2^n})$.

Let $X^{\mathcal{D}} = \cup_{n=0}^{\infty} X^{D_n}$. If $u \in X^{\mathcal{D}}$ then $u \in X^{D_n}$ for a unique $n \in \mathbb{N} \cup \{0\}$, denoted by $|u|$. Denote $\prec$ on $X^{\mathcal{D}}$ by $u \prec v$ if $|u| < |v|$ and for $k = |v| - |u|$, $u(t) = 2^{-k/p} \Sigma_{s \in D_k} v(t \cdot s)$.

Let $\overline{X}^{\delta}$ be the set of all $u \in X^{\mathcal{D}}$ such that

$$\delta(\Sigma_{t \in D_{|u|}} |c(t)|^p)^{1/p} \leq \| \Sigma_{t \in D_{|u|}} c(t)u(t) \|_X \leq (\Sigma_{t \in D_{|u|}} |c(t)|^p)^{1/p} \quad (1)$$

for all $c \in \mathbb{R}^{D_{|u|}}$.

Definition

Let $H_0^\delta(X) = \overline{X}^\delta$.
If $\alpha = \beta + 1$ and $H_\beta^\delta(X)$ has been defined, define

$$H_\alpha^\delta(X) = \{u \in H_\beta^\delta(X) : u \prec v \text{ for some } v \in H_\beta^\delta(X)\}.$$

If α is a limit ordinal define $H_\alpha^\delta(X) = \cap_{\beta<\alpha} H_\beta^\delta(X)$.

Let $h_p(\delta, X)$ be the least ordinal α such that $H_\alpha^\delta(X) = H_{\alpha+1}^\delta(X)$. If $L_p \not\hookrightarrow X$, define $h_p(X) = sup_{0<\delta\leq 1} h_p(\delta, X)$. If $L_p \hookrightarrow X$, define $h_p(X) = \omega_1$.

Theorem (Bourgain, Rosenthal, Schechtman 1981)

Let $1 < p < \infty$, $p \neq 2$. Then there is a strictly increasing function $\tau : \omega_1 \to \omega_1$ such that for $\alpha < \beta < \omega_1$, $X^p_{T_{\tau(\beta)}} \not\hookrightarrow X^p_{T_{\tau(\alpha)}}$.

$X^p_{T_\alpha}$ spaces are isomorphically distinct at limit ordinals (Alspach 1999).

Theorem (Dutta, K.)

Let $2 < p < \infty$ and X be an infinite dimensional subspace of L_p.

(a) $h_p(X) = \omega$, $h_p(X) = \omega \cdot 2$ or $h_p(X) \geq \omega^2$.

(b) $h_p(\ell_2) = \omega$.

(c) Let $X \not\sim \ell_2$. Then $h_p(X) = \omega$ if and only if $X \hookrightarrow \ell_p$.

(d) If $X \not\hookrightarrow X^p_{T_\omega}$ then $h_p(X) \geq \omega^2$.

Lemma

Let $2 < p < \infty$ and X be a subspace of L_p. If for some $0 < \delta \leq 1$ and $n \in \mathbb{N}$, $H^\delta_{\omega+n}(X) \neq \emptyset$ then there exists a constant C (depending on δ, p and X only) such that $X^p_{T_{\omega+n},0} \overset{C}{\hookrightarrow} (X)_{p,2,(1)}$.

Lemma

Let $2 < p < \infty$ and X be a subspace of L_p. If for some $0 < \delta \leq 1$ and $n \in \mathbb{N}$, $H^\delta_{\omega+n}(X) \neq \emptyset$ then there exists a constant C (depending on δ, p and X only) such that $X^p_{T_{\omega+n},0} \overset{C}{\hookrightarrow} (X)_{p,2,(1)}$.

Let X be a subspace of $L_p(\Omega, \mu)$. For any sequence (x_n) such that $x_n \in X$, let $||(x_n)||_{p,2,(1)} = max\{(\sum ||x_n||_p^p)^{1/p}, (\sum ||x_n||_2^2)^{1/2}\}$. Define $(X)_{p,2,(1)} = \{(x_n) : x_n \in X \text{ for all } n \text{ and } ||(x_n)||_{p,2,(1)} < \infty\}$.

Proof.

(Idea)

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(i) $X^p_{T_\omega} \sim X^p_{T_{\omega+k}}$ for all $k \in \mathbb{N}$. (Since $X^p_{T_\omega}$ is square)

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(*ii*) $h_p(X^p_{T_{\omega+k}}) \geq \omega + k + 1$. (BRS)

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(iii) Suppose $h_p(X^p_{T_\omega}) > \omega \cdot 2$.

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(iv) $X^p_{T_{\omega \cdot 2}} \overset{A}{\hookrightarrow} X^p_{T_\omega}$. (Using Lemma, this needs a bit of work!)

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- (iii) Suppose $h_p(X^p_{T_\omega}) > \omega \cdot 2$.
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This is a contradiction to $\ell_p(\ell_2) \hookrightarrow X^p_{T_{\omega \cdot 2}}$ as $\ell_p(\ell_2) \not\hookrightarrow X^p_{T_\omega}$.

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Thus $h_p(X^p_{T_\omega}) = \omega \cdot 2$. Using similar arguments as above (but one needs careful modifications) we can show that $h_p(\ell_p) = \omega$ and $h_p(\ell_2) = \omega$.

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If $X \not\hookrightarrow X^p_{T_\omega}$ then $h_p(X) \geq \omega^2$ needs some results of HOS(2011), Alspach(1999) and Schechtman(1975).

□