Orthomodular lattices, natural density and non-distributive L^p spaces

Relations Between Banach Space Theory and Geometric Measure Theory

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Abstract

The undercurrent in the paper involves orthomodular lattices and generalized measure algebras where one replaces Boolean algebra & a measure with a lattice & a submeasure.

In the first part of the talk we take a look at natural density of natural numbers and how it can be related to measure algebras.

The second part of the paper and talk are speculative in nature. We discuss how L^p spaces on lattices with submeasures 'should' look like. Then the 'supports' of simple functions do not behave distributively as in the Boolean case.

The preprint is available at ArXiv.

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- Relevance: Quantum theory.

Natural density sets

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- Important in number theory. Have been studied in Banach spaces too.
- Notoriously badly behaved: $A \cap B$, $A \cup B$ may fail to be density sets even if A and B are such.
- Singletons have density 0, thus σ-additivity of d fails.

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Theorem

Let $\mathcal{F} \subset \mathcal{D}$ be a family closed under finite intersections. Then there is a σ-algebra Σ order-isomorphically included in $\mathcal{D}/_{\sim}$ such that $\mathcal{F}/_{\sim} \subset \Sigma$ and $\hat{d}\colon \Sigma \to [0,1]$, $\hat{d}(K/_{\sim}) = d(K)$, is σ-additive.
Morerover, if $\mathcal{F}/_{\sim}$ is countable and the corresponding σ-generated measure algebra $(\Sigma, \hat{d})$ is atomless, then it is in fact isomorphic to the measure algebra on the unit interval.

Structure of the proof 1/2: σ-additivity of d on the cosets

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Lemma

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- The gist of the proof: 'lagged' subsets.
- Version of the main lemma in the paper in more general form: $\mathbb{N} \rightsquigarrow \mathcal{L}$, $[0,1] \rightsquigarrow G$, $\{1,\ldots,n\}$-averages $\rightsquigarrow \varphi_n$, $\lim \rightsquigarrow \lim_{\mathcal{F}}$.

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- The modification of the π-λ-lemma argument gives that Δ is essentially a σ-algebra.

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- We start with the space $c_{00}(\mathcal{L})$ and denote its canonical Hamel basis unit vectors by e_A, $A \in \mathcal{L}$.
- This vector space by itself is not 'realistic' model for 'simple functions' because there is a spike supported on $\mathbf{0}$ (empty set). Also, $c_{00}(\mathcal{L})$ does not recognize the possible overlap of the supports.

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  be the canonical quotient mapping.
- Write $X = c_{00}(\mathcal{L})/\Delta$ and we denote by

$$a \otimes A := q(ae_A) \in X, \quad a \in \mathbb{R},\ A \in \mathcal{L}.$$

  Note that X is the space of vectors of the form

$$\sum_{i\in I} a_i \otimes A_i, \quad a_i \in \mathbb{R},\ A_i \in \mathcal{L},\ I \text{ finite}.$$

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We define a semi-norm on X by

$$\rho\left(\sum_i a_i \otimes A_i\right) = \inf\left\{\left(\sum_k |b_k|^p \varphi(B_k)\right)^{\frac{1}{p}} : \pm \sum_i a_i \otimes A_i \sqsubseteq \sum_k |b_k| \otimes B_k\right\}.$$

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Indeed, it is easy to see that this is a semi-norm; the triangle inequality follows from the condition that $x + y \sqsubseteq v + w$ whenever $x \sqsubseteq v$ and $y \sqsubseteq w$.

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Theorem

Let $1 \leq p < \infty$, $\mathcal{L}$ be an orthomodular lattice with an order-preserving map $\varphi \colon \mathcal{L} \to [0,1]$, as above. Let (Ω, Σ, μ) be a probability space and $\Sigma_0 \subset \Sigma$ a Boolean algebra which σ-generates Σ. Let us assume that $\varphi(M \vee N) = \varphi(M) + \varphi(N)$ whenever $N \leq M^\perp$. Suppose that there is an order-embedding $\jmath \colon \Sigma_0 \to \mathcal{L}$ such that $\mu(M) = \varphi(\jmath M)$ for all $M \in \Sigma_0$. (We are not assuming here that $\jmath$ respects the orthocomplementation operation.) Then

$$\sum_i a_i [1_{A_i}]_{\underset{=}{a.e.}} \mapsto \sum_i a_i \otimes \jmath(A_i), \quad A_i \in \Sigma_0$$

extends to a linear (into) isometry $L^p(\Omega, \Sigma, \mu) \to L^p(\mathcal{L}, \varphi)$.

Some related references

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