Localization and projections on bi–parameter BMO

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Overview

1. A description of the problem class
2. One dimension
3. Two dimensions

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A description of the problem class

- Let X be a Banach space
- $T : X \to X$ a linear operator

Find conditions on T and X such that the identity on X factors through T (or $\mathrm{Id} - T$), i.e.

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}} & X \\ {\scriptstyle E}\downarrow & & \uparrow{\scriptstyle P} \\ X & \xrightarrow[T]{} & X \end{array} \qquad \|E\|\|P\| \le C.$$

- The problem has finite dimensional (quantitative) and infinite dimensional (qualitative) aspects.
- Classical examples include: ℓ^p (Pelczynski), $C([0,1])$ (Lindenstrauss-Pelczynski), L^p (Gamlen-Gaudet), L^1 (Enflo-Starbird), ℓ^p_n (Bourgain-Tzafriri).
- Generically, we expect that T has a large diagonal with respect to an unconditional basis in X.
- We consider special cases where $X = H^p(H^q)$ with the bi-parameter Haar system as its unconditional basis.

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Dyadic H^p

- $\mathscr{D} = \{[\frac{k-1}{2^n}, \frac{k}{2^n}[: \, k \geq 0, n \geq 0\}$ denotes the dyadic intervals on the unit interval,
- h_I the L^∞-normalized Haar function, $I \in \mathscr{D}$.
- Let $f = \sum_{I \in \mathscr{D}} a_I h_I$ be a finite linear combination,
- then the square function $\mathbb{S}(f)$ of f is given by

$$\mathbb{S}(f) = \Big(\sum_{I \in \mathscr{D}} a_I^2 h_I^2 \Big)^{1/2}.$$

- The norm of the one-parameter Hardy space H^p, $1 \leq p < \infty$ is defined by

$$\|f\|_{H^p} = \| \mathbb{S}(f) \|_{L^p} = \Big(\int_0^1 \Big(\sum_{I \in \mathscr{D}} a_I^2 h_I^2(x) \Big)^{p/2} \, \mathrm{d}x \Big)^{1/p}.$$

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Andrew 1D

Theorem

Let $1 < p < \infty$, $\delta > 0$ and $T : H^p \to H^p$ be a linear operator with large diagonal, i.e. $\langle Th_I, h_I \rangle \geq \delta |I|$. Then we have

$$\begin{CD} H^p @>{\mathrm{Id}}>> H^p \\ @V{E}VV @AA{P}A \\ H^p @>>{T}> H^p \end{CD} \qquad \|E\|\|P\| \leq C/\delta,$$

where the constant $C > 0$ is universal.

By Gamlen-Gaudet construction $(\mathscr{B}_I)$ and a random choice of signs ε_I there exists a block basis $b_I^{(\varepsilon)} = \sum_{K \in \mathscr{B}_I} \varepsilon_K h_K$ of the Haar system such that

$$Tb_I^{(\varepsilon)} = \alpha_I b_I^{(\varepsilon)} + \text{small error}, \qquad \alpha_I \geq \delta.$$

The orthogonal projection $Qf = \sum_{I \in \mathscr{D}} \frac{\langle f, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} b_I$ is bounded on H^p (Gamlen-Gaudet).

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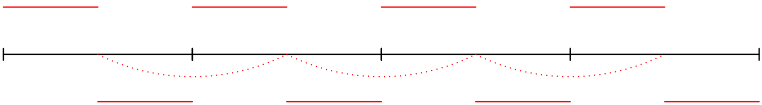
The Gamlen-Gaudet construction



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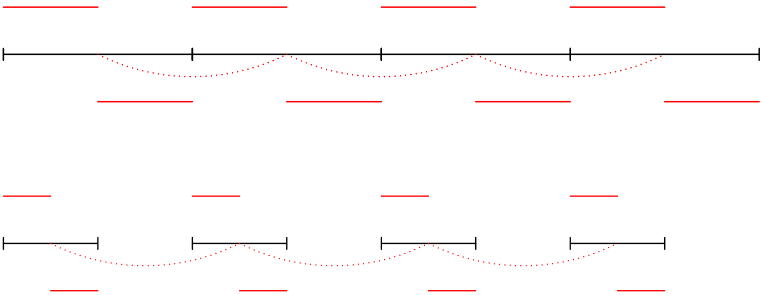

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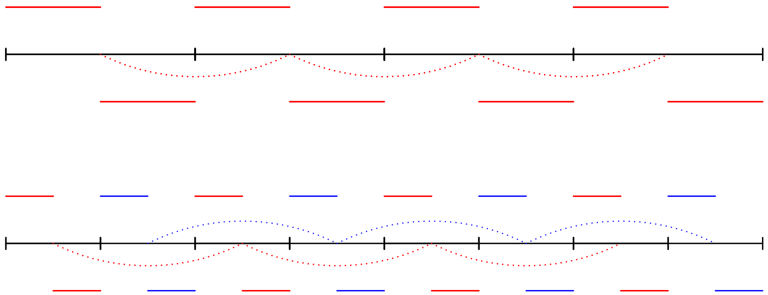

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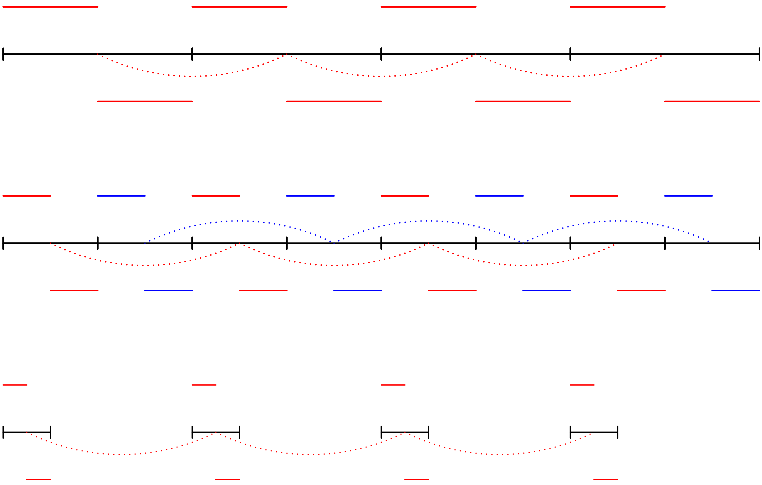

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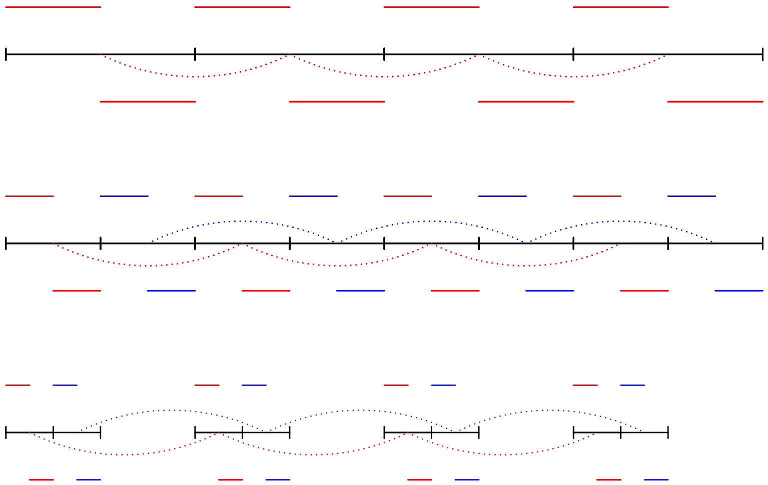

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- Andrew precedes the study of operators with property A (Johnson-Maurey-Schechtman-Tzafriri).
- The Rademacher system converges weakly to 0 in H^p.
- The operator T is preconditioned by multiplying the Haar system with highly oscillating Rademacher functions.
- This gives that $\langle Tb_I^{(\varepsilon)}, b_J^{(\varepsilon)} \rangle \approx 0$, if $I \neq J$.
- The second part consists of choosing signs ε_I such that $\langle Tb_I^{(\varepsilon)}, b_I^{(\varepsilon)} \rangle \geq \delta \|b_I^{(\varepsilon)}\|_2^2$.
- The block basis $\{b_I^{(\varepsilon)} : I \in \mathscr{D}\}$ is equivalent to the Haar system.
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Mixed-norm Hardy spaces $H^p(H^q)$

- $\mathscr{R} = \{I \times J : I, J \in \mathscr{D}\}$ denotes the dyadic rectangles on the unit square,
- $h_{I\times J}(x, y) = h_I(x)h_J(y)$ the L^∞-normalized tensor product Haar function, $I \times J \in \mathscr{R}$.
- Let $f = \sum_{I\times J\in\mathscr{R}} a_{I\times J} h_{I\times J}$ be a finite linear combination,
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$$\mathbb{S}(f) = \Big(\sum_{I\times J\in\mathscr{R}} a_{I\times J}^2 h_{I\times J}^2 \Big)^{1/2}.$$

- we define the norm of the bi-parameter Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ by

$$\|f\|_{H^p(H^q)} = \Big(\int_0^1 \Big(\int_0^1 \big(\sum_{I\in\mathscr{D}} a_{I\times J}^2 h_{I\times J}^2(x, y) \big)^{q/2} \mathrm{d}y \Big)^{p/q} \mathrm{d}x \Big)^{1/p}.$$

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Mixed-norm Hardy spaces $H^p(H^q)$

- $\mathscr{R} = \{I \times J \,:\, I, J \in \mathscr{D}\}$ denotes the dyadic rectangles on the unit square,
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Capon's theorem: $H^p(H^q)$ is primary (augmented)

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Let $1 < p, q < \infty$ or $p = q = 1$. For any operator T the identity on $H^p(H^q)$ factors through $H = T$ or $H = \mathrm{Id} - T$, i.e.

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$$H^1_N(H^1_N) = \operatorname{span}\{h_{I\times J} \,:\, |I|,|J| \geq 2^{-N}\} \subset H^1(H^1).$$

Theorem (R. L. & P. F. X. M.)

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Combinatorial covering lemma

Let $\tau, \delta > 0$,

$$x \in H^1(H^1) \qquad \text{and} \qquad \|x\| \leq 1.$$

There exists a large collection of pairwise disjoint dyadic intervals K such that

$$|\langle x, h_{K\times[0,1]}\rangle| \leq \tau|K| \quad \text{and} \quad |K| \geq \delta^2\tau^2 \quad (m \geq -\log_2 \delta^2\tau^2).$$

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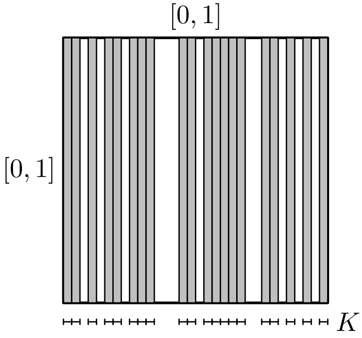

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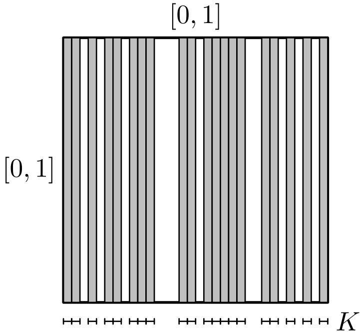

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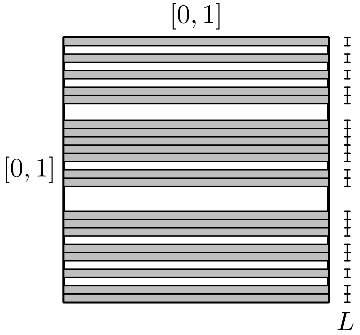

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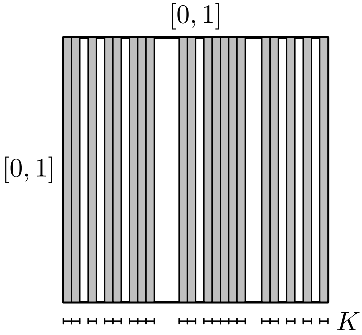

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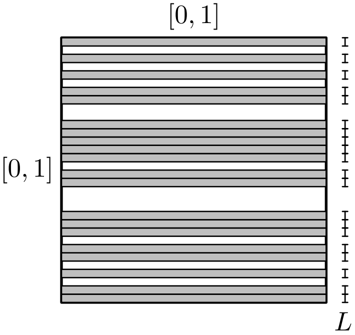

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Andrew 2D

Theorem (R. L. & P. F. X. M.)

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$$\begin{array}{ccc} H^p(H^q) & \xrightarrow{\mathrm{Id}} & H^p(H^q) \\ {\scriptstyle E}\downarrow & & \uparrow{\scriptstyle P} \\ H^p(H^q) & \xrightarrow[T]{} & H^p(H^q) \end{array} \qquad \|E\|\|P\| \le C/\delta,$$

where the constant $C > 0$ is universal.

By Capon's bi-parameter construction $(\mathscr{B}_{I\times J})$ and a random choice of signs $\varepsilon_{I\times J}$, there exists a block basis $b^{(\varepsilon)}_{I\times J} = \sum_{K \in \mathscr{B}_{I\times J}} \varepsilon_{K\times L} h_{K\times L}$ of the Haar system such that

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The orthogonal projection $Qf = \sum_{I \in \mathscr{D}} \frac{\langle f, b_{I\times J}\rangle}{\|b_{I\times J}\|_2^2} b_{I\times J}$ is bounded on $H^p(H^q)$.

Andrew 2D

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Summary

Let $T : H^p(H^q) \to H^p(H^q)$ be a linear operator. When does the identity factor through T (or $\mathrm{Id} - T$)?

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Outlook

- Full characterization in 2D for $H^p(H^q)$
  - 2D version of property A,
  - Projection Q onto more general block basis $b_{I\times J}$ in $H^p(H^q)$.

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Outlook

- Full characterization in 2D for $H^p(H^q)$
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  - Projection Q onto more general block basis [illegible]

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Outlook

- Full characterization in 2D for $H^p(H^q)$
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Summary

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Outlook

- Full characterization in 2D for $H^p(H^q)$
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Outlook

- Full characterization in 2D for $H^p(H^q)$
  - 2D version of property A,
  - Projection Q onto more general block basis (e.g. in $H^p(H^q)$)

Summary

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- Full characterization in 2D for $H^p(H^q)$
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Outlook

- Full characterization in 2D for $H^p(H^q)$
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- Full characterization in 2D for $H^p(H^q)$
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Outlook

- Full characterization in 2D for $H^p(H^q)$???
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Outlook

- Full characterization in 2D for $H^p(H^q)$?
  - 2D version of property A,
  - Projection Q onto more general block basis $b_{I \times J}$ in $H^p(H^q)$.

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  - 2D version of property A,
  - Projection Q onto more general block basis $b_{I\times J}$ in $H^p(H^q)$.