On products of nuclear operators

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Nuclear operators

An operator $T : X \to Y$ is nuclear if it is of the form

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$

for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k|| \, ||y_k|| < \infty$.
We use the notation $N(X, Y)$

If T is nuclear, then

$$T : X \to c_0 \to l_1 \to Y.$$

A. Grothendieck, Produits tensoriels topologiques et espases nucléaires, Mem. Amer. Math. Soc., Volume 16, 1955, 196 + 140.

Let A be a compact operator in H. Then A has the norm convergent expansion

$$A = \sum_{n=1}^{N} \mu_n(A)\,(f_n, \cdot)h_n,$$

where (f_n), (h_n) are ONS's, $\mu_1(A) \geq \mu_2(A) \geq \cdots > 0)$

The $\mu_n(A)$ are called the singular values of A. Notation $s_n(A)$ or just s_n.

Simon B., Trace ideals and their applications, London Math. Soc. Lecture Notes 35, Cambridge University Press, 1979.

R. Schatten and J. von Neumann

- $$A \in S_p(H): \sum s_n^p(A) < \infty, \; p > 0.$$

- $$S_p \circ S_q \subset S_r, \; 1/r = 1/p + 1/q;$$

  $p, q \in (0, \infty)$

  $$N(H) = S_1(H).$$

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s-nuclear operators – Applications de puissance s.éme sommable

- An operator $T : X \to Y$ is s-nuclear $(0 < s \leq 1)$ if it is of the form
$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$
for all $x \in X$, where
$(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k||^s \, ||y_k||^s < \infty$. We use the notation $N_s(X, Y)$.

- $$N_p(H) = S_p(H), 0 < p \leq 1.$$

- R. Oloff, p-normierte Operatorenideale, Beiträge Anal. 4, 105-108 (1972).

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A natural question (due to Boris Mitjagin):

- Is it true that a product of two nuclear operators in Banach spaces can be factored through a trace class (i.e., S_1-) operator in a Hilbert space?
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Explanation

f is Carleman's continuous function:
$\hat{f} \in l_2 \setminus \cup_{p<2} l_p$.

$$T : C \stackrel{*f}{\to} C.$$

T is nuclear.

Consider the product TT. Note that eigenvalues $(\lambda_k(TT)) \in l_1$ and not better.

Suppose, there is an S_1-operator $U \in S_1(H)$ so that

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Explanation (continued)

Consider

$$H \xrightarrow{B} C \xrightarrow{A} H \xrightarrow{U} H \xrightarrow{B} C.$$

Eigenvalues of UAB = eigenvalues of $TT = BUA$ (and, so, in l_1).

BUT:

$$A \in \Pi_2; \quad \text{so,} \quad AB \in S_2; \; U \in S_1.$$

Hence,

$$UAB \in S_{2/3}.$$

Contradiction.

- *Remark.* Sharp fact is that if $V \in NN$, then it factors through an operator $U \in S_2$.

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General situation

- Let $\alpha, \beta \in (0,1]$. If $T \in N_\alpha \circ N_\beta$, then it factors through an S_r-operator, where

$$\frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{3}{2}.$$

- Particular cases:

$$\alpha = 1, \beta = \frac{2}{3} \implies r = 1;$$

$$\alpha = 1, \beta = 1 \implies r = 2.$$

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M. I. Zelikin

To formulate the theorem, we need a definition:

- The spectrum of A is central-symmetric, if together with any eigenvalue $\lambda \neq 0$ it has the eigenvalue $-\lambda$ of the same multiplicity.

  It was proved in a paper by M. I. Zelikin

M. I. Zelikin, A criterion for the symmetry of a spectrum", Dokl. Akad. Nauk 418 (2008), no. 6, 737-740

- **Theorem.** The spectrum of a nuclear operator A acting on a separable Hilbert space is central-symmetric iff *trace* $A^{2n-1} = 0$, $n \in \mathbf{N}$.

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Generalization

We can proof:

- **Theorem.** Let Y be a subspace of a quotient (or a quotient of a subspace) of an L_p-space, $1 \leq p \leq \infty$ and $T \in N_s(Y, Y)$ (s-nuclear), where $1/s = 1 + |1/2 - 1/p|$, The spectrum of T is central-symmetric iff *trace* $T^{2n-1} = 0, n = 1, 2, \ldots.$
- *Remark*: In the theorem "trace" is well defined. The result is sharp.
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Thank you for your attention!