A Measure Zero Universal Differentiability Set in the Heisenberg Group

Gareth Speight

Scuola Normale Superiore di Pisa

Warwick, June 2015

Rademacher's Theorem

Theorem (Rademacher)

Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable Lebesgue almost everywhere.

Rademacher's Theorem

Theorem (Rademacher)

Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable Lebesgue almost everywhere.*

Theorem (Aronszajn, Christensen, Mankiewicz)

Let X *be separable,* Y *have the Radon-Nikodym property, and* $f : X \to Y$ *be Lipschitz. Then* f *is Gâteaux differentiable 'almost everywhere'.*

Rademacher's Theorem

Theorem (Rademacher)

Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable Lebesgue almost everywhere.*

Theorem (Aronszajn, Christensen, Mankiewicz)

Let X *be separable,* Y *have the Radon-Nikodym property, and* $f : X \to Y$ *be Lipschitz. Then* f *is Gâteaux differentiable 'almost everywhere'.*

Theorem (Preiss)

Let X^* *be separable and* $f : X \to \mathbb{R}$ *be Lipschitz. Then* f *is Fréchet differentiable on a dense set.*

Converse to Rademacher's Theorem

Theorem

If $N \subset \mathbb{R}^n$ is Lebesgue null then there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ which is differentiable at no point of N.

- The case $n = 1$ is relatively simple.
- The case $n = 2$ was proved by Alberti, Csörnyei and Preiss.
- The case $n > 2$ uses work of ACP together with a recent (unpublished) result on the structure of Lebesgue null sets by Csörnyei and Jones.

Differentiability in Small Sets

Let $n > 1$.

Theorem (Preiss)

There exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point of N.

Differentiability in Small Sets

Let $n > 1$.

Theorem (Preiss)

There exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point of N.

Theorem (Doré-Maleva, Dymond-Maleva)

*The **universal differentiability set** N above can be made compact and of Hausdorff dimension, or even upper Minkowski dimension, equal to one.*

Differentiability in Small Sets

Let $n > 1$.

Theorem (Preiss)

There exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point of N.

Theorem (Doré-Maleva, Dymond-Maleva)

*The **universal differentiability set** N above can be made compact and of Hausdorff dimension, or even upper Minkowski dimension, equal to one.*

Theorem (Preiss, S.)

There exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is differentiable at a point of N.

Maximality of Directional Derivatives

Let E be a Banach space.

Theorem (Fitzpatrick)

Suppose $f: E \to \mathbb{R}$ is Lipschitz and $f'(x,e) = \mathrm{Lip}(f)$ for some $x \in E$ and $e \in E$ with $\|e\| = 1$. If the norm of E is Fréchet differentiable at e with derivative e^*, then f is Fréchet differentiable at x and $f'(x) = \mathrm{Lip}(f)e^*$.

Maximality of Directional Derivatives

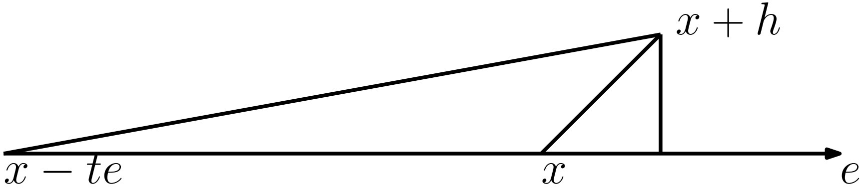
Let E be a Banach space.

Theorem (Fitzpatrick)

Suppose $f : E \to \mathbb{R}$ is Lipschitz and $f'(x,e) = \mathrm{Lip}(f)$ for some $x \in E$ and $e \in E$ with $\|e\| = 1$. If the norm of E is Fréchet differentiable at e with derivative e^*, then f is Fréchet differentiable at x and $f'(x) = \mathrm{Lip}(f)e^*$.

Suppose f is not differentiable at x - find $\varepsilon > 0$ and small h such that:

$$f(x+h) - f(x) > \mathrm{Lip}(f)e^*(h) + \varepsilon\|h\|.$$



Almost Maximality of Directional Derivatives

Let $D^f := \{(x, e) \in E \times E \colon \|e\| = 1, f'(x, e) \text{ exists}\}$.

Almost Maximality of Directional Derivatives

Let $D^f := \{(x,e) \in E \times E \colon \|e\| = 1, f'(x,e) \text{ exists}\}$.

Theorem (Preiss)

Suppose $f \colon E \to \mathbb{R}$ is Lipschitz and $(x_0, e_0) \in D^f$.

Almost Maximality of Directional Derivatives

Let $D^f := \{(x,e) \in E \times E \colon \|e\| = 1, f'(x,e) \text{ exists}\}$.

Theorem (Preiss)

Suppose $f\colon E \to \mathbb{R}$ is Lipschitz and $(x_0, e_0) \in D^f$. Let M denote the set of all pairs $(x,e) \in D^f$ such that $f'(x,e) \geq f'(x_0,e_0)$ and

$$|(f(x+te_0) - f(x)) - (f(x_0+te_0) - f(x_0))| \\ \leq 6|t|\sqrt{(f'(x,e) - f'(x_0,e_0))\mathrm{Lip}(f)}$$

for every $t \in \mathbb{R}$.

Almost Maximality of Directional Derivatives

Let $D^f := \{(x,e) \in E \times E \colon \|e\| = 1, f'(x,e) \text{ exists}\}$.

Theorem (Preiss)

Suppose $f\colon E \to \mathbb{R}$ is Lipschitz and $(x_0, e_0) \in D^f$. Let M denote the set of all pairs $(x,e) \in D^f$ such that $f'(x,e) \geq f'(x_0,e_0)$ and

$$|(f(x+te_0)-f(x)) - (f(x_0+te_0)-f(x_0))| \leq 6|t|\sqrt{(f'(x,e)-f'(x_0,e_0))\mathrm{Lip}(f)}$$

for every $t \in \mathbb{R}$. If the norm is Fréchet differentiable at e_0 and

$$\lim_{\delta \downarrow 0} \sup\{f'(x,e)\colon (x,e) \in M \text{ and } \|x-x_0\| \leq \delta\} \leq f'(x_0,e_0),$$

then f is Fréchet differentiable at x_0.

Heisenberg Group

Definition

The **Heisenberg group** $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(\langle x, y'\rangle - \langle y, x'\rangle)).$$

Heisenberg Group

Definition

The **Heisenberg group** $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(\langle x, y'\rangle - \langle y, x'\rangle)).$$

Left invariant **horizontal vector fields** on $\mathbb{H}^n$ are defined by:

$$X_i(x, y, t) = \partial_{x_i} + 2y_i\partial_t, \quad Y_i(x, y, t) = \partial_{y_i} - 2x_i\partial_t, \quad 1 \leq i \leq n.$$

Heisenberg Group

Definition

The **Heisenberg group** $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(\langle x, y'\rangle - \langle y, x'\rangle)).$$

Left invariant **horizontal vector fields** on $\mathbb{H}^n$ are defined by:

$$X_i(x, y, t) = \partial_{x_i} + 2y_i\partial_t, \quad Y_i(x, y, t) = \partial_{y_i} - 2x_i\partial_t, \quad 1 \leq i \leq n.$$

- The **Haar measure** on $\mathbb{H}^n$ is $\mathcal{L}^{2n+1}$.

Heisenberg Group

Definition

The **Heisenberg group** $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(\langle x, y' \rangle - \langle y, x' \rangle)).$$

Left invariant **horizontal vector fields** on $\mathbb{H}^n$ are defined by:

$$X_i(x, y, t) = \partial_{x_i} + 2y_i \partial_t, \quad Y_i(x, y, t) = \partial_{y_i} - 2x_i \partial_t, \quad 1 \leq i \leq n.$$

- The **Haar measure** on $\mathbb{H}^n$ is $\mathcal{L}^{2n+1}$.
- **Dilations** are defined by $\delta_r(x, y, t) = (rx, ry, r^2 t)$. They satisfy

$$\delta_r(ab) = \delta_r(a)\delta_r(b)$$

and

$$\mathcal{L}^{2n+1}(\delta_r(A)) = r^{2n+2}\mathcal{L}^{2n+1}(A).$$

Horizontal Curves

A curve $\gamma\colon [a,b] \to \mathbb{H}^n$ is **absolutely continuous** if it is differentiable almost everywhere and the fundamental theorem of calculus holds.

Horizontal Curves

A curve $\gamma \colon [a, b] \to \mathbb{H}^n$ is **absolutely continuous** if it is differentiable almost everywhere and the fundamental theorem of calculus holds.

Definition

An absolutely continuous curve $\gamma \colon [a, b] \to \mathbb{H}^n$ is **horizontal** if there exists $h \colon [a, b] \to \mathbb{R}^{2n}$ such that for almost every t:

$$\gamma'(t) = \sum_{i=1}^{n} h_i(t) X_i(\gamma(t)) + h_{i+n}(t) Y_i(\gamma(t)).$$

Horizontal Curves

A curve $\gamma\colon [a,b] \to \mathbb{H}^n$ is **absolutely continuous** if it is differentiable almost everywhere and the fundamental theorem of calculus holds.

Definition

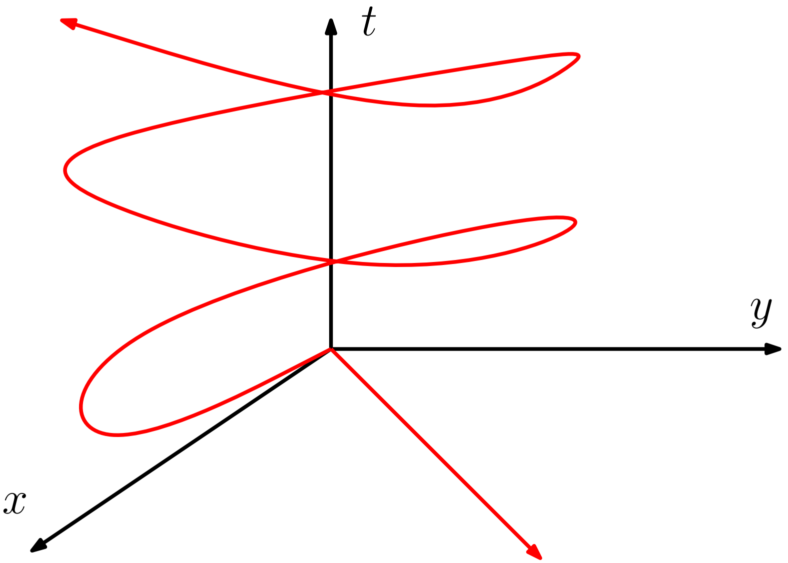
An absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{H}^n$ is **horizontal** if there exists $h\colon [a,b] \to \mathbb{R}^{2n}$ such that for almost every t:

$$\gamma'(t) = \sum_{i=1}^{n} h_i(t) X_i(\gamma(t)) + h_{i+n}(t) Y_i(\gamma(t)).$$

Define the **horizontal length** of such a curve by:

$$L(\gamma) = \int_a^b |h|.$$

Horizontal Curves



Carnot-Caratheodory Distance

Definition

Define the **Carnot-Caratheodory distance** d_{cc} on $\mathbb{H}^n$ by:

$$d_{cc}(x, y) = \inf\{L(\gamma) \colon \gamma \text{ horizontal and joins } x \text{ to } y\}.$$

Carnot-Caratheodory Distance

Definition

Define the **Carnot-Caratheodory distance** d_{cc} on $\mathbb{H}^n$ by:

$$d_{cc}(x, y) = \inf\{L(\gamma) \colon \gamma \text{ horizontal and joins } x \text{ to } y\}.$$

Theorem (Chow)

Any two points of $\mathbb{H}^n$ are connected by a horizontal curve. Further, the topology induced by d_{cc} coincides with the Euclidean topology.

Carnot-Caratheodory Distance

Definition

Define the **Carnot-Caratheodory distance** d_{cc} on $\mathbb{H}^n$ by:

$$d_{cc}(x, y) = \inf\{L(\gamma) \colon \gamma \text{ horizontal and joins } x \text{ to } y\}.$$

Theorem (Chow)

Any two points of $\mathbb{H}^n$ are connected by a horizontal curve. Further, the topology induced by d_{cc} coincides with the Euclidean topology.

- $d_{cc}(gx, gy) = d_{cc}(x, y)$ and $d_{cc}(\delta_r(x), \delta_r(y)) = r d_{cc}(x, y)$.

Carnot-Caratheodory Distance

Definition

Define the **Carnot-Caratheodory distance** d_{cc} on $\mathbb{H}^n$ by:

$$d_{cc}(x, y) = \inf\{L(\gamma) \colon \gamma \text{ horizontal and joins } x \text{ to } y\}.$$

Theorem (Chow)

Any two points of $\mathbb{H}^n$ are connected by a horizontal curve. Further, the topology induced by d_{cc} coincides with the Euclidean topology.

- $d_{cc}(gx, gy) = d_{cc}(x, y)$ and $d_{cc}(\delta_r(x), \delta_r(y)) = r d_{cc}(x, y)$.
- $\mathcal{L}^{2n+1}(B(0, r)) = r^{2n+2} B(0, 1)$. The Hausdorff dimension of $\mathbb{H}^{2n+1}$ is $2n + 2$ and the topological dimension is $2n + 1$.

Carnot-Caratheodory Distance

Definition

Define the **Carnot-Caratheodory distance** d_{cc} on $\mathbb{H}^n$ by:

$$d_{cc}(x,y) = \inf\{L(\gamma) \colon \gamma \text{ horizontal and joins } x \text{ to } y\}.$$

Theorem (Chow)

Any two points of $\mathbb{H}^n$ are connected by a horizontal curve. Further, the topology induced by d_{cc} coincides with the Euclidean topology.

- $d_{cc}(gx, gy) = d_{cc}(x,y)$ and $d_{cc}(\delta_r(x), \delta_r(y)) = r d_{cc}(x,y)$.
- $\mathcal{L}^{2n+1}(B(0,r)) = r^{2n+2} B(0,1)$. The Hausdorff dimension of $\mathbb{H}^{2n+1}$ is $2n+2$ and the topological dimension is $2n+1$.
- Carnot-Caratheodory distance is **not** Lipschitz equivalent to the Euclidean distance.

Lusin Approximation in $\mathbb{H}^n$

Theorem (Classical Lusin Approximation)

Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ *is absolutely continuous and* $\varepsilon > 0$*. Then there is a* C^1 *curve* $\Gamma\colon [a,b] \to \mathbb{R}^n$ *such that:*

$$\mathcal{L}^1\{t\colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

Lusin Approximation in $\mathbb{H}^n$

Theorem (Classical Lusin Approximation)

Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is absolutely continuous and $\varepsilon > 0$. Then there is a C^1 curve $\Gamma\colon [a,b] \to \mathbb{R}^n$ such that:

$$\mathcal{L}^1\{t\colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

Theorem (S.)

Let $\varepsilon > 0$ and $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve. Then there is a C^1 horizontal curve $\Gamma\colon [0,1] \to \mathbb{H}^n$ such that:

$$\mathcal{L}^1\{t\colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

Lusin Approximation in $\mathbb{H}^n$

Theorem (Classical Lusin Approximation)

Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ *is absolutely continuous and* $\varepsilon > 0$. *Then there is a* C^1 *curve* $\Gamma\colon [a,b] \to \mathbb{R}^n$ *such that:*

$$\mathcal{L}^1\{t\colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

Theorem (S.)

Let $\varepsilon > 0$ *and* $\gamma\colon [0,1] \to \mathbb{H}^n$ *be an absolutely continuous horizontal curve. Then there is a* C^1 *horizontal curve* $\Gamma\colon [0,1] \to \mathbb{H}^n$ *such that:*

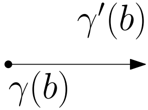
$$\mathcal{L}^1\{t\colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

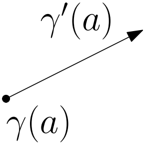
The same result holds in all step 2 Carnot groups (Le Donne, S.) but not in the Engel group which has step 3 (S.).

Euclidean Case

Idea:

1. Measure theory: restrict to a large compact set where the starting function γ is well approximated by a continuous derivative.
2. Geometry: use nice smooth curves to interpolate in the gaps (a, b).





$$\gamma'(b) \approx \gamma'(a)$$

$$\gamma(b) \approx \gamma(a) + (b-a)\gamma'(a)$$

Horizontal Lift

Lemma (Horizontal Lift)

An absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{H}^n$ *is horizontal if and only if*

$$\gamma_{2n+1}(t) = \gamma_{2n+1}(a) + 2\sum_{i=1}^{n} \int_a^t (\gamma_i' \gamma_{n+i} - \gamma_{n+i}' \gamma_i)$$

for every $t \in [a,b]$.

Horizontal Lift

Lemma (Horizontal Lift)

An absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{H}^n$ *is horizontal if and only if*

$$\gamma_{2n+1}(t) = \gamma_{2n+1}(a) + 2\sum_{i=1}^{n} \int_a^t (\gamma_i' \gamma_{n+i} - \gamma_{n+i}' \gamma_i)$$

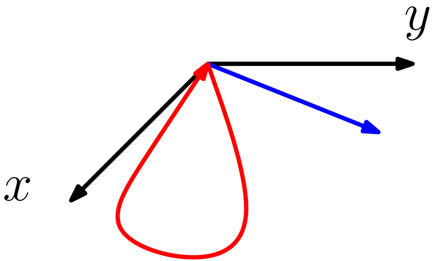
for every $t \in [a,b]$.

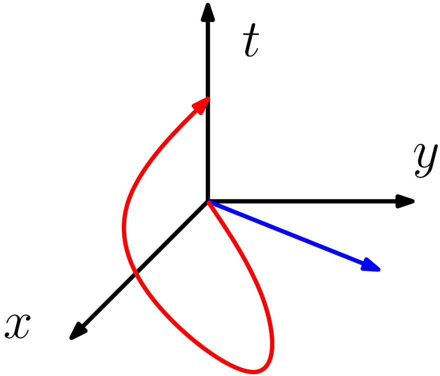
Lemma (Height-Area Interpretation)

Suppose $\sigma\colon [a,b] \to \mathbb{R}^2$ *is a smooth curve with* $\sigma(a) = 0$*. Let* A_σ *denote the signed area of the region enclosed by* σ *and the straight line* $[0, \sigma(b)]$*. Then*

$$A_\sigma = \frac{1}{2}\int_a^b (\sigma_1 \sigma_2' - \sigma_2 \sigma_1').$$

Horizontal Lift





Construction of a C^1 Horizontal Approximation

Let $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve.

Construction of a C^1 Horizontal Approximation

Let $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve.

1. Find a compact set $K \subset [0,1]$ of large measure such that:
   - $\gamma'|_K$ is uniformly continuous,
   - each point of K is a Lebesgue point of γ' with averages converging uniformly.

Construction of a C^1 Horizontal Approximation

Let $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve.

1. Find a compact set $K \subset [0,1]$ of large measure such that:
   - $\gamma'|_K$ is uniformly continuous,
   - each point of K is a Lebesgue point of γ' with averages converging uniformly.
2. Use the description of horizontal curves in $\mathbb{H}^n$ to obtain tighter control on the non-horizontal component of γ.

Construction of a C^1 Horizontal Approximation

Let $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve.

1. Find a compact set $K \subset [0,1]$ of large measure such that:
   - $\gamma'|_K$ is uniformly continuous,
   - each point of K is a Lebesgue point of γ' with averages converging uniformly.
2. Use the description of horizontal curves in $\mathbb{H}^n$ to obtain tighter control on the non-horizontal component of γ.
3. Construct C^1 curves in the plane which trace out a given area, subject to boundary conditions on the position and velocity.

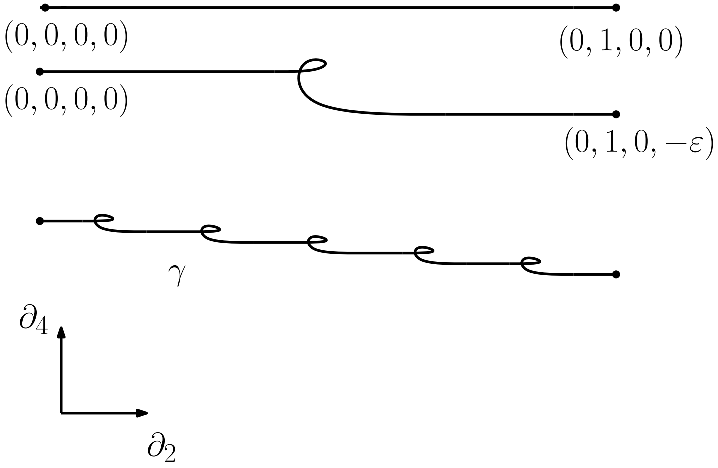
Construction of a C^1 Horizontal Approximation

Let $\gamma\colon [0,1] \to \mathbb{H}^n$ be an absolutely continuous horizontal curve.

1. Find a compact set $K \subset [0,1]$ of large measure such that:
   - $\gamma'|_K$ is uniformly continuous,
   - each point of K is a Lebesgue point of γ' with averages converging uniformly.
2. Use the description of horizontal curves in $\mathbb{H}^n$ to obtain tighter control on the non-horizontal component of γ.
3. Construct C^1 curves in the plane which trace out a given area, subject to boundary conditions on the position and velocity.
4. Lift these curves from the plane into $\mathbb{H}^n$ to redefine γ in the intervals $(a,b) \subset [0,1] \setminus K$.

Engel Group

The **Engel group** is a step 3 Carnot group with horizontal vector fields $X_1(x) = \partial_1$ and $X_2(x) = \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4$.



Pansu Differentiability

Definition

A function $L\colon \mathbb{H}^n \to \mathbb{R}$ is called $\mathbb{H}$-linear if $L(xy) = L(x) + L(y)$ and $L(\delta_r(x)) = rL(x)$ for all $x, y \in \mathbb{H}^n$ and $r > 0$.

Pansu Differentiability

Definition

A function $L\colon \mathbb{H}^n \to \mathbb{R}$ is called $\mathbb{H}$-linear if $L(xy) = L(x) + L(y)$ and $L(\delta_r(x)) = rL(x)$ for all $x, y \in \mathbb{H}^n$ and $r > 0$.

Definition

A function $f\colon \mathbb{H}^n \to \mathbb{R}$ is **Pansu differentiable** at $x \in \mathbb{H}^n$ if there is a $\mathbb{H}$-linear map $L\colon \mathbb{H}^n \to \mathbb{R}$ such that:

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(x^{-1}y)|}{d_{cc}(x, y)} = 0.$$

Pansu's Theorem and Universal Differentiability Sets

Theorem (Pansu)

Every Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable Lebesgue almost everywhere.

Pansu's Theorem and Universal Differentiability Sets

Theorem (Pansu)

Every Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable Lebesgue almost everywhere.

Corollary (Semmes)

There is no bilipschitz embedding of $\mathbb{H}^n$ into any Euclidean space.

Pansu's Theorem and Universal Differentiability Sets

Theorem (Pansu)

Every Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable Lebesgue almost everywhere.

Corollary (Semmes)

There is no bilipschitz embedding of $\mathbb{H}^n$ into any Euclidean space.

Theorem (Pinamonti, S.)

There is a Lebesgue measure zero set $N \subset \mathbb{H}^n$ such that every Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable at a point of N.

A Universal Differentiability Set in $\mathbb{H}^n$

Idea:

1. Fix a Lebesgue measure zero G_δ set S containing all horizontal lines joining pairs of points in $\mathbb{Q}^{2n+1}$.

A Universal Differentiability Set in $\mathbb{H}^n$

Idea:

1. Fix a Lebesgue measure zero G_δ set S containing all horizontal lines joining pairs of points in $\mathbb{Q}^{2n+1}$.
2. Find an 'almost maximal' directional derivative $Ef(x)$, where we consider $x \in S$ and horizontal vector fields E of unit length.

A Universal Differentiability Set in $\mathbb{H}^n$

Idea:

1. Fix a Lebesgue measure zero G_δ set S containing all horizontal lines joining pairs of points in $\mathbb{Q}^{2n+1}$.
2. Find an 'almost maximal' directional derivative $Ef(x)$, where we consider $x \in S$ and horizontal vector fields E of unit length.
3. Show that if $x \in S$ and $Ef(x)$ is 'almost maximal' then f is Pansu differentiable at x.

Directional Derivatives in $\mathbb{H}^n$

Let $V = \mathrm{Span}\{X_i, Y_i \colon 1 \leq i \leq n\}$. Define an inner product norm N on V by declaring X_i, Y_i to be orthonormal.

Directional Derivatives in $\mathbb{H}^n$

Let $V = \mathrm{Span}\{X_i, Y_i : 1 \leq i \leq n\}$. Define an inner product norm N on V by declaring X_i, Y_i to be orthonormal.

Definition

Let $f : \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $E \in V$. Define $Ef(x) := (f \circ \gamma)'(t)$ whenever it exists, where γ is any Lipschitz horizontal curve with $\gamma(t) = x$ and $\gamma'(t) = E(x)$.

Directional Derivatives in $\mathbb{H}^n$

Let $V = \mathrm{Span}\{X_i, Y_i : 1 \leq i \leq n\}$. Define an inner product norm N on V by declaring X_i, Y_i to be orthonormal.

Definition

Let $f : \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $E \in V$. Define $Ef(x) := (f \circ \gamma)'(t)$ whenever it exists, where γ is any Lipschitz horizontal curve with $\gamma(t) = x$ and $\gamma'(t) = E(x)$.

Lemma

Let $f : \mathbb{H}^n \to \mathbb{R}$ be Lipschitz. Then:

$$\mathrm{Lip}_{\mathbb{H}}(f) = \sup\{|Ef(x)| : x \in \mathbb{H}^n, E \in V, N(E) = 1, Ef(x) \textit{ exists } \}.$$

A Useful Horizontal Curve

Lemma

Let $a, b \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Suppose* $(a,b) \neq (0,0)$ *and let* $L = |(a,b)|$. *Define* $\gamma\colon [0,1] \to \mathbb{H}^n$ *by:*

$$\gamma(t) = \begin{cases} t\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) & 0 \leq t \leq 1/2, \\ \frac{1}{2}\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) + \left(t - \frac{1}{2}\right)\left(a + \frac{bc}{L^2}, b - \frac{ac}{L^2}, 2c\right) & 1/2 < t \leq 1. \end{cases}$$

A Useful Horizontal Curve

Lemma

Let $a, b \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Suppose* $(a, b) \neq (0, 0)$ *and let* $L = |(a, b)|$. *Define* $\gamma \colon [0, 1] \to \mathbb{H}^n$ *by:*

$$\gamma(t) = \begin{cases} t\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) & 0 \leq t \leq 1/2, \\ \frac{1}{2}\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) + \left(t - \frac{1}{2}\right)\left(a + \frac{bc}{L^2}, b - \frac{ac}{L^2}, 2c\right) & 1/2 < t \leq 1. \end{cases}$$

Then:

1. γ *is a Lipschitz horizontal curve joining* $(0, 0, 0) \in \mathbb{H}^n$ *to* $(a, b, c) \in \mathbb{H}^n$,

A Useful Horizontal Curve

Lemma

Let $a, b \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Suppose* $(a, b) \neq (0, 0)$ *and let* $L = |(a, b)|$. *Define* $\gamma\colon [0, 1] \to \mathbb{H}^n$ *by:*

$$\gamma(t) = \begin{cases} t\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) & 0 \leq t \leq 1/2, \\ \frac{1}{2}\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) + \left(t - \frac{1}{2}\right)\left(a + \frac{bc}{L^2}, b - \frac{ac}{L^2}, 2c\right) & 1/2 < t \leq 1. \end{cases}$$

Then:

1. γ *is a Lipschitz horizontal curve joining* $(0, 0, 0) \in \mathbb{H}^n$ *to* $(a, b, c) \in \mathbb{H}^n$,
2. $\mathrm{Lip}_{\mathbb{H}}(\gamma) \leq L\left(1 + \frac{c^2}{L^4} + \frac{4c^2}{L^2}\right)^{\frac{1}{2}}$,

A Useful Horizontal Curve

Lemma

Let $a, b \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Suppose* $(a, b) \neq (0, 0)$ *and let* $L = |(a, b)|$. *Define* $\gamma\colon [0, 1] \to \mathbb{H}^n$ *by:*

$$\gamma(t) = \begin{cases} t\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) & 0 \leq t \leq 1/2, \\ \frac{1}{2}\left(a - \frac{bc}{L^2}, b + \frac{ac}{L^2}, 0\right) + \left(t - \frac{1}{2}\right)\left(a + \frac{bc}{L^2}, b - \frac{ac}{L^2}, 2c\right) & 1/2 < t \leq 1. \end{cases}$$

Then:

1. γ *is a Lipschitz horizontal curve joining* $(0, 0, 0) \in \mathbb{H}^n$ *to* $(a, b, c) \in \mathbb{H}^n$,
2. $\mathrm{Lip}_{\mathbb{H}}(\gamma) \leq L\left(1 + \frac{c^2}{L^4} + \frac{4c^2}{L^2}\right)^{\frac{1}{2}}$,
3. $\gamma'(t)$ *exists and* $|\gamma'(t) - (a, b, 0)| \leq \frac{c}{L}(1 + 4L^2)^{\frac{1}{2}}$ *for* $t \in [0, 1] \setminus \{1/2\}$.

Maximality implies Differentiability

Lemma

Fix $u_1, u_2 \in \mathbb{R}^n$ not both zero and let $u = (u_1, u_2, 0) \in \mathbb{H}^n$. Then:

1. $d_{cc}(uz, 0) \geq d_{cc}(u, 0) + \langle z, u/d_{cc}(u, 0)\rangle$ *for any* $z \in \mathbb{H}^n$,
2. $d_{cc}(uz, 0) = d_{cc}(u, 0) + \langle z, u/d_{cc}(u, 0)\rangle + o(d_{cc}(z, 0))$ *as* $z \to 0$.
   That is, the Pansu derivative of $d_{cc}(\cdot, 0)$ *at* u *is* $x \mapsto \langle x, u/d_{cc}(u, 0)\rangle$.

Maximality implies Differentiability

Lemma

Fix $u_1, u_2 \in \mathbb{R}^n$ *not both zero and let* $u = (u_1, u_2, 0) \in \mathbb{H}^n$. *Then:*

1. $d_{cc}(uz, 0) \geq d_{cc}(u, 0) + \langle z, u/d_{cc}(u, 0)\rangle$ *for any* $z \in \mathbb{H}^n$,
2. $d_{cc}(uz, 0) = d_{cc}(u, 0) + \langle z, u/d_{cc}(u, 0)\rangle + o(d_{cc}(z, 0))$ *as* $z \to 0$.
   That is, the Pansu derivative of $d_{cc}(\cdot, 0)$ *at* u *is* $x \mapsto \langle x, u/d_{cc}(u, 0)\rangle$.

Theorem

Let $f : \mathbb{H}^n \to \mathbb{R}$ *be Lipschitz,* $x \in \mathbb{H}^n$ *and* $E \in V$ *with* $N(E) = 1$. *Suppose* $Ef(x)$ *exists and* $Ef(x) = \mathrm{Lip}_{\mathbb{H}}(f)$. *Then* f *is Pansu differentiable at* x *with derivative* $x \mapsto \mathrm{Lip}_{\mathbb{H}}(f)\langle x, E(0)\rangle$.

Almost Maximality implies Differentiability

Let $D^f := \{(x, E) \in S \times V \colon N(E) = 1,\ Ef(x) \text{ exists}\}$.

Almost Maximality implies Differentiability

Let $D^f := \{(x, E) \in S \times V \colon N(E) = 1,\ Ef(x) \text{ exists}\}$.

Theorem

Let $f \colon \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $(x_0, E_0) \in D^f$.

Almost Maximality implies Differentiability

Let $D^f := \{(x, E) \in S \times V \colon N(E) = 1,\ Ef(x) \text{ exists}\}$.

Theorem

Let $f \colon \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $(x_0, E_0) \in D^f$. Let M denote the set of pairs $(x, E) \in D^f$ such that $Ef(x) \geq E_0 f(x_0)$ and

$$\begin{aligned} &|(f(x + tE_0(x)) - f(x)) - (f(x_0 + tE_0(x_0)) - f(x_0))| \\ &\leq 6|t|((Ef(x) - E_0 f(x_0))\mathrm{Lip}_{\mathbb{H}}(f))^{\frac{1}{4}} \end{aligned}$$

for every $t \in (-1, 1)$.

Almost Maximality implies Differentiability

Let $D^f := \{(x, E) \in S \times V \colon N(E) = 1,\ Ef(x) \text{ exists}\}$.

Theorem

Let $f \colon \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $(x_0, E_0) \in D^f$. Let M denote the set of pairs $(x, E) \in D^f$ such that $Ef(x) \geq E_0 f(x_0)$ and

$$|(f(x + tE_0(x)) - f(x)) - (f(x_0 + tE_0(x_0)) - f(x_0))| \leq 6|t|((Ef(x) - E_0 f(x_0))\mathrm{Lip}_{\mathbb{H}}(f))^{\frac{1}{4}}$$

for every $t \in (-1, 1)$. If

$$\lim_{\delta \downarrow 0} \sup\{Ef(x) \colon (x, E) \in M \text{ and } d_{cc}(x, x_0) \leq \delta\} \leq E_0 f(x_0),$$

then f is Pansu differentiable at x_0 with Pansu derivative $x \mapsto E_0 f(x_0)\langle x, E_0(0)\rangle$.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.
- The Heisenberg group $\mathbb{H}^n$ is a metric measure space admitting translations and dilations. Distances are measured using lengths of 'horizontal curves'.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.
- The Heisenberg group $\mathbb{H}^n$ is a metric measure space admitting translations and dilations. Distances are measured using lengths of 'horizontal curves'.
- Absolutely continuous horizontal curves in $\mathbb{H}^n$ admit a Lusin approximation by C^1 horizontal curves. The same is true in all step 2 Carnot groups, but not in those of higher step.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.
- The Heisenberg group $\mathbb{H}^n$ is a metric measure space admitting translations and dilations. Distances are measured using lengths of 'horizontal curves'.
- Absolutely continuous horizontal curves in $\mathbb{H}^n$ admit a Lusin approximation by C^1 horizontal curves. The same is true in all step 2 Carnot groups, but not in those of higher step.
- Lipschitz functions $\mathbb{H}^n \to \mathbb{R}$ are 'Pansu differentiable' almost everywhere.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.
- The Heisenberg group $\mathbb{H}^n$ is a metric measure space admitting translations and dilations. Distances are measured using lengths of 'horizontal curves'.
- Absolutely continuous horizontal curves in $\mathbb{H}^n$ admit a Lusin approximation by C^1 horizontal curves. The same is true in all step 2 Carnot groups, but not in those of higher step.
- Lipschitz functions $\mathbb{H}^n \to \mathbb{R}$ are 'Pansu differentiable' almost everywhere.
- There are measure zero 'universal differentiability sets' in $\mathbb{H}^n$, which contain points of Pansu differentiability for every Lipschitz function $\mathbb{H}^n \to \mathbb{R}$.

Key Points

- A converse to Rademacher's theorem holds for Lipschitz functions $\mathbb{R}^n \to \mathbb{R}^m$ if and only if $n \leq m$.
- The Heisenberg group $\mathbb{H}^n$ is a metric measure space admitting translations and dilations. Distances are measured using lengths of 'horizontal curves'.
- Absolutely continuous horizontal curves in $\mathbb{H}^n$ admit a Lusin approximation by C^1 horizontal curves. The same is true in all step 2 Carnot groups, but not in those of higher step.
- Lipschitz functions $\mathbb{H}^n \to \mathbb{R}$ are 'Pansu differentiable' almost everywhere.
- There are measure zero 'universal differentiability sets' in $\mathbb{H}^n$, which contain points of Pansu differentiability for every Lipschitz function $\mathbb{H}^n \to \mathbb{R}$.

Thank you for listening!