Random walks on the East model

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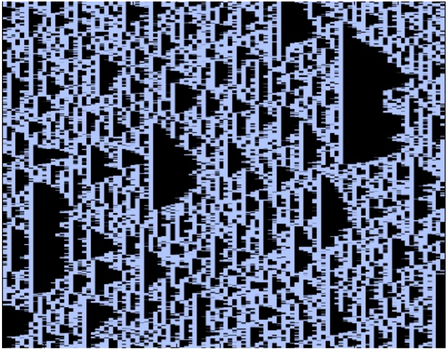
East model

Markov process $(\xi_t)_{t\in\mathbb{R}_+}$ on $\{0,1\}^{\mathbb{Z}}$. $\rho \in (0,1)$ density parameter.

At site x, $\begin{cases} 0 & \xrightarrow{\rho} & 1 \\ 1 & \xrightarrow{1-\rho} & 0 \end{cases}$ <u>if</u> $x+1$ (the East neighbor of x) is empty.

$$L_E f(\xi) = \sum_x (1-\xi(x+1))\big(\rho(1-\xi(x)) + (1-\rho)\xi(x)\big)\left[f(\xi^x) - f(\xi)\right]$$



$\rho = 1/2$; space horizontal, time goes down, $1 = \bullet$, $0 = \bullet$.

A few properties of the East model

- Reversible w.r.t. $\mu := \mathrm{Ber}(\rho)^{\otimes \mathbb{Z}}$.
- Mixing in $L^2(\mu)$ [Aldous-Diaconis '00]:

  $\forall \rho \in (0,1),\ \exists \gamma > 0$ s.t. $\forall f, g \in L^2(\mu)$

$$|\mathbb{E}_\mu [f(\eta_0) g(\eta_t)] - \mu(f)\mu(g)| \leq C(f,g) e^{-\gamma t}.$$

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Some non-properties of the East model

Not attractive.
No uniform or cone-mixing property.

RW in (dynamic) random environment

$(Z_t)_{t\geq 0}$ process on $\mathbb{Z}^d$ such that

$$\mathbb{P}(Z_{t+dt} = x + y | Z_t = x) = r(y, \tau_x \xi_t) dt,$$

where $(\xi_t)_{t\geq 0}$ is the random environment, with values in $\{0,1\}^{\mathbb{Z}^d}$.

Today: ξ is the East model.

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Environment seen from the walker

$$\eta_t := \tau_{Z_t} \xi_t.$$

has generator

$$Lf(\eta) = L_E f(\eta) + \sum_y r(y, \eta) \left[f(\tau_y \eta) - f(\eta) \right]$$

Jumps of $(X_t)_{t \geq 0} \longleftrightarrow$ spatial shifts for $(\eta_t)_{t \geq 0}$.
Z_t depends on $(\xi_s)_{s \leq t}$.

Random walks

X (*unperturbed walker*) and $X^{(\epsilon)}$ (*perturbed walker*) with continuous rates $r(y,\eta), r_\epsilon(y,\eta) \geq 0$ such that

- $r_\epsilon(y,\eta) = r(y,\eta) + \hat{r}_\epsilon(y,\eta)$.
- $\eta_t := \tau_{X_t}\xi_t$ has invariant measure μ.
  In particular: $(\tau_{X_t}\xi_t)_{t\geq 0}$ has same invariant measure and mixing property as $(\xi_t)_{t\geq 0}$.
- $2\epsilon = \|\hat{L}_\epsilon\| < \gamma$.
- $\sum_y |y|^4 \sup_\eta (r(y,\eta) + |\hat{r}_\epsilon(y,\eta)|) < \infty$.
- $r(y,\eta) > 0 \Longrightarrow r_\epsilon(y,\eta)$.

where

$$
\begin{aligned}
L_\epsilon f(\eta) &= L_E f(\eta) + \sum_y r_\epsilon(y,\eta)\,[f(\tau_y\eta) - f(\eta)] \\
L f(\eta) &= L_E f(\eta) + \sum_y r(y,\eta)\,[f(\tau_y\eta) - f(\eta)] \\
\hat{L}_\epsilon f(\eta) &= \sum_y \hat{r}_\epsilon(y,\eta)\,[f(\tau_y\eta) - f(\eta)]
\end{aligned}
$$

Examples

Driven ghost probe [Jack-Kelsey-Garrahan-Chandler PRE 2008]

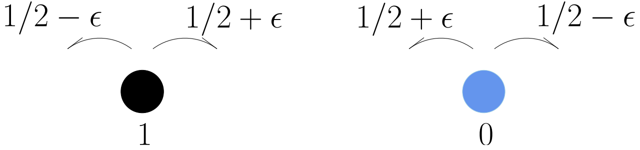
$$\begin{aligned} r(\pm 1,\eta) &= (1-\eta(0))(1-\eta(\pm 1)) \\ r_\epsilon(\pm 1,\eta) &= (1-\eta(0))(1-\eta(\pm 1))\tilde{r}_\epsilon(\pm 1), \end{aligned}$$

where $\tilde{r}_\epsilon(1) = 2/(1+e^{-\epsilon}) = e^{\epsilon}\tilde{r}_\epsilon(-1)$.

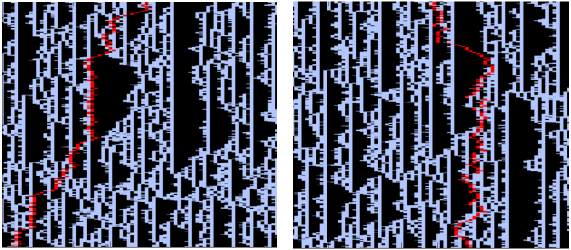
Examples

ϵ-RW

For $\epsilon \in (0, 1/2)$, jump rates given by



N.B.: In the pictures, $p = 1/2 + \epsilon$.



$p = 0.2$ $\quad$ $p = 0.8$

General results

Theorem

1. *The process seen from the walker* $\eta_t^{(\epsilon)} = \tau_{X_t^{(\epsilon)}} \xi_t$*, started from* $\nu \ll \mu$*, converges to an ergodic invariant measure* μ_ϵ *such that* $\mu_\epsilon \sim \mu$ *and for all* $f \in L^2(\mu)$

$$\mu_\epsilon(f) = \mu(f) + \sum_{n=0}^{\infty} \int_0^{\infty} \mu\big(\hat{L}_\epsilon S^{(n)}(s) f\big)\, ds,$$

where $S^{(0)}(t) = e^{tL}$*,* $S^{n+1}(t) = \int_0^t S^{(0)}(t-s) \hat{L}_\epsilon S^{(n)}(s)\, ds$*.*

2. *For all* $f, g \in L^\infty(\mu)$

$$|\mathbb{E}_{\mu_\epsilon}[f(\eta_0) g(\eta_t)] - \mu_\epsilon(f)\mu_\epsilon(g)| \leq C(f,g) e^{-\frac{\gamma - 2\epsilon}{2} t}.$$

3. *If* r_ϵ *have finite range and bounded support,* $\exists \delta > 0$ *s.t.*

$$|\mu_\epsilon(\eta(x)) - \rho| \leq C e^{-\delta |x|}.$$

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3. *If* r_ϵ *have finite range and bounded support,* $\exists \delta > 0$ *s.t.*
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N.B.: Point 1 proved in [Komorowski-Olla '05] when $d\nu/d\mu \in L^2(\mu)$, via a series expansion of the densities rather than semigroups. Perturbation $\hat{L}_\epsilon$ can be unbounded, should satisfy a sector condition.

General results

Theorem

1. ***LLN.***

$$\frac{1}{t}X_t^{(\epsilon)} \underset{t\to\infty}{\longrightarrow} v(\epsilon)$$

2. ***Invariance principle.***

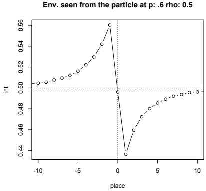
$$\frac{X_{nt}^{(\epsilon)} - v(\epsilon)nt}{\sqrt{n}} \to \sqrt{2D_\epsilon}B_t,$$

3. *If the environment seen from the unperturbed walker $(\eta_t)_{t\geq 0}$ is reversible w.r.t. μ, $D_\epsilon > 0$ for ϵ small enough.*

More on the ϵ-RW

Theorem

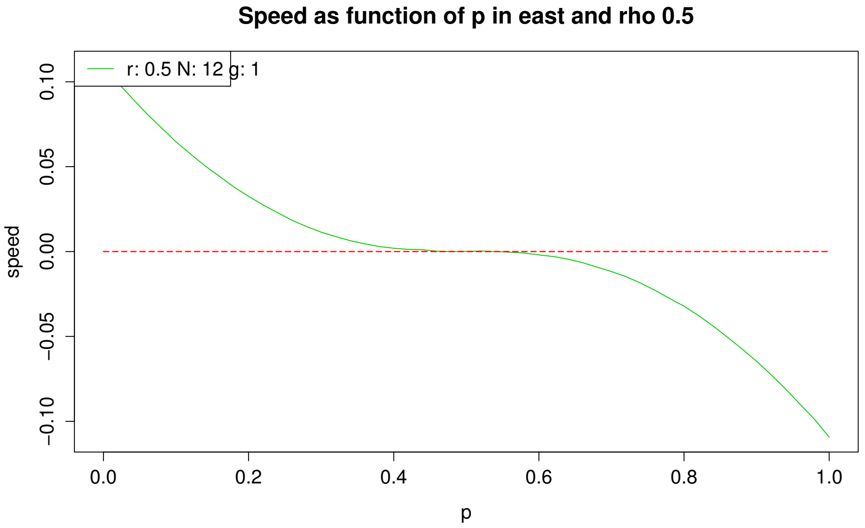
1. *If $\epsilon > 0$ small enough, $\mu_\epsilon(\eta(1)) < \rho < \mu_\epsilon(\eta(-1))$ (and the reverse if $\epsilon > 0$).*



Numerics by P. Thomann.

2. $v(\epsilon) = 2\epsilon(2\rho - 1) + \epsilon^3 \kappa_\rho + O(\epsilon^5)$.
   In particular, if $\rho \neq 1/2$, $|\epsilon| \neq 0$ small enough, $\mathrm{sgn}(v(\epsilon)) = \mathrm{sgn}(\epsilon(2\rho - 1))$ (the sign of the speed is given by the dominating type of site in the environment).

What if $\rho = 1/2$?



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More on the speed $v(\epsilon)$

Conjecture

If $\rho = 1/2$, $\epsilon \cdot v(\epsilon) < 0$.
(Weaker: $\kappa_{1/2} < 0$.)

A consequence of the orientation choice?

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Theorem

If $2\epsilon < \gamma$,

$$v(-\epsilon) = -v(\epsilon).$$

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Theorem

If $2\epsilon < \gamma$,

$$v(-\epsilon) = -v(\epsilon).$$

In particular,

$$v_{\mathrm{East}}(\epsilon) = v_{\mathrm{West}}(\epsilon).$$

Rather, a signature of the space-time correlated bubbles.
N.B.: The antisymmetry holds for any reversible environment with L^2 exponential mixing.

A degenerate version of the ϵ-RW

"Infinite rates".

$(Y_t)_{t\geq 0}$ lives on edges of $\mathbb{Z}$ of type $\bullet - \bullet$.

When the particle on the left is erased, jumps to the right-most such edge.

When the hole on the right is filled, jumps to the left-most such edge (one step to the right).

$$\limsup Y_t/t < 0$$

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Three-point correlation function

Proposition

If for any $y \geq 1$, for any $s, t \geq 0$,

$$\mathbb{E}_{\rho}^{\mathrm{East}}\left[\xi_0(0)(2\xi_t(y) - 1)\xi_{t+s}(0)\right] > 0,$$

then $\kappa_\rho < 0$.

Thank you for your attention.