Bayesian Level Set Method for Piecewise Geometry Reconstruction

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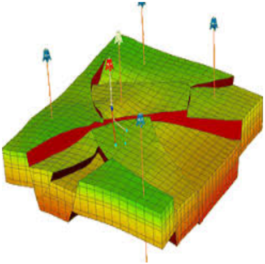
Uncertainty in Complex Computer Models

February 2, 2015

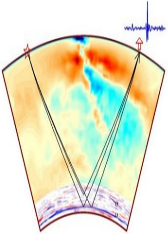
Outline

Enabling Quantification of EQUIP Uncertainty for Inverse Problems

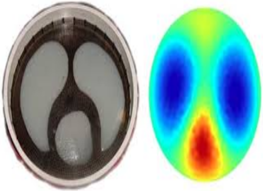
Geometric Inverse Problems



Petroleum reservoir[1]



Seismic imaging[2]



EIT[3]

[1] http://www.coatsengineering.com

[2] http://gmig.math.purdue.edu

[3] http://www.siltanen-research.net

Model Problems: Source Problem

Forward Model

Given $f = \mathbf{1}_D$ with $D \subset \tilde{D}$, find $v \in H_0^1(\tilde{D})$ satisfying

$$-\Delta v = f \text{ in } \tilde{D}, \quad v = 0 \text{ on } \partial\tilde{D}$$

Inverse Problem

Given the noisy boundary measurement data

$$y = L(v) + \eta, \text{ with } L(v) = \left\{ \ell_j \left(\frac{\partial v}{\partial \nu}\Big|_{\partial\tilde{D}} \right) \right\}_{j=1}^{N}$$

and noise η, find the domain D.

Model Problems: Groundwater Flow Problem

Darcy Flow Model

$$-\nabla \cdot (\kappa \nabla p) = f, \quad x \in D$$
$$p = 0, \quad x \in \partial D$$

with permeability $\kappa = \sum_{i=1}^{n} \kappa_i \mathbf{1}_{D_i}$. Given $f \in H^{-1}(D)$ and κ, find $p \in H_0^1(D)$.

Inverse Problem

Given noisy data

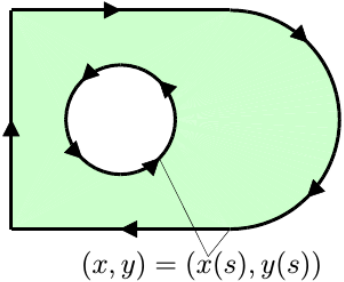
$$y = L(p) + \eta, \text{ with } L(p) = \{\ell_j(p)\}_{j=1}^{N}$$

with noise η, find the domains $\{D_i\}_{i=1}^{N}$.

Geometry Representations

Explicit Geometry

- Boundaries by parameterization



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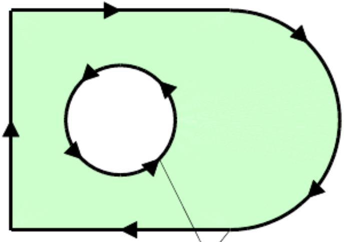
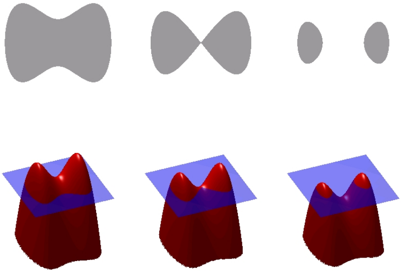


$$(x, y) = (x(s), y(s))$$

Implicit Geometry

- Boundaries as zero level set



Inversion Methods

1. **Optimization method (shape derivative + regularization)**
   - Geometry parameterization
     Hettlich-Rundell 1996, Hohage 1997.
   - Level set representation
     Santosa 1996, Burger 2001, Iglesias and McLaughlin 2011.

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2. **Bayesian inference**
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   - Level set representation

Bayesian Levl Set !

Level Set Formulation

Given constants $-\infty = c_0 < c_1 < \cdots < c_n = \infty$, let $\overline{D} = \cup_{i=1}^n \overline{D_i}$ where

$$D_i = \{x \in D \mid c_{i-1} \leq u(x) < c_i\}, i = 1, \cdots, n,$$
$$D_i^0 = \overline{D_i} \cap \overline{D_{i+1}} = \{x \in D \mid u(x) = c_i\}$$

where u is the **level set function**. The **level set map** F is defined by

$$F : u(x) \rightarrow f(x) = \sum_{i=1}^{n} f_i \, \mathbf{1}_{D_i}(x)$$

which maps the level set function to the physical parameter in the model. Here $f_i, i = 1, \cdots, n$ are constants known a priori.

Bayesian Level Set Method

Geometric Inverse problem: determine u from noisy data y

$$y = \mathcal{G}(u) + \eta$$

where the observational operator $\mathcal{G} : U \to Y := \mathbb{R}^J$ takes the form

$$\mathcal{G} = L \circ G \circ F$$

with geometric operator $F : U \to X$, forward operator $G : X \to V$ and observation operator $L : V \to Y$.

Bayesian approach:

Prior measure μ_0 on $u +$ the data likelihood $y|u$

$$\overset{\text{Bayes}}{\Longrightarrow} \text{Posterior measure } \mu^y$$

Uncertainty for Inverse Problems

Theoretical Foundations

- **Existence**

  Given the Gaussian Prior $u \sim \mu_0$, noise $\eta \sim \mathbb{Q}_0 = \mathcal{N}(0, \Gamma)$, let $\Phi(u; y) = \frac{1}{2}|y - \mathcal{G}(u)|_\Gamma^2$, then the posterior

$$\mu^y(du) \propto \exp\left(-\Phi(u; y)\right) \mu_0(du)$$

- **Well-posedness**

  μ^y is locally Lipschitz with respect to y, in Hellinger distance.

Minimization Versus Bayesian

- **Minimization Approach**

  Find $\overline{u}$ such that

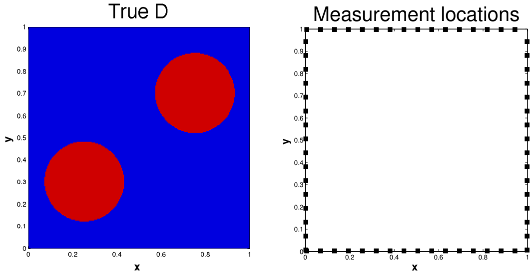
  $$\overline{u} = \arg\min_{u}\{\Phi(u; y) + \lambda R(u)\}$$

  Classial minimization technique fails to work because $u \mapsto \Phi(u; y)$ is discontinuous.

- **Bayesian Approach**

  Putting a Gaussian prior on u, the map $u \mapsto \Phi(u; y)$ is continuous almost surely, i.e. the discontinuity set of $\Phi(u; y)$ has prior measure zero.

Inverse Source Problem



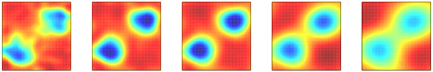
Prior $\mu_0 = \mathcal{N}(0, \mathcal{C})$ where the covariance $\mathcal{C}$ is given by

$$\mathcal{C}\varphi(x) = \int_D K(x,y)\varphi(y)dy \text{ with } K(x,y) = \exp\left(-\frac{|x-y|^2}{2\varepsilon^2}\right)$$

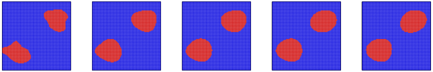
Inverse Source Problem: Effect of Prior Length-Scale I

From left to right prior length-scale: 0.1, 0.15, 0.2, 0.3, 0.4.

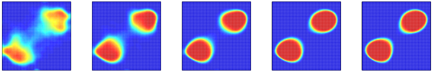
Mean of level set functions



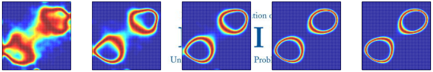
Map forward onto f



Mean of f



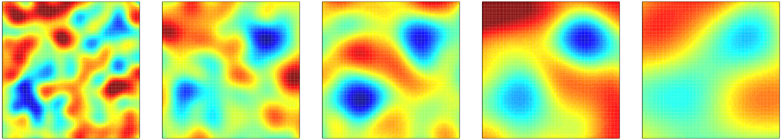
Variance of f



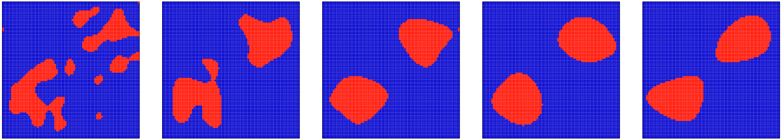
Inverse Source Problem: Effect of Prior Length-Scale II

From left to right prior length-scale: $0.1, 0.15, 0.2, 0.3, 0.4$.

An arbitrary (level-set function) sample from the posterior

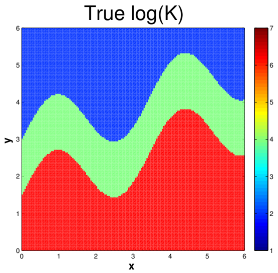


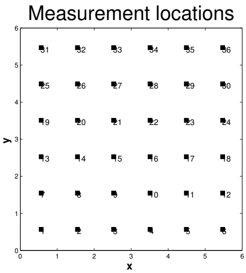
push forward of (level-set function) sample from the posterior



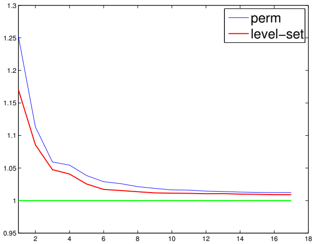
Enabling Quantification of EQUIP Uncertainty for Inverse Problems

Numerical Experiments: Groundwater Flow





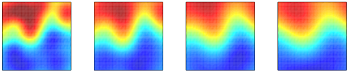
Gelman-Rubin inter-chain statistic:



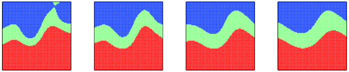
Groundwater Flow: Effect of Prior Length-Scale I

From left to right prior length-scale: $0.2, 0.3, 0.4, 0.5$.

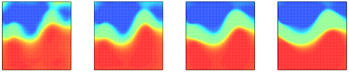
Mean of level set functions



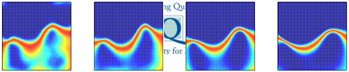
Map forward onto κ



Mean of κ



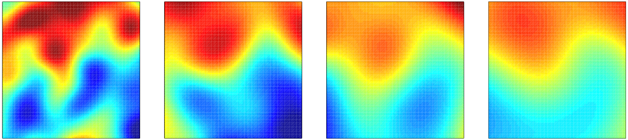
Variance of κ



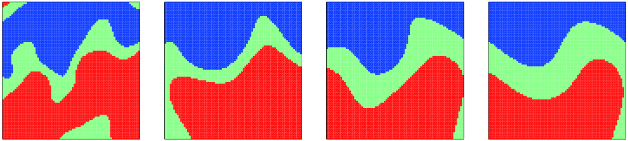
Groundwater Flow: Effect of Prior Length-Scale II

From left to right prior length-scale: 0.2, 0.3, 0.4, 0.5.

Samples from the posterior on level-set function



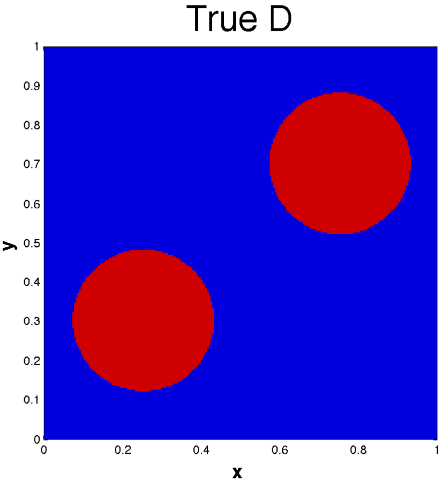
and corresponding permeability functions



Enabling Quantification of

EQUIP

Uncertainty for Inverse Problems


True D
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y
0
0.2
0.4
0.6
0.8
1
x

References

T. Bui-Thanh, and O. Ghattas.
An analysis of infinite dimensional Bayesian inverse shape acoustic scattering and its numerical approximation.
SIAM/ASA Journal on Uncertainty Quantification 2.1 (2014): 203-222.

M.A. Iglesias, K. Lin and A. M. Stuart
Well-posed Bayesian geometric inverse problems arising in subsurface flow.
Inverse Problems, 30 (2014) 114001.

M. A. Iglesias, Y. Lu and A. M. Stuart
A level-set approach to Bayesian geometric inverse problems.
In Preparation, 2015.

F. Santosa
A level-set approach for inverse problems involving obstacles.
ESAIM: Control, Optimisation and Calculus of Variations 1 (1996): 17-33.