Closed Ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Th. Schlumprecht
(joint with A. Zsák)

Warwick, June 2015

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called closed ideal in $\mathcal{L}(X, Y)$ if:

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called closed ideal in $\mathcal{L}(X, Y)$ if:

- $\mathcal{J}$ linear subspace of $\mathcal{L}(X, Y)$,

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called closed ideal in $\mathcal{L}(X, Y)$ if:

- $\mathcal{J}$ linear subspace of $\mathcal{L}(X, Y)$,
- If $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$, and $T \in \mathcal{J}$ then $A \circ T \circ B \in \mathcal{J}$,

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called closed ideal in $\mathcal{L}(X, Y)$ if:

- $\mathcal{J}$ linear subspace of $\mathcal{L}(X, Y)$,
- If $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$, and $T \in \mathcal{J}$ then $A \circ T \circ B \in \mathcal{J}$,
- $\mathcal{J}$ closed in operator norm.

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called **closed ideal in $\mathcal{L}(X, Y)$** if:

- $\mathcal{J}$ linear subspace of $\mathcal{L}(X, Y)$,
- If $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$, and $T \in \mathcal{J}$ then $A \circ T \circ B \in \mathcal{J}$,
- $\mathcal{J}$ closed in operator norm.

$\mathcal{J}$ is called **proper** if $\mathcal{J} \subsetneq \mathcal{L}(X, Y)$, **non trivial** if $\{0\} \subsetneq \mathcal{J} \subsetneq \mathcal{L}(X, Y)$.

Definition

For two Banach spaces X and Y

$\mathcal{L}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

$\mathcal{J} \subset \mathcal{L}(X, Y)$ is called closed ideal in $\mathcal{L}(X, Y)$ if:

- $\mathcal{J}$ linear subspace of $\mathcal{L}(X, Y)$,
- If $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$, and $T \in \mathcal{J}$ then $A \circ T \circ B \in \mathcal{J}$,
- $\mathcal{J}$ closed in operator norm.

$\mathcal{J}$ is called proper if $\mathcal{J} \subsetneq \mathcal{L}(X, Y)$, non trivial if $\{0\} \subsetneq \mathcal{J} \subsetneq \mathcal{L}(X, Y)$.

Main Problem

The closed ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$, $1 < p < q < \infty$.

Examples of closed ideals

Examples of closed ideals

$\mathcal{K}(X, Y) = \{T :\in \mathcal{L}(X, Y) : T \text{ compact}\}$

Examples of closed ideals

$\mathcal{K}(X, Y) = \{T :\in \mathcal{L}(X, Y) : T \text{ compact}\}$

$\mathcal{S}(X, Y) = \{T :\in \mathcal{L}(X, Y) : T \text{ strictly singular}\}$

Examples of closed ideals

$\mathcal{K}(X, Y) = \{T :\in \mathcal{L}(X, Y) : T \text{ compact}\}$

$\mathcal{S}(X, Y) = \{T :\in \mathcal{L}(X, Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{F}S(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{F}S(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

$T : X \to Y$ finitely strictly singular if

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{F}S(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

$T : X \to Y$ finitely strictly singular if

$\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \ \forall F \hookrightarrow X, \dim(F) = n \ \exists x \in S_F \quad \|T(x)\| \leq \varepsilon$

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{FS}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

$T : X \to Y$ finitely strictly singular if

$$\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \forall F \hookrightarrow X, \dim(F) = n\ \exists x \in S_F \quad \|T(x)\| \le \varepsilon$$

Let U, V Banach spaces and $S \in \mathcal{L}(U,V)$, the ideal in $\mathcal{L}(X,Y)$ generated by S is the smallest closed ideal $\mathcal{J}^S$ in $\mathcal{L}(X,Y)$, containing $A \circ S \circ B$, for all $A \in \mathcal{L}(V,Y)$ and $B \in \mathcal{L}(X,U)$,

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{F}S(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

$T : X \to Y$ finitely strictly singular if

$\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \forall F \hookrightarrow X, \dim(F) = n\ \exists x \in S_F \quad \|T(x)\| \le \varepsilon$

Let U, V Banach spaces and $S \in \mathcal{L}(U,V)$, the ideal in $\mathcal{L}(X,Y)$ generated by S is the smallest closed ideal $\mathcal{J}^S$ in $\mathcal{L}(X,Y)$, containing $A \circ S \circ B$, for all $A \in \mathcal{L}(V,Y)$ and $B \in \mathcal{L}(X,U)$, *i.e.,*

$$\mathcal{J}^S(X,Y) = \overline{\Big\{ \sum_{j=1}^n A_j \circ S \circ B_j : A_j \in \mathcal{L}(V,Y), B_j \in \mathcal{L}(X,U) \Big\}}.$$

Examples of closed ideals

$\mathcal{K}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ compact}\}$

$\mathcal{S}(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ strictly singular}\}$

$T : X \to Y$ strictly singular if $\forall Z \hookrightarrow X, \dim(Z) = \infty \quad T|_Z$ not isom.

$\mathcal{F}S(X,Y) = \{T :\in \mathcal{L}(X,Y) : T \text{ finitely strictly singular}\}$

$T : X \to Y$ finitely strictly singular if

$\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \forall F \hookrightarrow X, \dim(F) = n\ \exists x \in S_F \quad \|T(x)\| \le \varepsilon$

Let U, V Banach spaces and $S \in \mathcal{L}(U,V)$, the ideal in $\mathcal{L}(X,Y)$ generated by S is the smallest closed ideal $\mathcal{J}^S$ in $\mathcal{L}(X,Y)$, containing $A \circ S \circ B$, for all $A \in \mathcal{L}(V,Y)$ and $B \in \mathcal{L}(X,U)$, *i.e.*,

$$\mathcal{J}^S(X,Y) = \overline{\Big\{ \sum_{j=1}^n A_j \circ S \circ B_j : A_j \in \mathcal{L}(V,Y), B_j \in \mathcal{L}(X,U) \Big\}}.$$

If $U = V$ and S identity we write $\mathcal{J}^U(X,Y)$, closed ideal generated by the operators factoring through the space U.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Assume $1 < p < q < \infty$.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Assume $1 < p < q < \infty$.

$\mathcal{K}(\ell_p)$ only non trivial ideal of $\mathcal{L}(\ell_p)$ (Gohberg, Markus, Feldman '60)

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Assume $1 < p < q < \infty$.

$\mathcal{K}(\ell_p)$ only non trivial ideal of $\mathcal{L}(\ell_p)$ (Gohberg, Markus, Feldman '60)

$\mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$ and $\mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$ (Pitt '31)

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Assume $1 < p < q < \infty$.

$\mathcal{K}(\ell_p)$ only non trivial ideal of $\mathcal{L}(\ell_p)$ (Gohberg, Markus, Feldman '60)

$\mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$ and $\mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$ (Pitt '31)

Let $T \in \mathcal{L}(\ell_p \oplus \ell_q)$, and write

$$T = \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix}$$

with $T_{(1,1)} \in \mathcal{L}(\ell_p)$, $T_{(1,2)} = \mathcal{K}(\ell_q, \ell_p)$, $T_{(2,1)} \in \mathcal{S}(\ell_p, \ell_q)$, $T_{(2,2)} \in \mathcal{L}(\ell_q)$

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Assume $1 < p < q < \infty$.

$\mathcal{K}(\ell_p)$ only non trivial ideal of $\mathcal{L}(\ell_p)$ (Gohberg, Markus, Feldman '60)

$\mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$ and $\mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$ (Pitt '31)

Let $T \in \mathcal{L}(\ell_p \oplus \ell_q)$, and write

$$T = \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix}$$

with $T_{(1,1)} \in \mathcal{L}(\ell_p)$, $T_{(1,2)} = \mathcal{K}(\ell_q, \ell_p)$, $T_{(2,1)} \in \mathcal{S}(\ell_p, \ell_q)$, $T_{(2,2)} \in \mathcal{L}(\ell_q)$

It follows that (Volkmann '76):

$\mathcal{J}^{\ell_p} = \{T \in \mathcal{L}(\ell_p \oplus \ell_q) : T_{(2,2)} \in \mathcal{K}(\ell_q)\}$ and

$\mathcal{J}^{\ell_q} = \{T \in \mathcal{L}(\ell_p \oplus \ell_q) : T_{(1,1)} \in \mathcal{K}(\ell_p)\}$

are the only two maximal proper closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$.

Reduction to ideals in $\mathcal{L}(\ell_p, \ell_q)$

All other closed ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ are of the form

$$\tilde{\mathcal{J}} = \{T \in \mathcal{L}(\ell_p \oplus \ell_q) : T_{(1,1)} \in \mathcal{K}(\ell_p), T_{(2,2)} \in \mathcal{K}(\ell_q), T_{(2,1)} \in \mathcal{J}\}$$

where $\mathcal{J}$ is a closed ideal in $\mathcal{L}(\ell_p, \ell_q)$, and the map $\mathcal{J} \to \tilde{\mathcal{J}}$ is a bijection between the closed ideals of $\mathcal{L}(\ell_p, \ell_q)$ and the non maximal proper closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$.

Reduction to ideals in $\mathcal{L}(\ell_p, \ell_q)$

All other closed ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ are of the form

$$\tilde{\mathcal{J}} = \{T \in \mathcal{L}(\ell_p \oplus \ell_q) : T_{(1,1)} \in \mathcal{K}(\ell_p), T_{(2,2)} \in \mathcal{K}(\ell_q), T_{(2,1)} \in \mathcal{J}\}$$

where $\mathcal{J}$ is a closed ideal in $\mathcal{L}(\ell_p, \ell_q)$, and the map $\mathcal{J} \to \tilde{\mathcal{J}}$ is a bijection between the closed ideals of $\mathcal{L}(\ell_p, \ell_q)$ and the non maximal proper closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$.
Thus, the study of the closed ideals $\mathcal{L}(\ell_p \oplus \ell_q)$ reduces to the study of the closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Previous Results

Previous Results

Theorem (Milman '70)

Previous Results

Theorem (Milman '70)

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subsetneq \mathcal{L}(\ell_p, \ell_q)$$

where $I_{p,q} : \ell_p \to \ell_q$ *is the formal identity map.*

Previous Results

Theorem (Milman '70)

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{F}S \subsetneq \mathcal{L}(\ell_p, \ell_q)$$

where $I_{p,q} : \ell_p \to \ell_q$ is the formal identity map.

Previous Results

Theorem (Milman '70)

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$$

where $I_{p,q} : \ell_p \to \ell_q$ is the formal identity map.

Sketch:

$\mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$

Pełczyński: ℓ_p isomorphic to $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$.

Consider: $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p} \ni (x_n) \mapsto (x_n) \in (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_q}$

Previous Results

Theorem (Milman '70)

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$$

where $I_{p,q} : \ell_p \to \ell_q$ is the formal identity map.

Sketch:

$\mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$

Pełczyński: ℓ_p isomorphic to $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$.

Consider: $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p} \ni (x_n) \mapsto (x_n) \in (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_q}$

Lemma: If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)

Previous Results

Theorem (Milman '70)

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$$

where $I_{p,q} : \ell_p \to \ell_q$ is the formal identity map.

Sketch:

$\mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q)$

Pełczyński: ℓ_p isomorphic to $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$.

Consider: $(\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p} \ni (x_n) \mapsto (x_n) \in (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_q}$

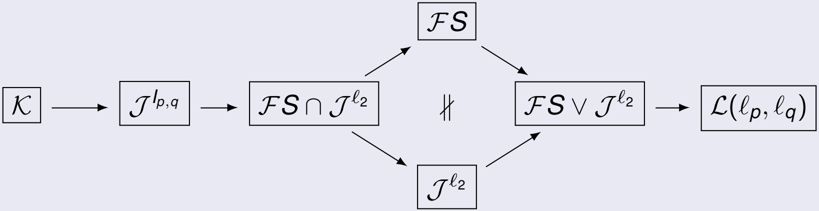
Lemma: If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)

Thus in each n-dimensional subspace of ℓ_p there is a vector $x \neq 0$ with $\|x\|_q^q \leq \frac{1}{n^{q-p}} \|x\|_p^p$. Thus $I_{p,q} \in \mathcal{FS}$.

Assume $1 < p < 2 < q < \infty$

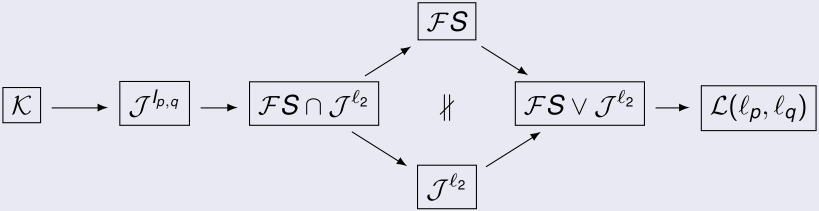
Assume $1 < p < 2 < q < \infty$

Theorem (Sari, S, Tomczak-Jaegerman, Troitsky '07)

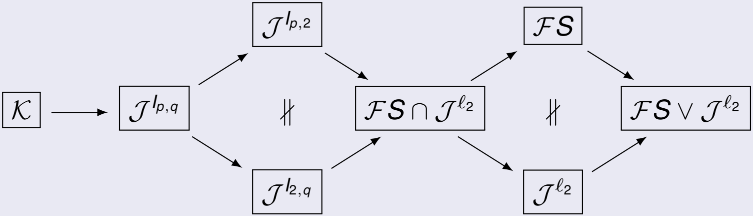


Assume $1 < p < 2 < q < \infty$

Theorem (Sari, S, Tomczak-Jaegerman, Troitsky '07)



Theorem (S '11)



Main Result

Answering a question by Pietsch '78

Theorem (B. Wallis, 2015)

The same conclusion is true if $1 < p < 2$ *and* ℓ_q *is replaced by* c_0*, and if* $p = 1$ *and* $2 < q < \infty$*.*

Main Result

Answering a question by Pietsch '78

Theorem (Zsák & S)

There are infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Theorem (B. Wallis, 2015)

The same conclusion is true if $1 < p < 2$ and ℓ_q is replaced by c_0, and if $p = 1$ and $2 < q < \infty$.

Main Result

Answering a question by Pietsch '78

Theorem (Zsák & S)

There are infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$. More precisely: There is a family $(T_r : r \in [0, 1))$ of operators so that

$$\mathcal{J}^{I_{p,q}} \subsetneq \mathcal{J}^{T_r} \subsetneq \mathcal{J}^{T_s} \subsetneq \mathcal{F}S, \text{ if } r < s.$$

and if $1 < p < 2 < q < \infty$ then

$$\mathcal{J}^{I_{p,q}} \subsetneq \mathcal{J}^{T_r} \subsetneq \mathcal{J}^{T_s} \subsetneq \mathcal{F}S \cap \mathcal{J}^{\ell_2}, \text{ if } r < s.$$

Theorem (B. Wallis, 2015)

The same conclusion is true if $1 < p < 2$ and ℓ_q is replaced by c_0, and if $p = 1$ and $2 < q < \infty$.

Generic Construction

Generic Construction

W.l.o.g. $1 < p < 2$

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q)$ $\iff$ $\mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q)$ $\iff$ $\mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Generic Construction

W.l.o.g. $1 < p < 2$

$(\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'}))$

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_q}$, $k_n = 3^n$ $(L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n})$

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

$\dim(F_n) = n$,

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q)$ $\iff$ $\mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^\infty \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^\infty \ell_q^n)_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,
- has c-unconditional normalized basis $(f_j^{(n)})_{j=1}^n$.

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,
- has c-unconditional normalized basis $(f_j^{(n)})_{j=1}^n$.

Khintchin:

$$\frac{1}{C(c,p)} \Big(\sum_{j=1}^n |a_j|^2 \Big)^{1/2} \le \Big\| \sum_{j=1}^n a_j f_j^{(n)} \Big\| \le C(c,p) \Big(\sum_{j=1}^n |a_j|^p \Big)^{1/p}.$$

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,
- has c-unconditional normalized basis $(f_j^{(n)})_{j=1}^{n}$.

Khintchin:

$$\frac{1}{C(c,p)} \Big(\sum_{j=1}^{n} |a_j|^2 \Big)^{1/2} \leq \Big\| \sum_{j=1}^{n} a_j f_j^{(n)} \Big\| \leq C(c,p) \Big(\sum_{j=1}^{n} |a_j|^p \Big)^{1/p}.$$

The "formal identity" $I_F : (\oplus F_n)_{\ell_p} \to (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_q} \qquad f_j^{(n)} \mapsto e_j^{(n)}$

Generic Construction

W.l.o.g. $1 < p < 2$

$(\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'}))$

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ $(L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n})$

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,
- has c-unconditional normalized basis $(f_j^{(n)})_{j=1}^n$.

Khintchin:

$$\frac{1}{C(c,p)} \Big(\sum_{j=1}^n |a_j|^2 \Big)^{1/2} \leq \Big\| \sum_{j=1}^n a_j f_j^{(n)} \Big\| \leq C(c,p) \Big(\sum_{j=1}^n |a_j|^p \Big)^{1/p}.$$

The "formal identity" $I_F : (\oplus F_n)_{\ell_p} \to (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_q}$ $\quad f_j^{(n)} \mapsto e_j^{(n)}$ is bounded, and factors through $I_{2,q}$ and thus

Generic Construction

W.l.o.g. $1 < p < 2$

($\mathcal{J}$ cl. ideal in $\mathcal{L}(\ell_p, \ell_q) \iff \mathcal{J}^* = \{T^* : T \in \mathcal{J}\}$ cl. ideal in $\mathcal{L}(\ell_{q'}, \ell_{p'})$)

For simplicity we also assume $1 < p < 2 < q < \infty$.

Write $\ell_p = (\oplus_{n=1}^{\infty} \ell_p^{k_n})_{\ell_p}$ and $\ell_q = (\oplus_{n=1}^{\infty} \ell_q^{n})_{\ell_q}$, $k_n = 3^n$ ($L_p(\{-1,0,1\}^n) \equiv \ell_p^{k_n}$)

Assume $F = (F_n)$ with $F_n \hookrightarrow \ell_p^{k_n}$ has following property

- $\dim(F_n) = n$,
- uniformly (with resp. to $n \in \mathbb{N}$) complemented in $\ell_p^{k_n}$,
- has c-unconditional normalized basis $(f_j^{(n)})_{j=1}^n$.

Khintchin:

$$\frac{1}{C(c,p)} \Big(\sum_{j=1}^n |a_j|^2 \Big)^{1/2} \leq \Big\| \sum_{j=1}^n a_j f_j^{(n)} \Big\| \leq C(c,p) \Big(\sum_{j=1}^n |a_j|^p \Big)^{1/p}.$$

The "formal identity" $I_F : (\oplus F_n)_{\ell_p} \to (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_q} \qquad f_j^{(n)} \mapsto e_j^{(n)}$ is bounded, and factors through $l_{2,q}$ and thus

$\forall \varepsilon \geq 0 \, \exists \delta \geq 0$ If $x \in (\oplus F_n)_{\ell_p}$, $\|x\| \leq 1$ & $\|x\|_\infty < \delta$, then $\|I_F(x)\| < \varepsilon$. $(*)$.

Wanted: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$

Wanted: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$

Assume that also $G = (G_n)$ is a sequence of spaces G_n, $\dim(G_n) = n$, with $G_n \hookrightarrow \ell_p^{k_n}$, uniformly complemented, and G_n has a c-unconditional and normalized basis $(g_j^{(n)})_{j=1}^n$.

Wanted: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$

Assume that also $G = (G_n)$ is a sequence of spaces G_n, $\dim(G_n) = n$, with $G_n \hookrightarrow \ell_p^{k_n}$, uniformly complemented, and G_n has a c-unconditional and normalized basis $(g_j^{(n)})_{j=1}^n$.

Want: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$.

Wanted: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$

Assume that also $G = (G_n)$ is a sequence of spaces G_n, $\dim(G_n) = n$, with $G_n \hookrightarrow \ell_p^{k_n}$, uniformly complemented, and G_n has a c-unconditional and normalized basis $(g_j^{(n)})_{j=1}^n$.

Want: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$.

Note: If $(f_j^{(n)})$ uniformly (in n) dominates $(g_j^{(n)})$, and thus operator

$$I_{F,G} : (\oplus F_n)_{\ell_p} \to (\oplus G_n)_{\ell_p}, \quad f_j^{(n)} \mapsto g_j^{(n)}$$

is bounded, and $I_F = I_G \circ I_{G,F}$, it follows that $\mathcal{J}^{I_F} \subset \mathcal{J}^{I_G}$.

Assume that also $G = (G_n)$ is a sequence of spaces G_n, $\dim(G_n) = n$, with $G_n \hookrightarrow \ell_p^{k_n}$, uniformly complemented, and G_n has a c-unconditional and normalized basis $(g_j^{(n)})_{j=1}^n$.

Want: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$.

Note: If $(f_j^{(n)})$ uniformly (in n) dominates $(g_j^{(n)})$, and thus operator

$$I_{F,G} : (\oplus F_n)_{\ell_p} \to (\oplus G_n)_{\ell_p}, \quad f_j^{(n)} \mapsto g_j^{(n)}$$

is bounded, and $I_F = I_G \circ I_{G,F}$, it follows that $\mathcal{J}^{I_F} \subset \mathcal{J}^{I_G}$.

Need: Operator $T \in \mathcal{J}^{I_G}$, but $T \notin \mathcal{J}^{I_F}$, and thus a functional $\Phi \in \mathcal{L}^*(\ell_p, \ell_q)$ so that

$$\Phi(T) = 1 \text{ and } \Phi|_{\mathcal{J}^{I_F}} \equiv 0.$$

Wanted: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$

Assume that also $G = (G_n)$ is a sequence of spaces G_n, $\dim(G_n) = n$, with $G_n \hookrightarrow \ell_p^{k_n}$, uniformly complemented, and G_n has a c-unconditional and normalized basis $(g_j^{(n)})_{j=1}^n$.

Want: $\mathcal{J}^{I_F} \subsetneq \mathcal{J}^{I_G}$.

Note: If $(f_j^{(n)})$ uniformly (in n) dominates $(g_j^{(n)})$, and thus operator

$$I_{F,G} : (\oplus F_n)_{\ell_p} \to (\oplus G_n)_{\ell_p}, \quad f_j^{(n)} \mapsto g_j^{(n)}$$

is bounded, and $I_F = I_G \circ I_{G,F}$, it follows that $\mathcal{J}^{I_F} \subset \mathcal{J}^{I_G}$.

Need: Operator $T \in \mathcal{J}^{I_G}$, but $T \notin \mathcal{J}^{I_F}$, and thus a functional $\Phi \in \mathcal{L}^*(\ell_p, \ell_q)$ so that

$$\Phi(T) = 1 \text{ and } \Phi|_{\mathcal{J}^{I_F}} \equiv 0.$$

Φ will be an accumulation point of elements Φ_i in $\ell_p \hat{\otimes} \ell_{q'} \equiv (\oplus_{n=1}^\infty \ell_p^{k_n})_{\ell_p} \hat{\otimes} (\oplus_{n=1}^\infty \ell_{q'}^n)_{\ell_{q'}}$.

Definition of T and (Φ_n)

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^{\infty} P_n : \big(\oplus_{n=1}^{\infty} \ell_p^{k_n} \big)_{\ell_p} \to \Big(\oplus_{n=1}^{\infty} G_n \Big)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.
Choose $T = I_G \circ P$.

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^{\infty} P_n : \left(\oplus_{n=1}^{\infty} \ell_p^{k_n} \right)_{\ell_p} \to \left(\oplus_{n=1}^{\infty} G_n \right)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.

Choose $T = I_G \circ P$.

For $n \in \mathbb{N}$ let $\Phi_n \in \ell_{q'} \otimes \ell_p$

$$\Phi_n = \frac{1}{n} \sum_{j=1}^{n} e_j^{(n)*} \otimes g_j^{(n)}, \quad (e_j^{(n)*}) \text{ unit basis of } \ell_{q'}^n.$$

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^{\infty} P_n : \big(\oplus_{n=1}^{\infty} \ell_p^{k_n}\big)_{\ell_p} \to \Big(\oplus_{n=1}^{\infty} G_n\Big)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.
Choose $T = I_G \circ P$.
For $n \in \mathbb{N}$ let $\Phi_n \in \ell_{q'} \otimes \ell_p$

$$\Phi_n = \frac{1}{n} \sum_{j=1}^{n} e_j^{(n)*} \otimes g_j^{(n)}, \quad (e_j^{(n)*}) \text{ unit basis of } \ell_{q'}^n.$$

Thus the Φ_n acts on a general $S \in L_p(\ell_p, \ell_q) = L(\oplus \ell_p^{k_n} . \oplus \ell_q^n)$ as follows:

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^{\infty} P_n : \left(\oplus_{n=1}^{\infty} \ell_p^{k_n}\right)_{\ell_p} \to \left(\oplus_{n=1}^{\infty} G_n \right)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.
Choose $T = I_G \circ P$.
For $n \in \mathbb{N}$ let $\Phi_n \in \ell_{q'} \otimes \ell_p$

$$\Phi_n = \frac{1}{n} \sum_{j=1}^{n} e_j^{(n)*} \otimes g_j^{(n)}, \quad (e_j^{(n)*}) \text{ unit basis of } \ell_{q'}^n.$$

Thus the Φ_n acts on a general $S \in L_p(\ell_p, \ell_q) = L(\oplus \ell_p^{k_n}. \oplus \ell_q^n)$ as follows: We restrict S to $F_n \subset \ell_p^{k_n}$ and then project the image onto ℓ_q^n). In terms of the bases $(f_j^{(n)})$ and $e_j^{(n)}$ we obtain a matrix, and we take the normalized trace of that matrix.

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^\infty P_n : \big(\oplus_{n=1}^\infty \ell_p^{k_n}\big)_{\ell_p} \to \Big(\oplus_{n=1}^\infty G_n\Big)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.
Choose $T = I_G \circ P$.
For $n \in \mathbb{N}$ let $\Phi_n \in \ell_{q'} \otimes \ell_p$

$$\Phi_n = \frac{1}{n} \sum_{j=1}^{n} e_j^{(n)*} \otimes g_j^{(n)}, \quad (e_j^{(n)*}) \text{ unit basis of } \ell_{q'}^n.$$

Thus the Φ_n acts on a general $S \in L_p(\ell_p, \ell_q) = L(\oplus \ell_p^{k_n}. \oplus \ell_q^n)$ as follows: We restrict S to $F_n \subset \ell_p^{k_n}$ and then project the image onto ℓ_q^n). In terms of the bases $(f_j^{(n)})$ and $e_j^{(n)}$ we obtain a matrix, and we take the normalized trace of that matrix. Thus
$\langle \Phi_n, T\rangle = \frac{1}{n}\sum_{j=1}^n \langle e_j^{(n)*}, T(g_j^{(n)})\rangle = 1.$

Definition of T and (Φ_n)

Let $P_n : \ell_p^{k_n} \to G_n$, linear projection with $\sup \|P_n\| < \infty$ and
$P = \oplus_{n=1}^{\infty} P_n : \big(\oplus_{n=1}^{\infty} \ell_p^{k_n}\big)_{\ell_p} \to \Big(\oplus_{n=1}^{\infty} G_n\Big)_{\ell_p}, \quad (x_n) \mapsto (P_n(x_n))$.
Choose $T = I_G \circ P$.
For $n \in \mathbb{N}$ let $\Phi_n \in \ell_{q'} \otimes \ell_p$

$$\Phi_n = \frac{1}{n} \sum_{j=1}^{n} e_j^{(n)*} \otimes g_j^{(n)}, \quad (e_j^{(n)*}) \text{ unit basis of } \ell_{q'}^n.$$

Thus the Φ_n acts on a general $S \in L_p(\ell_p, \ell_q) = L(\oplus \ell_p^{k_n}. \oplus \ell_q^n)$ as follows: We restrict S to $F_n \subset \ell_p^{k_n}$ and then project the image onto ℓ_q^n). In terms of the bases $(f_j^{(n)})$ and $e_j^{(n)}$ we obtain a matrix, and we take the normalized trace of that matrix. Thus
$$\langle \Phi_n, T\rangle = \tfrac{1}{n} \textstyle\sum_{j=1}^n \langle e_j^{(n)*}, T(g_j^{(n)})\rangle = 1.$$
Still Needed: Sufficient conditions for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$.

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \le 1$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \leq 1$

$$\big|\langle \Phi_n, A \circ I_F \circ B \rangle\big|$$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \le 1$

$$\big|\langle \Phi_n, A \circ I_F \circ B \rangle\big| = \frac{1}{n}\Big|\sum_{j=1}^{n} \langle e_j^{(n)*}, A \circ I_F \circ B(g_j^{(n)})\rangle\Big|$$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \le 1$

$$\begin{aligned} \big|\langle \Phi_n, A \circ I_F \circ B\rangle\big| &= \frac{1}{n}\Big|\sum_{j=1}^{n} \langle e_j^{(n)*}, A \circ I_F \circ B(g_j^{(n)})\rangle\Big| \\ &\le \frac{1}{n}\sum_{j=1}^{n} \big|\langle A^*(e_j^{(n)*}), I_F \circ B(g_j^{(n)})\rangle\big| \end{aligned}$$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \le 1$

$$\begin{aligned}
\big|\langle \Phi_n, A \circ I_F \circ B\rangle\big| &= \frac{1}{n}\Big|\sum_{j=1}^{n} \langle e_j^{(n)*}, A \circ I_F \circ B(g_j^{(n)})\rangle\Big| \\
&\le \frac{1}{n}\sum_{j=1}^{n} \big|\langle A^*(e_j^{(n)*}), I_F \circ B(g_j^{(n)})\rangle\big| \\
&\le \frac{1}{n}\sum_{j=1}^{n} \big\|I_F \circ B_n(g_j^{(n)})\big\|_{\ell_q} \text{ with } B_n = B|_{G_n}
\end{aligned}$$

Sufficient Condition for $\Phi_n \to 0$ pointwise on $\mathcal{J}^{I_F}$

Let $A \in \mathcal{L}(\ell_q, \ell_q)$ and $B \in \mathcal{L}\big(\ell_p, (\oplus_{n=1}^{\infty} F_n)_{\ell_p}\big)$, with $\|A\|, \|B\| \le 1$

$$
\begin{aligned}
\big|\langle \Phi_n, A \circ I_F \circ B\rangle\big| &= \frac{1}{n}\Big|\sum_{j=1}^{n} \langle e_j^{(n)*}, A \circ I_F \circ B(g_j^{(n)})\rangle\Big| \\
&\le \frac{1}{n}\sum_{j=1}^{n} \big|\langle A^*(e_j^{(n)*}), I_F \circ B(g_j^{(n)})\rangle\big| \\
&\le \frac{1}{n}\sum_{j=1}^{n} \big\|I_F \circ B_n(g_j^{(n)})\big\|_{\ell_q} \text{ with } B_n = B|_{G_n}
\end{aligned}
$$

Conclusion: By (*) it suffices to ensure that

$$\frac{1}{n}\sum_{j=1}^{n} \big\|B_n(g_j^{(n)})\big\|_{\infty} \to 0 \text{ if } (B_n) \text{ uniformly bounded } B_n : G_n \to \big(\oplus F_n\big)_{\ell_p}.$$

with $\|y\|_\infty = \sup_{n\in\mathbb{N}, j=1,2\ldots n} |a_j^{(n)}|$ for $y = \sum_{n=1}^{\infty}\sum_{j=1}^{n} a_j^{(n)} f_j^{(n)} \in \big(\oplus F_n\big)_{\ell_p}$.

Key Lemma

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup \left\{ \Big\| \sum_{j \in A} e_j \Big\| : |A| \le k \right\}$$ upper Fundamental function of (e_i)

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \le k\Big\}$$ upper Fundamental function of (e_i)

$$\lambda_X(k) = \inf\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \ge k\Big\}$$ lower Fundamental function of (e_i)

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup\left\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \le k\right\}$$ upper Fundamental function of (e_i)

$$\lambda_X(k) = \inf\left\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \ge k\right\}$$ lower Fundamental function of (e_i)

Y infinite-dimensional Banach space with normalized unconditional basis (e_j).

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \le k\Big\} \text{ upper Fundamental function of } (e_i)$$

$$\lambda_X(k) = \inf\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \ge k\Big\} \text{ lower Fundamental function of } (e_i)$$

Y infinite-dimensional Banach space with normalized unconditional basis (e_j). For $m \in \mathbb{N}$ let G_m be an m-dimensional Banach space with a normalized, c-unconditional basis $\{g_i^{(m)} : 1 \le i \le m\}$ for some $c \ge 1$.

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \le k\Big\} \text{ upper Fundamental function of } (e_i)$$

$$\lambda_X(k) = \inf\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \ge k\Big\} \text{ lower Fundamental function of } (e_i)$$

Y infinite-dimensional Banach space with normalized unconditional basis (e_j). For $m \in \mathbb{N}$ let G_m be an m-dimensional Banach space with a normalized, c-unconditional basis $\{g_i^{(m)} : 1 \le i \le m\}$ for some $c \ge 1$.

If $B_m\colon G_m \to Y$, with $\sup_m \|B_m\| \le 1$, then

$$\frac{1}{m}\sum_{j=1}^m \big\|B_m(g_j^{(m)})\big\|_\infty \to 0 \quad \text{as} \quad m \to \infty\,.$$

Key Lemma

For (finite or infinite dimensional) Banach space X with normalized unconditional basis (e_j) we define for $k \in \mathbb{N}$

$$\phi_X(k) = \sup\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \le k\Big\} \text{ upper Fundamental function of } (e_i)$$

$$\lambda_X(k) = \inf\Big\{\Big\|\sum_{j\in A} e_j\Big\| : |A| \ge k\Big\} \text{ lower Fundamental function of } (e_i)$$

Y infinite-dimensional Banach space with normalized unconditional basis (e_j). For $m \in \mathbb{N}$ let G_m be an m-dimensional Banach space with a normalized, c-unconditional basis $\{g_i^{(m)} : 1 \le i \le m\}$ for some $c \ge 1$. Assume that

$$\lim_{k\to\infty} \sup_{m\ge k} \frac{\phi_{G_m}(k)}{k} = 0\,, \quad \text{and} \lim_{m\to\infty} \frac{\phi_{G_m}(m)}{\lambda_Y(cm)} = 0 \quad \text{for all } c > 0\,.$$

If $B_m\colon G_m \to Y$, with $\sup_m \|B_m\| \le 1$, then

$$\frac{1}{m}\sum_{j=1}^m \big\|B_m(g_j^{(m)})\big\|_\infty \to 0 \quad \text{as} \quad m \to \infty\,.$$

Problem

For $r \in [0,1)$ *we will need* $(F_n^{(r)})_{n=1}^{\infty}$, *with* $F_n^{(r)} \hookrightarrow \ell_p^{k_n}$, *uniformly complemented in* $\ell_p^{k_n}$, $\dim F_n^{(r)} = n$ *and has c-unconditional basis* $(f_j^{(r,n)})$, *so that letting* $Y_r = (\oplus F_n^{(r)})_{\ell_p}$ *we need for* $s > r$

(2)

Problem

For $r \in [0,1)$ *we will need* $(F_n^{(r)})_{n=1}^{\infty}$, *with* $F_n^{(r)} \hookrightarrow \ell_p^{k_n}$, *uniformly complemented in* $\ell_p^{k_n}$, $\dim F_n^{(r)} = n$ *and has* c*-unconditional basis* $(f_j^{(r,n)})$, *so that letting* $Y_r = (\oplus F_n^{(r)})_{\ell_p}$ *we need for* $s > r$

$$F_n^{(r)} \to F_n^{(s)}, \qquad f_j^{(r,n)} \mapsto f_j^{(s,n)}, \text{ is uniformly bounded} \tag{1}$$

$$\lim_{m\to\infty} \frac{\phi_{F_m^{(s)}}(m)}{\lambda_{Y_r}(cm)} = 0 \text{ for all } c > 0. \tag{2}$$

Rosenthal's $X_{p,w}$ spaces

Rosenthal's $X_{p,w}$ spaces

Let $2 < p' < \infty$, let $0 < w < 1$, $n \in \mathbb{N}$ and let $F^* = F^*(w, n) = \text{span}(f_j^* : j = 1, 2 \ldots n)$, with f_j^*, $j = 1, 2 \ldots, n$ identically, independently distributed, 3-valued and symmetric random variables in L_p, with

$$\|f_j^*\|_{L_{p'}} = 1 \text{ and } \|f_j^*\|_{L_2} = w.$$

Rosenthal's $X_{p,w}$ spaces

Let $2 < p' < \infty$, let $0 < w < 1$, $n \in \mathbb{N}$ and let
$F^* = F^*(w,n) = \mathrm{span}(f_j^* : j = 1,2\ldots n)$, with f_j^*, $j = 1,2\ldots,n$
identically, independently distributed, 3-valued and symmetric random variables in L_p, with

$$\|f_j^*\|_{L_{p'}} = 1 \text{ and } \|f_j^*\|_{L_2} = w.$$

Properties: There is a constant , only depending on p, so that

Rosenthal's $X_{p,w}$ spaces

Let $2 < p' < \infty$, let $0 < w < 1$, $n \in \mathbb{N}$ and let $F^* = F^*(w,n) = \text{span}(f_j^* : j = 1,2\ldots n)$, with f_j^*, $j = 1,2\ldots,n$ identically, independently distributed, 3-valued and symmetric random variables in L_p, with

$$\|f_j^*\|_{L_{p'}} = 1 \text{ and } \|f_j^*\|_{L_2} = w.$$

Properties: There is a constant , only depending on p, so that

- $(f_j^*)_{j=1}^n$ c-unconditional,

Rosenthal's $X_{p,w}$ spaces

Let $2 < p' < \infty$, let $0 < w < 1$, $n \in \mathbb{N}$ and let $F^* = F^*(w, n) = \text{span}(f_j^* : j = 1, 2 \ldots n)$, with f_j^*, $j = 1, 2 \ldots, n$ identically, independently distributed, 3-valued and symmetric random variables in L_p, with

$$\|f_j^*\|_{L_{p'}} = 1 \text{ and } \|f_j^*\|_{L_2} = w.$$

Properties: There is a constant , only depending on p, so that

- $(f_j^*)_{j=1}^n$ c-unconditional,
- F^* is c-complemented in L_p (and thus in $\ell_{p'}^{k_n}$),

Rosenthal's $X_{p,w}$ spaces

Let $2 < p' < \infty$, let $0 < w < 1$, $n \in \mathbb{N}$ and let $F^* = F^*(w, n) = \text{span}(f_j^* : j = 1, 2 \ldots n)$, with f_j^*, $j = 1, 2 \ldots, n$ identically, independently distributed, 3-valued and symmetric random variables in L_p, with

$$\|f_j^*\|_{L_{p'}} = 1 \text{ and } \|f_j^*\|_{L_2} = w.$$

Properties: There is a constant , only depending on p, so that

- $(f_j^*)_{j=1}^n$ c-unconditional,
- F^* is c-complemented in L_p (and thus in $\ell_{p'}^{k_n}$),
- $\Big\| \sum_{j=1}^n a_j f_j^* \Big\| \sim_c \Big(\sum_{j=1}^n |a_j|^{p'} \Big)^{1/p'} \vee w \Big(\sum |a_j|^2 \Big)^{1/2} =: \big\| (a_j)_{j=1}^n \big\|_{w,p}$.

Let $F(w,n)$ be dual of $F^*(w,n)$, where $F^*(w,n)$ is renormed (uniformly equivalently) by

$$\Big\|\sum_{j=1}^n a_j f_j^*\Big\|_{w,p} = \Big(\sum_{j=1}^n |a_j|^{p'}\Big)^{1/p'} \vee w\Big(\sum |a_j|^2\Big)^{1/2}.$$

$(f_j)_{j=1}^n$ dual basis to $(f_j^*)_{j=1}^n$.

Let $F(w,n)$ be dual of $F^*(w,n)$, where $F^*(w,n)$ is renormed (uniformly equivalently) by

$$\Big\|\sum_{j=1}^n a_j f_j^*\Big\|_{w,p} = \Big(\sum_{j=1}^n |a_j|^{p'}\Big)^{1/p'} \vee w\Big(\sum |a_j|^2\Big)^{1/2}.$$

$(f_j)_{j=1}^n$ dual basis to $(f_j^*)_{j=1}^n$.

Properties

Let $F(w,n)$ be dual of $F^*(w,n)$, where $F^*(w,n)$ is renormed (uniformly equivalently) by

$$\Big\|\sum_{j=1}^n a_j f_j^*\Big\|_{w,p} = \Big(\sum_{j=1}^n |a_j|^{p'}\Big)^{1/p'} \vee w\Big(\sum |a_j|^2\Big)^{1/2}.$$

$(f_j)_{j=1}^n$ dual basis to $(f_j^*)_{j=1}^n$.

Properties

- $(f_j)_{j=1}^n$ 1-unconditional basis of F,

Let $F(w,n)$ be dual of $F^*(w,n)$, where $F^*(w,n)$ is renormed (uniformly equivalently) by

$$\|\sum_{j=1}^n a_j f_j^*\|_{w,p} = \Big(\sum_{j=1}^n |a_j|^{p'}\Big)^{1/p'} \vee w\Big(\sum |a_j|^2\Big)^{1/2}.$$

$(f_j)_{j=1}^n$ dual basis to $(f_j^*)_{j=1}^n$.

Properties

- $(f_j)_{j=1}^n$ 1-unconditional basis of F,
- F unif. equivalent to unif. complemented subspace of $\ell_p^{k_n}$,

Let $F(w,n)$ be dual of $F^*(w,n)$, where $F^*(w,n)$ is renormed (uniformly equivalently) by

$$\Big\|\sum_{j=1}^n a_j f_j^*\Big\|_{w,p} = \Big(\sum_{j=1}^n |a_j|^{p'}\Big)^{1/p'} \vee w\Big(\sum |a_j|^2\Big)^{1/2}.$$

$(f_j)_{j=1}^n$ dual basis to $(f_j^*)_{j=1}^n$.

Properties

- $(f_j)_{j=1}^n$ 1-unconditional basis of F,
- F unif. equivalent to unif. complemented subspace of $\ell_p^{k_n}$,
- $\Big\|\sum_{j\in A} f_j\Big\| = \Big\|\sum_{j=1}^k f_j\Big\| = k^{1/p} \wedge \frac{1}{w}k^{1/2}$ if $|A| = k \le n$.

Definition of (F_n)

Definition of (F_n)

Our sequences of spaces (F_n) will be $F_n = F(w_n, n)$ with w_n decreasing to 0 and $w_n \geq n^{\frac{1}{2}-\frac{1}{p}}$, $n \in \mathbb{N}$.

Definition of (F_n)

Our sequences of spaces (F_n) will be $F_n = F(w_n, n)$ with w_n decreasing to 0 and $w_n \geq n^{\frac{1}{2}-\frac{1}{p}}$, $n \in \mathbb{N}$.

Then

$$\phi_{F_m}(m) = \frac{1}{w_m} m^{1/2}$$

Definition of (F_n)

Our sequences of spaces (F_n) will be $F_n = F(w_n, n)$ with w_n decreasing to 0 and $w_n \geq n^{\frac{1}{2}-\frac{1}{p}}$, $n \in \mathbb{N}$.

Then

$$\phi_{F_m}(m) = \frac{1}{w_m} m^{1/2}$$

and a little computation shows (let $Y = (\oplus_{m=1}^{\infty} F_m)_{\ell_p}$) that for some $C > 0$

$$\lambda_Y(k) \geq C \cdot \frac{k^{1/2}}{w_{\sqrt{k}}}.$$

Definition of (F_n)

Our sequences of spaces (F_n) will be $F_n = F(w_n, n)$ with w_n decreasing to 0 and $w_n \geq n^{\frac{1}{2}-\frac{1}{p}}$, $n \in \mathbb{N}$.

Then

$$\phi_{F_m}(m) = \frac{1}{w_m} m^{1/2}$$

and a little computation shows (let $Y = (\oplus_{m=1}^{\infty} F_m)_{\ell_p}$) that for some $C > 0$

$$\lambda_Y(k) \geq C \cdot \frac{k^{1/2}}{w_{\sqrt{k}}}.$$

Therefore we need a family of sequences $(w_n^{(r)})_{n\in\mathbb{N}}$, $r \in [0,1)$, with $w_n^{(r)} \geq n^{\frac{1}{2}-\frac{1}{p}}$, $n \in \mathbb{N}$, and so that for all $d > 0$ and $0 \leq r < s < 1$

$$0 = \lim_{m\to\infty} \frac{\phi_{F_m^{(s)}}(m)}{\lambda_{Y_r}(dm)} \leq C \lim_{m\to\infty} \frac{w^{(r)}_{\sqrt{dm}}}{w_m^{(s)}}.$$

Choice of $(w_n^{(r)})$

Choice of $(w_n^{(r)})$

For example using Dedekind cuts for $\mathbb{Q}$, we choose $(N_r : r \in [0,1))$ with $N_r \subset \mathbb{N}$, so that

$$N_s \subset N_r \text{ and } \big|N_r \setminus N_s\big| = \infty \text{ if } r < s.$$

Choice of $(w_n^{(r)})$

For example using Dedekind cuts for $\mathbb{Q}$, we choose $(N_r : r \in [0,1))$ with $N_r \subset \mathbb{N}$, so that

$$N_s \subset N_r \text{ and } \big|N_r \setminus N_s\big| = \infty \text{ if } r < s.$$

Then if we write N_r as increasing sequence (k_j) we put $w^{(r)}(1) = 1$, and

$$w^{(r)}\big(2^{3^{k_j}}\big) = 2^{j(\frac{1}{2}-\frac{1}{p})} \text{ for each } j \in \mathbb{N},$$

and then extend the definition of $w^{(r)}(n)$ to the rest of $\mathbb{N}$ by linear interpolation.

Open Questions

Open Questions

Problem

Are there closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ which are not generated by one (or countably many) many operators? Is $\mathcal{F}S$ such a closed ideal?

Open Questions

Problem

Are there closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ which are not generated by one (or countably many) many operators? Is $\mathcal{FS}$ such a closed ideal?

Problem

How many pairwise incomparable closed ideals are in $\mathcal{L}(\ell_p, \ell_q)$? (for the moment we know only that this number is at least 2)

Open Questions

Problem

Are there closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ which are not generated by one (or countably many) many operators? Is $\mathcal{F}S$ such a closed ideal?

Problem

How many pairwise incomparable closed ideals are in $\mathcal{L}(\ell_p, \ell_q)$? (for the moment we know only that this number is at least 2)

Problem

What is the cardinality of the closed ideals in $\mathcal{L}(\ell_p, \ell_q)$? (This cardinality must be between c and 2^c)