A weak*-topological dichotomy in the dual unit ball of the Banach space of continuous functions on the first uncountable ordinal, with applications in operator theory

Niels Laustsen

Lancaster University, UK

Relations between Banach Space Theory and Geometric Measure Theory

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Joint work with
Tomasz Kania (Lancaster) and Piotr Koszmider (IMPAN, Warsaw)

$C(K)$-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{K} : f \text{ is continuous}\} \qquad (\text{where } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}).$$

Fact. *$C(K)$ separable $\iff$ K metrizable.*

Classification. *Let K be a compact metric space. Then:*

- *K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;*
- (Milutin) *K is uncountable $\iff$ $C(K) \cong C[0,1]$;*
- (Bessaga and Pełczyński) *K is countably infinite $\iff$ $C(K) \cong C[0, \omega^{\omega^\alpha}]$ for a unique countable ordinal α.*

Here, for an ordinal σ, the interval $[0, \sigma] = \{\alpha \text{ ordinal} : \alpha \leqslant \sigma\}$ is equipped with the *order topology*, which is determined by the basis

$$[0, \beta), \qquad (\alpha, \beta), \qquad (\alpha, \sigma] \qquad (0 \leqslant \alpha < \beta \leqslant \sigma).$$

Note: $C[0, \omega_1]$, where ω_1 is the first uncountable ordinal, is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Fact. *Each $f \in C[0, \omega_1]$ is eventually constant.*

Theorem (Semadeni 1960). $C[0, \omega_1] \not\cong C[0, \omega_1] \oplus C[0, \omega_1]$.

The topological dichotomy

For convenience, we work with the hyperplane

$$C_0[0,\omega_1) = \{f \in C[0,\omega_1] : f(\omega_1) = 0\}$$

instead of $C[0,\omega_1]$.

Theorem (Kania–Koszmider–L). *Let K be a weak*-compact subset of $C_0[0,\omega_1)^*$. Then exactly one of the following two alternatives holds:*

- *K is* uniformly Eberlein compact, *that is, homeomorphic to a weakly compact subset of a Hilbert space;*
- *K contains a homeomorphic copy of $[0,\omega_1]$ of the form*

$$\{\rho + \lambda\delta_\alpha : \alpha \in D\} \cup \{\rho\},$$

where $\rho \in C_0[0,\omega_1)^*$, $\lambda \in \mathbb{K} \setminus \{0\}$, δ_α is the Dirac measure at α, and D is a closed and unbounded subset of $[0,\omega_1)$.

Note:

(i) $[0,\omega_1]$ is not contained in any uniformly Eberlein compact space;

(ii) the unit ball of $C_0[0,\omega_1)^*$ in the weak* top. contains a homeomorphic copy of every uniformly Eberlein compact space of density at most $\aleph_1$.

An operator-theoretic dichotomy

Idea: for an operator (= bounded, linear map) T from $C_0[0,\omega_1)$ into some Banach space X, apply the topological dichotomy to the weak*-compact set

$$K = T^*(\text{the unit ball of } X^*).$$

Definition. A Banach space X is *Hilbert-generated* if there exists an operator with norm-dense range from a Hilbert space into X.

Relevance:

- *$C(K)$ is Hilbert-generated $\iff$ K is uniformly Eberlein compact;*
- *a Banach space X embeds in a Hilbert-generated Banach space $\iff$ the unit ball of X^* is uniformly Eberlein compact in the weak* topology.*

Theorem (Kania–Koszmider–L). *Let X be a Banach space, and suppose that there exists a bounded, linear surjection $T\colon C_0[0,\omega_1) \to X$. Then exactly one of the following two alternatives holds:*

- *X embeds in a Hilbert-generated Banach space;*
- *$I_{C_0[0,\omega_1)}$ factors through T, and X is isomorphic to the direct sum of $C_0[0,\omega_1)$ and a subspace of a Hilbert-generated Banach space.*

Characterizations of the operators not factoring the identity

Theorem (Kania–Koszmider–L). *Let $T \in \mathscr{B}(C_0[0,\omega_1))$. Then TFAE:*

(a) $I_{C_0[0,\omega_1)}$ *does not factor through* T: $I_{C_0[0,\omega_1)} \neq STR$ *for all* $R, S \in \mathscr{B}(C_0[0,\omega_1))$;

(b) T *does not fix a copy of* $C_0[0,\omega_1)$;

(c) T *is a* Semadeni operator, *in the sense that* T^{**} *maps the subspace*

$$\{\Lambda \in C_0[0,\omega_1)^{**} : \langle \lambda_n, \Lambda\rangle \to 0 \text{ as } n \to \infty \\ \text{for every weak}^*\text{-null sequence } (\lambda_n) \text{ in } C_0[0,\omega_1)^*\}$$

into the canonical copy of $C_0[0,\omega_1)$ *in its bidual;*

(d) *there is a closed, unbounded subset* D *of* $[0,\omega_1)$ *such that*

$$(Tf)(\alpha) = 0 \qquad (f \in C_0[0,\omega_1),\ \alpha \in D);$$

(e) T *factors through the Banach space* $\left(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\right)_{c_0}$;

(f) *the range of* T *is contained in a Hilbert-generated subspace of* $C_0[0,\omega_1)$;

(g) *the range of* T *is contained in a* weakly compactly generated subspace *of* $C_0[0,\omega_1)$; *that is, there exist a reflexive Banach space* X *and an operator* $U\colon X \to C_0[0,\omega_1)$ *such that* $T(C_0[0,\omega_1)) \subseteq \overline{U(X)}$.

Some consequences: bounded left approximate identities

Let

$$\mathscr{M} = \{T \in \mathscr{B}(C_0[0,\omega_1)) : \forall\, R, S \in \mathscr{B}(C_0[0,\omega_1)) \colon I_{C_0[0,\omega_1)} \neq STR\}.$$

This is an ideal of $\mathscr{B}(C_0[0,\omega_1))$ by the theorem above. It is then automatically the unique maximal ideal (Dosev–Johnson). We call it the *Loy–Willis ideal* because it was first studied (in a different guise) by Loy and Willis (1989).

Loy and Willis' key result. *$\mathscr{M}$ has a bounded right approximate identity; that is, $\mathscr{M}$ contains a norm-bounded net (U_j) such that $TU_j \to T$ for each $T \in \mathscr{M}$.*

Question: does $\mathscr{M}$ also have a bounded left approximate identity, that is, does $\mathscr{M}$ contain a norm-bounded net (U_j) such that $U_jT \to T$ for each $T \in \mathscr{M}$?

Answer: Yes! — In fact more is true:

Theorem (Kania–Koszmider–L). *$\mathscr{M}$ contains a net (Q_j) of projections with $\|Q_j\| \leqslant 2$ such that*

$$\forall\, T \in \mathscr{M}\ \exists\, j_0\ \forall\, j \geqslant j_0 \colon\ Q_jT = T.$$

Corollary (using Dixon 1973). *$\mathscr{M}$ has a bounded two-sided approximate identity.*

Set $E_{\omega_1} = \left(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\right)_{c_0}$, and recall that

$$T \in \mathscr{M} \iff T \text{ factors through } E_{\omega_1}.$$

Theorem (Kania–L). *Let $T \in \mathscr{B}(C_0[0,\omega_1))$. Then TFAE:*

(a) *T fixes a copy of E_{ω_1};*

(b) *$I_{E_{\omega_1}}$ factors through T;*

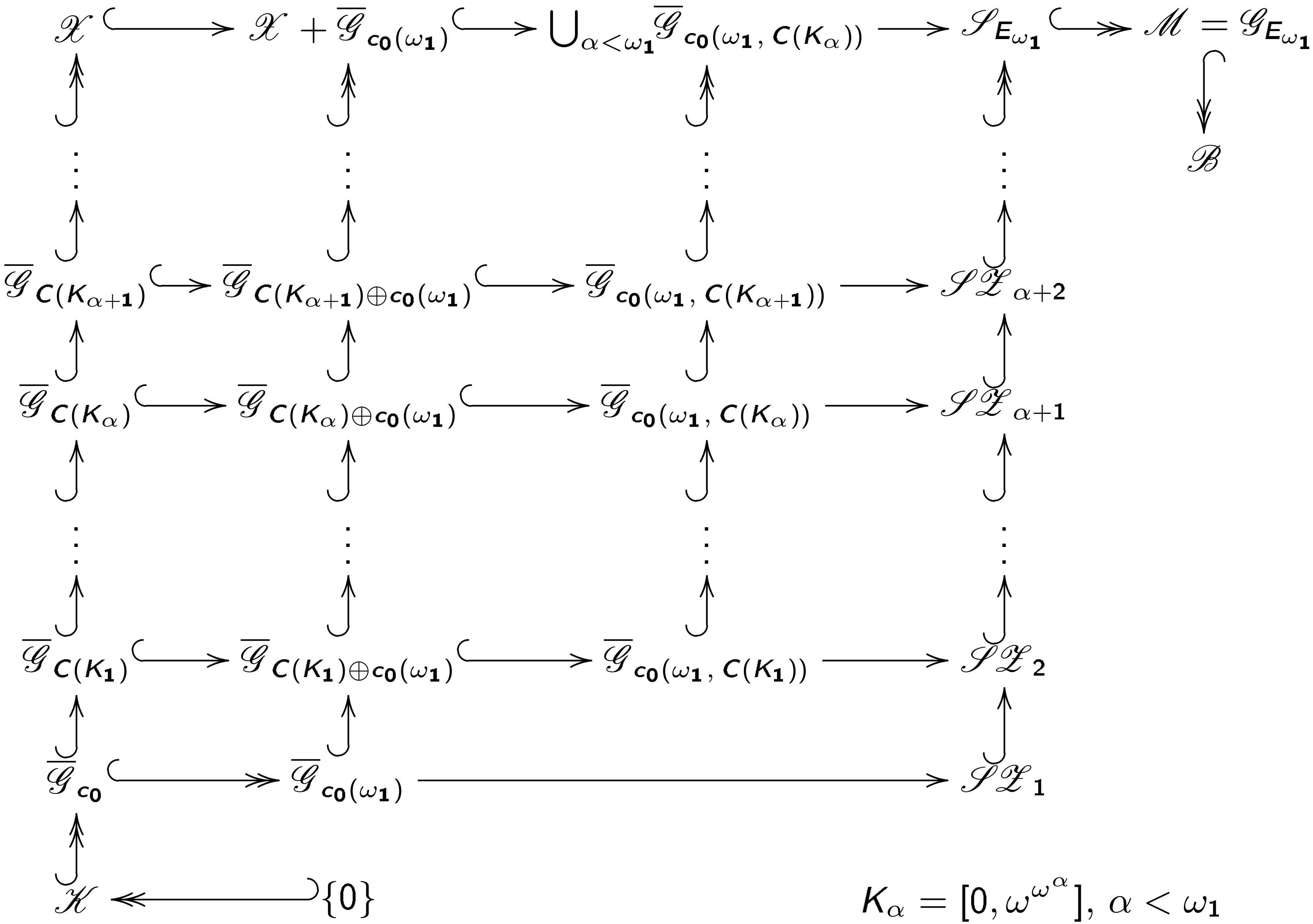
(c) *the Szlenk index of T is uncountable.*

Corollary. *The set*

$$\begin{aligned}\mathscr{S}_{E_{\omega_1}}(C_0[0,\omega_1)) &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : T \text{ does not fix a copy of } E_{\omega_1}\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \forall R \in \mathscr{B}(E_{\omega_1}, C_0[0,\omega_1)), \\ &\qquad \forall S \in \mathscr{B}(C_0[0,\omega_1), E_{\omega_1}) : I_{E_{\omega_1}} \neq STR\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \operatorname{Sz} T < \omega_1\}\end{aligned}$$

is the second-largest proper closed ideal of $\mathscr{B}(C_0[0,\omega_1))$: for each proper ideal $\mathscr{I}$ of $\mathscr{B}(C_0[0,\omega_1))$, either $\mathscr{I} = \mathscr{M}$ or $\mathscr{I} \subseteq \mathscr{S}_{E_{\omega_1}}(C_0[0,\omega_1))$.

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C_0[0, \omega_1))$



$K_\alpha = [0, \omega^{\omega^\alpha}]$, $\alpha < \omega_1$

Conventions

- We suppress $C_0[0,\omega_1)$ everywhere, thus writing $\mathscr{K}$ instead of $\mathscr{K}(C_0[0,\omega_1))$ for the ideal of compact operators on $C_0[0,\omega_1)$, *etc.*;
- $\mathscr{I} \hookrightarrow \mathscr{J}$ means that the ideal $\mathscr{I}$ is properly contained in the ideal $\mathscr{J}$;
- $\mathscr{I} \hookrightarrow\!\!\!\!\!\twoheadrightarrow \mathscr{J}$ indicates that there are no closed ideals between $\mathscr{I}$ and $\mathscr{J}$;
- $\mathscr{G}_X$ denotes the set of operators that factor through the Banach space X and $\overline{\mathscr{G}_X}$ its closure;
- $c_0(\omega_1, X)$ denotes the c_0-direct sum of ω_1 copies of the Banach space X, and $c_0(\omega_1) := c_0(\omega_1, \mathbb{K})$;
- $\mathscr{X}$ denotes the ideal of operators with separable range.

A few references (in chronological order)

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