Metastability in the reversible inclusion process

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Work in progress jointly with

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Inclusion process

Interacting particle system with N particles

Vertex set S with $|S| < \infty$

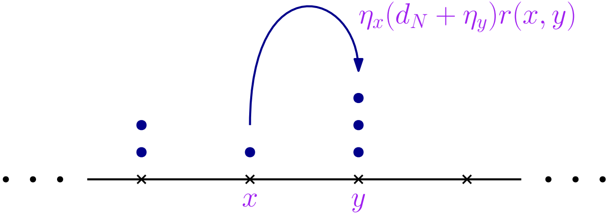
Configuration $\eta = (\eta_x)_{x \in S} \in \{0, \dots, N\}^S$, $\eta_x = \#\text{particles on } x \in S$

Underlying random walk on S with transition rates $r(x, y)$

Inclusion process is continuous time Markov process with generator

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} \eta_x (d_N + \eta_y) r(x, y) \left[f(\eta^{x,y}) - f(\eta) \right]$$

Particle jump rates



Particle jump rates can be split into

$\eta_x \, d_N \, r(x,y)$	independent random walkers	diffusion
$\eta_x \, \eta_y \, r(x,y)$	attractive interaction	inclusion

Comparison with other processes:

$\eta_x \, (1 - \eta_y) \, r(x,y)$	exclusion process
$g(\eta_x) \, r(x,y)$	zero range process

Motivation

Symmetric IP on $\mathbb{Z}$ introduced as dual of Brownian momentum process
Giardinà, Kurchan, Redig, Vafayi, 2007–2010

Natural bosonic counterpart to the (fermionic) exclusion process

Interesting dynamical behavior: condensation / metastability
In symmetric IP: Grosskinsky, Redig, Vafayi 2011, 2013

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Can we analyze this using the martingale approach?
Beltrán, Landim, 2010

Successfully used for reversible zero range process Beltrán, Landim, 2012

Can we generalize results to the reversible IP?

Reversible inclusion process

Random walk $r(\cdot,\cdot)$ reversible w.r.t. some measure $m(\cdot)$:

$$m(x)r(x,y) = m(y)r(y,x) \qquad \forall\, x, y \in S$$

Normalized such that

$$\max_{x\in S} m(x) = 1$$

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Then, also inclusion process reversible w.r.t. probability measure

$$\mu_N(\eta) = \frac{1}{Z_N}\prod_{x\in S} m(x)^{\eta_x} w_N(\eta_x)$$

where Z_N is a normalization constant and

$$w_N(k) = \frac{\Gamma(d_N + k)}{k!\Gamma(d_N)}$$

Condensation

Let $S_\star = \{x \in S : m(x) = 1\}$ and $\eta^{x,N}$ the configuration η with $\eta_x = N$

Proposition

Suppose that $d_N \log N \to 0$ *as* $N \to \infty$. *Then*

$$\lim_{N\to\infty} \mu_N(\eta^{x,N}) = \frac{1}{|S_\star|} \qquad \forall\, x \in S_\star$$

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Assumption on d_N such that

$$\frac{N}{d_N} w_N(N) = \frac{1}{d_N \Gamma(d_N)} \frac{\Gamma(N + d_N)}{(N-1)!} = \frac{1}{\Gamma(d_N + 1)} e^{d_N \log N}(1 + o(1)) \to 1$$

(using Stirling's approximation)

Movement of the condensate

Consider the following process on $S_\star \cup \{0\}$:

$$X_N(t) = \sum_{x \in S_\star} x \, \mathbb{1}_{\{\eta_x(t)=N\}}$$

Theorem (Bianchi, D., Giardinà, 2016)

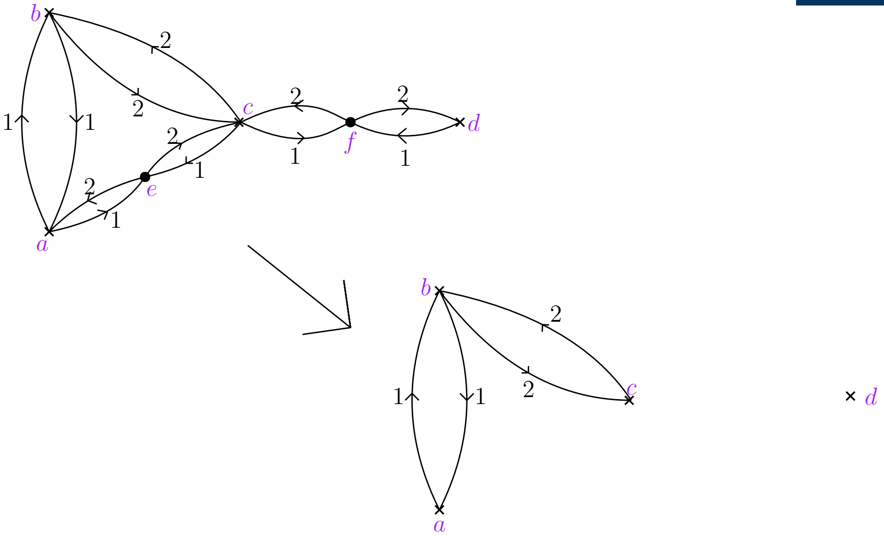
Suppose that $d_N \log N \to 0$ *as* $N \to \infty$ *and that* $\eta_y(0) = N$ *for some* $y \in S_\star$. *Then*

$$X_N(t/d_N) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

where $x(t)$ *is a Markov process on* $S_\star$ *with* $x(0) = y$ *and transition rates*

$$p(x,y) = r(x,y)$$

Example



Zero range process Beltrán, Landim, 2012

Underlying reversible random walk $r(\cdot,\cdot)$

Transition rates for a particle to move from x to y

$$\left(\frac{\eta_x}{\eta_x - 1}\right)^{\alpha} r(x,y), \qquad \alpha > 1$$

Condensate consists of at least $N - \ell_N$ particles, $\ell_N = o(N)$

At timescale $tN^{\alpha+1}$ the condensate moves from $x \in S_\star$ to $y \in S_\star$ at rate

$$p(x,y) = C_\alpha \mathrm{cap}(x,y)$$

where $\mathrm{cap}(x,y)$ is the capacity of the underlying random walk between x and y.

Proof strategy

If $r(\cdot, \cdot)$ is symmetric ($S = S_\star$), cite Grosskinsky, Redig, Vafayi, 2013
They analyze directly rescaled generator

Otherwise, martingale approach Beltrán, Landim, 2010
Potential theory combined with martingale arguments

Successfully applied to zero range process Beltrán, Landim, 2012

Martingale approach Beltrán, Landim, 2010

To prove the theorem we need to check the following three hypotheses:

(H0) $\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to p(x,y) = r(x,y)$

where $p_N(\eta^{x,N}, \eta^{y,N})$ rate to go from $\eta^{x,N}$ to $\eta^{y,N}$ in original process

(H1) All states in each metastable set are visited before exiting.

(H2) $$\lim_{N\to\infty} \frac{\mu_N(\eta : \nexists y \in S_\star : \eta_y = N)}{\mu_N(\eta^{x,N})} = 0 \qquad \forall x \in S_\star$$

Martingale approach Beltrán, Landim, 2010

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Capacities

Capacity satisfy

$$\mathrm{Cap}_N(A,B) = \inf\{D_N(F) : F(\eta) = 1\ \forall \eta \in A, F(\xi) = 0\ \forall \eta \in B\}$$

where $D_N(F)$ is the Dirichlet form

$$D_N(F) = \frac{1}{2}\sum_{\eta} \mu_N(\eta) \sum_{x,y\in S} \eta_x(d_N + \eta_y) r(x,y)[F(\eta^{x,y}) - F(\eta)]^2$$

Lemma (Beltrán, Landim, 2010)

$$\mu_N(\eta^{x,N}) p_N(\eta^{x,N}, \eta^{y,N})$$
$$= \frac{1}{2}\Big\{\mathrm{Cap}_N\Big(\{\eta^{x,N}\}, \bigcup_{z\in S_\star, z\neq x} \{\eta^{z,N}\}\Big) + \mathrm{Cap}_N\Big(\{\eta^{y,N}\}, \bigcup_{z\in S_\star, z\neq y} \{\eta^{z,N}\}\Big)$$
$$- \mathrm{Cap}_N\Big(\{\eta^{x,N}, \eta^{y,N}\}, \bigcup_{z\in S_\star, z\neq x,y} \{\eta^{z,N}\}\Big)\Big\}$$

Capacities in inclusion process

Proposition

Let $S_\star^1 \subsetneq S_\star$ *and* $S_\star^2 = S_\star \setminus S_\star^1$. *Then, for* $d_N \log N \to 0$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x\in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y\in S_\star^2} \{\eta^{y,N}\}\Big) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y)$$

Combining this proposition and the previous lemma indeed gives

$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to r(x,y)$$

Lower bound on Dirichlet form

Fix a function F such that $F(\eta^{x,N}) = 1 \; \forall\, x \in S_\star^1$ and $F(\eta^{y,N}) = 0 \; \forall\, y \in S_\star^2$

Sufficient to show that

$$D_N(F) \geq d_N \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)\big(1 + o(1)\big)$$

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For lower bound we can throw away terms in the Dirichlet form

$$\begin{aligned} D_N(F) &= \frac{1}{2} \sum_{\eta} \mu_N(\eta) \sum_{x,y \in S} \eta_x (d_N + \eta_y) r(x,y) [F(\eta^{x,y}) - F(\eta)]^2 \\ &\geq \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y) \sum_{\eta_x + \eta_y = N} \mu_N(\eta) \eta_x (d_N + \eta_y) [F(\eta^{x,y}) - F(\eta)]^2 \end{aligned}$$

If condensates jumps from x to y all particles will move from x to y

Lower bound on Dirichlet form (continued)

Fix $x \in S^1_\star, y \in S^2_\star$. If $\eta_x + \eta_y = N$ it is sufficient to know how many particles are on x

$$\sum_{\eta_x+\eta_y=N} \mu_N(\eta)\eta_x(d_N + \eta_y)[F(\eta^{x,y}) - F(\eta)]^2$$

$$= \sum_{k=1}^{N} \frac{w_N(k)w_N(N-k)}{Z_N} k(d_N + N - k)[G(k-1) - G(k)]^2$$

where $G(k) = F(\eta_x = k, \eta_y = N - k)$ and where we used $m(x) = m(y) = 1$ since $x \in S^1_\star, y \in S^2_\star$.

Lower bound on $w_N(k)$

Recall

$$w_N(k) = \frac{\Gamma(d_N + k)}{k!\Gamma(d_N)} = \prod_{\ell=1}^{k} \frac{\ell - 1 + d_N}{\ell}$$

$$\geq d_N \prod_{\ell=2}^{k} \frac{\ell - 1}{\ell} = \frac{d_N}{k}$$

Hence

$$w_N(k)k \geq d_N \qquad \forall\, k \geq 1$$

and

$$w_N(k)(d_N + k) \geq d_N \qquad \forall\, k \geq 0$$

Lower bound on Dirichlet form (continued)

We can now bound

$$\sum_{k=1}^{N} \frac{w_N(k)w_N(N-k)}{Z_N} k(d_N + N - k)[G(k-1) - G(k)]^2$$

$$\geq \frac{1}{Z_N} d_N^2 \sum_{k=1}^{N} [G(k-1) - G(k)]^2$$

Since $G(N) = 1$ and $G(0) = 0$ we can use resistance of linear chain to bound

$$\sum_{k=1}^{N} [G(k-1) - G(k)]^2 \geq 1/N$$

because the minimizer of this over all such G is $G(k) = k/N$

Lower bound on Dirichlet form (conclusion)

So far we proved that

$$D_N(F) \geq \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y) \frac{1}{Z_N} d_N^2 \frac{1}{N}$$

We know that

$$Z_N = |S_\star| w_N(N)(1 + o(1)) = \frac{d_N}{N}(1 + o(1))$$

Hence,

$$\frac{1}{d_N} D_N(F) \geq \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)(1 + o(1))$$

Taking infimum and limit on both sides indeed proves that

$$\lim_{N \to \infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x \in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y \in S_\star^2} \{\eta^{y,N}\} \Big) \geq \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)$$

Upper bound on Dirichlet form

Need to construct test function $F(\eta)$

Good guess inside tubes $\eta_x + \eta_y = N$: $F(\eta) \approx \eta_x / N$

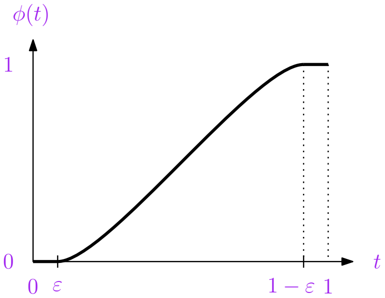
In fact better to choose smooth monotone function $\phi(t), t \in [0,1]$ with

$\phi(t) = 1 - \phi(1-t) \; \forall \, t \in [0,1]$
$\phi(t) = 0$ if $t \leq \varepsilon$

and set $F(\eta) = \phi\left(\eta_x / N\right)$

For general η we set

$$F(\eta) = \sum_{x \in S^1_\star} \phi\left(\eta_x / N\right)$$



Observations for upper bound on $D_N(F)$

$$D_N(F) = \frac{1}{2}\sum_{\eta}\mu_N(\eta)\sum_{x,y\in S}\eta_x(d_N+\eta_y)r(x,y)[F(\eta^{x,y})-F(\eta)]^2$$

For $\varepsilon N \leq \eta_x \leq (1-\varepsilon)N$ we can use $w_N(\eta_x)\eta_x \approx d_N$

By construction particles moving from $x \in S_\star^1$ to $y \in S_\star^2$ give correct contribution

If numbers of particles on sites in $S_\star^1$ don't change, F is constant

If particles move between sites in $S_\star^1$, F is also constant

Unlikely to be in config. with particles on three sites / sites not in $S_\star$

Unlikely for a particle to escape from a tube

Capacities in inclusion process (conclusion)

Combining the lower and upper bound indeed this proposition follows

Proposition

Let $S_\star^1 \subsetneq S_\star$ *and* $S_\star^2 = S_\star \setminus S_\star^1$. *Then, for* $d_N \log N \to 0$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x\in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y\in S_\star^2} \{\eta^{y,N}\} \Big) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y)$$

And the transition rates indeed satisfy

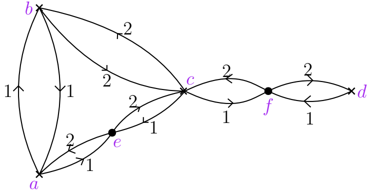
$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to r(x,y)$$

proving the theorem

Open problems / future work

What if vertices in $S_\star$ are not connected?
Longer timescale(s)?

Can we compute relaxation time?



Can we compute thermodynamic limit?
Zero-range process: Armendáriz, Grosskinsky, Loulakis, 2015

Can we say something about the formation of the condensate?
Studied for SIP in Grosskinsky, Redig, Vafayi, 2013

Can we obtain similar results for non-reversible dynamics?
e.g. (T)ASIP on $\mathbb{Z}/L\mathbb{Z}$. Heuristics: Cao, Chleboun, Grosskinsky, 2014