Currents in infinite dimensions

Bernd Kirchheim(jointly with Luigi Ambrosio)
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- Caccioppoli sets, $n-1$-dimensional "boundaries" ∂E in $\mathbb{R}^n$ understood as distributional deriv. of χ_E (De Giorgi)
- theory of k-dim. currents $\mathcal{D}_k$ in $\mathbb{R}^n$ (Federer-Fleming). **FF**-currents

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Lipschitz forms Let E be a complete metric space, $\mathcal{D}^k(E)$ are all $(k+1)$-ples $\omega = (f, \pi_1, \ldots, \pi_k)$ of Lipschitz real valued functions in E with the first function f in $\mathrm{Lip}_b(E)$

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mapping $\mathcal{D}^k(E)$ into $\mathcal{D}^{k+1}(E)$, note $dd\omega \neq 0$?? and for $\varphi \in \mathrm{Lip}(E, F)$ a **pull back operator**

$$\varphi^{\#}\omega = \varphi^{\#}(f, \pi_1, \ldots, \pi_k) = (f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_k \circ \varphi)$$

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iv) $\exists \|T\|$ (minimal) finite measure s.t.

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v) **boundary** $\partial T : \omega \in \mathcal{D}^{k+1} \to T(d\omega) = T(1, f, \pi_1, \dots, \pi_k)$ in $\mathbf{N}_{k+1}(E)$, i.e. has also finite mass.

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v) **boundary** $\partial T : \omega \in \mathcal{D}^{k+1} \to T(d\omega) = T(1, f, \pi_1, \dots, \pi_k)$ in $\mathbf{N}_{k+1}(E)$, i.e. has also finite mass. Note $\partial\partial T = 0$ by ii) meaning of these conditions becomes clear looking at "**pushforwards**" under $\varphi \in \mathrm{Lip}(E, F)$

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$$T(f\, d\pi_1 \wedge \ldots \wedge d\pi_k) + T(\pi_1\, df \wedge \ldots \wedge d\pi_k) = T(1\, d(f\pi_1) \wedge \ldots \wedge d\pi_k)$$

whenever $f, \pi_1 \in \mathrm{Lip}_b(E)$, and

$$T(f\, d\psi_1(\pi) \wedge \ldots \wedge d\psi_k(\pi)) = T(f \det \nabla\psi(\pi)\, d\pi_1 \wedge \ldots \wedge d\pi_k)$$

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In particular T alternating in the π_j's.
So rather use notation $\omega = f\, d\pi = f\, d\pi_1 \wedge \ldots \wedge d\pi_k$, call f weight and $d\pi$ the differential part.

The (outer) Hausdorff k-dimensional measure of $B \subset E$ is

$$\mathcal{H}^k(B) := \lim_{\delta \downarrow 0} \frac{\omega_k}{2^k} \inf \left\{ \sum_{i=0}^{\infty} [\mathrm{diam}(B_i)]^k : \ B \subset \bigcup_{i=0}^{\infty} B_i, \ \mathrm{diam}(B_i) < \delta \right\}$$

where ω_k is the Lebesgue measure of the unit ball of $\mathbb{R}^k$.
$\mathcal{H}^k$-measurable set $S \subset E$ is **countably $\mathcal{H}^k$-rectifiable** if there exist sets $A_i \subset \mathbb{R}^k$ and Lipschitz functions $f_i : A_i \to E$ such that

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Finally, we say (normal) T is rectifiable, $T \in \mathcal{R}_k(E)$ if there is a k-rectifiable B such that $\|T\|(E \setminus B) = 0$ " **concentrated on** B".
Note $\|T\| \ll \mathcal{H}^k$ for all $T \in \mathbf{N}_k(E)$

We also have the following parametric presentation: $T \in \mathcal{R}_k(E)$, iff there are compacts $K_i \subset \mathbb{R}^k$, functions $\theta_i \in L^1(\mathbb{R}^k)$ with $\operatorname{supp} \theta_i \subset K_i$ and bi-Lipschitz maps $f_i : K_i \to E$ such that

$$T = \sum_{i=0}^{\infty} f_{i\#}[\![\theta_i]\!] \qquad \text{and} \qquad \sum_{i=0}^{\infty} \mathbf{M}(f_{i\#}[\![\theta_i]\!]) = \mathbf{M}(T),$$

$\mathbf{M}(T) = \|T\|(E)$ and $[\![\theta]\!](fd\pi) = \int_{\mathbb{R}^k} f\theta \det(D\pi)\, dx$ current in $\mathbb{R}^k$.

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$\mathbf{M}(T) = \|T\|(E)$ and $[\![\theta]\!](f d\pi) = \int_{\mathbb{R}^k} f \theta \det(D\pi)\, dx$ current in $\mathbb{R}^k$. We say, that T is integer rectifiable or integral (since $T \in \mathbf{N}_k(E)$) if the θ_i's can be choosen integervalued. This is equivalent to $T \in \mathcal{R}_k(E)$ and for all $\varphi \in \mathrm{Lip}(E, \mathbb{R}^k)$ and any open set $A \subset E$ we have $\varphi_\#(T \llcorner A) = [\![\theta]\!]$ for some $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$.

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Crucial question Is $\mathbf{I}_k(E)$ closed under weak (i.e. pointwise $\forall\omega$) convergence, is this compact if E not (locally) compact?

It turns out, that a normal current T can not concentrate on $\mathcal{H}^k$-(σ)-finite purely k-unrectifiable sets M, i.e. $\mathcal{H}^k(M \cap S) = 0$ for all k-rectifiable k.

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This argument has difficulties if E not sufficient homogeneous (how to rewrite lipschitz forms) but also in ℓ^2 as unrectifiable sets do not have generic zero projections [de Pauw]. Rather use behaviour on slices.

Let $\omega = (g, \tau_1, \ldots, \tau_m) \in \mathcal{D}^m(E)$, with $m \leq k$ ($\omega = g$ if $m = 0$). We define the $(k-m)$-dimensional restriction of T denoted $T \llcorner \omega$, by setting

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We can show the existence of slices of any normal T w.r.t. any $\pi \in \mathrm{Lip}(E, \mathbb{R}^m)$, i.e. existence of a family $T_x = \langle T, \pi, x \rangle \in \mathbf{N}_{k-m}(E)$ for $x \in \mathbb{R}^m$ such that

$$\int_{\mathbb{R}^m} \langle T, \pi, x \rangle \psi(x)\, dx = T \llcorner (\psi \circ \pi)\, d\pi \qquad \forall \psi \in C_c(\mathbb{R}^k).$$

moreover T_x and ∂T_x is concentrated on $\pi^{-1}(x) \cap L$ if T and ∂T are so and the mass-measure also disintegrates

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Using the parametric presentation, slices of rectifiable/integral currents are rectifiable/integral.

crucial observation, inspired by R.Jerrard - but see also [Fed, 5.3.5(1)], the map $x \mapsto T_x$ of bounded variation in for proper metric on the currents, has geometric meaning if $m = k$, i.e. s T_x 0-dimensional.

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$$\mathbf{F}(T) = \sup\{\{T(\phi) : \phi \in \mathrm{Lip}_b(E),\ \mathrm{Lip}(\phi) + \|\phi\|_\infty \leq 1\}$$

For Dirac measures $F(\delta_a - \delta_b) \sim \mathrm{dist}(a,b)$ locally, so since BV-maps from $\mathbb{R}^k$ into metric spaces are σ-lipschitz, we see that the set of all atoms of all 0-dim slices is k-rectifiable. So T is rectifiable provided its 0-dim slices are rectifiable, i.e. discrete measures.

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For Dirac measures $F(\delta_a - \delta_b) \sim \mathrm{dist}(a, b)$ locally, so since BV-maps from $\mathbb{R}^k$ into metric spaces are σ-lipschitz, we see that the set of all atoms of all 0-dim slices is k-rectifiable. So T is rectifiable provided its 0-dim slices are rectifiable, i.e. discrete measures. Controlling the masses of T_j, ∂T_j and the $\mathcal{H}^k(S_j)$, where S_j is the set on which T_j lives, the (weak) limit T is again normal and the slices are discrete (slices of T_j could not smear out). Hence T is rectfiable.

The closure theorem, just mentioned, give compactness of $\mathbf{I}_k(E)$ if E is compact. In ∞-dimensions this does not help ... look at diracs.
What do we really want to do? Find

$$\min \mathbf{M}(T) \text{ over all } T \in \mathbf{I}_k(E) \text{ such that } \partial T = S \in \mathbf{I}_{k-1}(E)$$

i.e. solve the Plateau problem. Here $\partial S = 0$, $\mathrm{spt}(S)$ compact supposed. We need somehow to localize the T_s and produce limits. For the pointwise convergence of the currents, we restrict ourself to w^*-continuous lipschitz forms. These capture enough information because measure live in general on σ-compact sets and we have

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$$\|T_j\|(B(x,r)) \geq \varepsilon_0 r^k \text{ for } x \in \operatorname{spt}(T_j)$$

By Ekland-Bishop-Phelps find nearly-minimizers, so we only need a standart argument... use isoperimetric inequality.
We reproved these isoperimetric inequalities using an idea of Gromov, which gives a uniform c_k such that for all finite dimensional V and $S \in \mathbf{I}_{k-1}(V)$ a compact cycle in V some filling $T \in \mathbf{I}_k(V)$ with $\partial T = S$ and

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The same estimate for general banach spaces ?? Only if there was some finite approximation property. Stefan Wenger could establish them using a more intrinsic argument.