Strong pseudoconvexity in Banach spaces

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Relations between Banach space theory
and geometric measure theory
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Convexity and pseudoconvexity

Convexity with respect to a class of functions:

Definition

Let U be an open subset of a complex Banach space X. Suppose that $\mathcal{F}$ is a class of upper semicontinuous real-valued functions on U.

The $\mathcal{F}$-convex hull of a compact set $K \subset U$, denoted $\hat{K}_{\mathcal{F}}$, is the set of all points of U that cannot be separated from K by a function in the class $\mathcal{F}$.

The open set U is called $\mathcal{F}$-convex if for every compact subset K of U, the $\mathcal{F}$-convex hull $\hat{K}_{\mathcal{F}}$ is a compact subset of U.

Remark (For open subsets of a complex Banach space)

Geometric convexity:
Convexity with respect to (the real part of) affine linear functions.

Pseudoconvexity:
Convexity with respect to plurisubharmonic functions.

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$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

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Another characterization of pseudoconvexity (Mujica, ’86):

Proposition

An open subset U of a complex Banach space X is pseudoconvex if and only if the function $-\log d_U$ is plurisubharmonic on U (where d_U denotes the distance to the boundary of U).

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Strict pseudoconvexity in the case of C^2 boundary

What is strong (or strict) pseudoconvexity?

Definition

Let U be an open subset of a complex Banach space X, and let $f : U \to \mathbb{R}$ be an $\mathbb{R}$-differentiable mapping. Let $Df(a)$ denote the real differential of f at a. Then let $D'f(a)$ and $D''f(a)$ be defined by

$$D'f(a)(t) = 1/2[Df(a)(t) - iDf(a)(it)],$$

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for every $t \in X$.

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A function $f \in C^2(U, \mathbb{R})$ is plurisubharmonic iff for each $a \in U$ and $b \in X$ we have that $D'D''f(a)(b,b) \geq 0$.

With that in mind, and the 2-homogeneity of $b \mapsto D'D''f(a)(b,b)$, strict plurisubharmonicity is defined as follows (Mujica, '86):

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Let U be an open subset of a complex Banach space X. A function $f \in C^2(U, \mathbb{R})$ is said to be strictly plurisubharmonic if for each $a \in U$ and $b \in S_X$ we have that $D'D''f(a)(b,b) > 0$.

Strict pseudoconvexity of an open set U with C^2 boundary is defined as having $-\log d_U$ strictly plurisubharmonic.

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Just as local uniform convexity and uniform convexity are special cases of strict convexity, we similarly define local uniform pseudoconvexity and uniform pseudoconvexity:

Definition

Let U be an open subset of a complex Banach space X, with C^2 boundary. U is said to be locally uniformly pseudoconvex at $a \in U$ if $-\log d_U$ is locally uniformly plurisubharmonic at a, i.e $\delta_U(a) := \inf\{D'D''(-\log d_U)(a)(b,b) : b \in S_X\} > 0$.

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(Local) uniform pseudoconvexity

Interesting phenomenon:

Proposition (O.-C., '14)

U open subset of a complex Banach space X, $f \in C^2(U, \mathbb{R})$.

Then f is locally uniformly plurisubharmonic at $a \in U$ iff $\exists\, C(a) > 0$ such that

$$C(a)\|b\|^2/4 \leq \frac{1}{2\pi}\int_0^{2\pi}(f(a+e^{i\theta}b) - f(a))d\theta \tag{1}$$

for each $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$.

Similarly, f is uniformly plurisubharmonic iff $\exists\, C > 0$ such that

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Now we can extend (local) uniform pseudoconvexity to any open and bounded set.

Definition

Let U open and bounded in X complex Banach space. U is said to be (locally) uniformly pseudoconvex (at $a \in U$) if $-\log d_U$ is (locally) uniformly plurisubharmonic (at a).

We can also extend strict pseudoconvexity to any open and bounded set, using the known fact that, in the finite-dimensional setting, strict plurisubharmonicity coincides with local uniform plurisubharmonicity.

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Let us now exhibit some Banach spaces whose unit ball is uniformly pseudoconvex:

Definition (Davis, Garling, Tomczak-Jaegermann, '84)

If $0 < q < \infty$, a Banach space X is 2-uniformly PL-convex if there exists $\lambda > 0$ such that

$$(\frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta}b\|^q d\theta)^{1/q} \geq (\|a\|^2 + \lambda\|b\|^2)^{1/2}$$

for all a and b in X.

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Davis, Garling and Tomczak-Jaegermann proved that for $p \in [1, 2]$, $L_p(\Sigma, \Omega, \mu)$ is 2-uniformly PL-convex, and the following result implies that it has uniformly pseudoconvex unit ball.

Theorem (O.-C., '14)

If X is a 2-uniformly PL-convex Banach space then B_X is uniformly pseudoconvex.

In contrast, the following theorem gives us that, for $p > 2$, the unit balls of ℓ_p and L_p are not even strictly pseudoconvex.

Theorem (O.-C., '14)

If $n \geq 2$ and $p > 2$, the unit ball of ℓ_p^n is not strictly pseudoconvex.

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If $n \geq 2$ and $p > 2$, the unit ball of ℓ_p^n is not strictly pseudoconvex.

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