Universal Fluctuation Formulae for one-cut β-ensembles

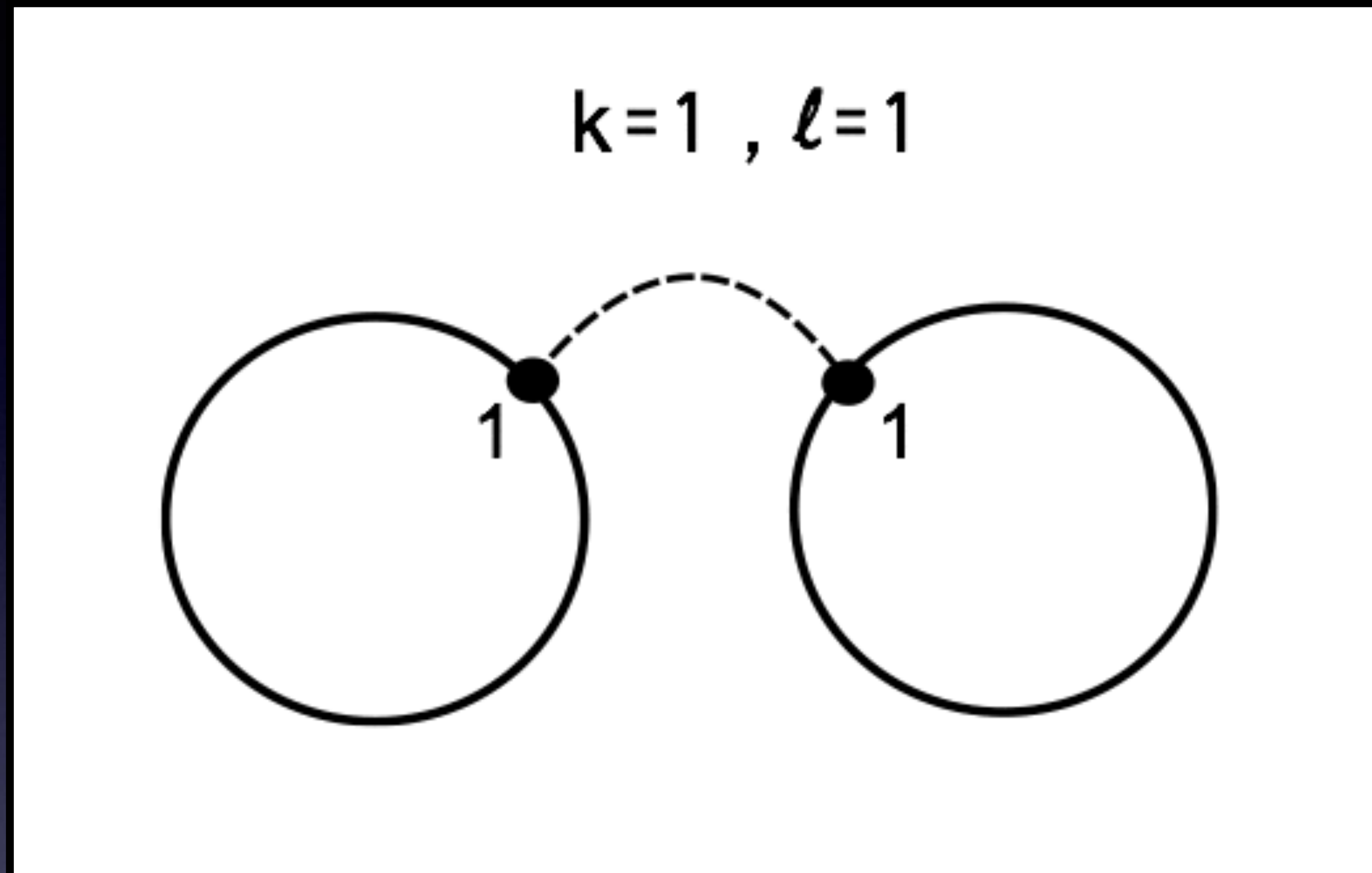
with a combinatorial touch

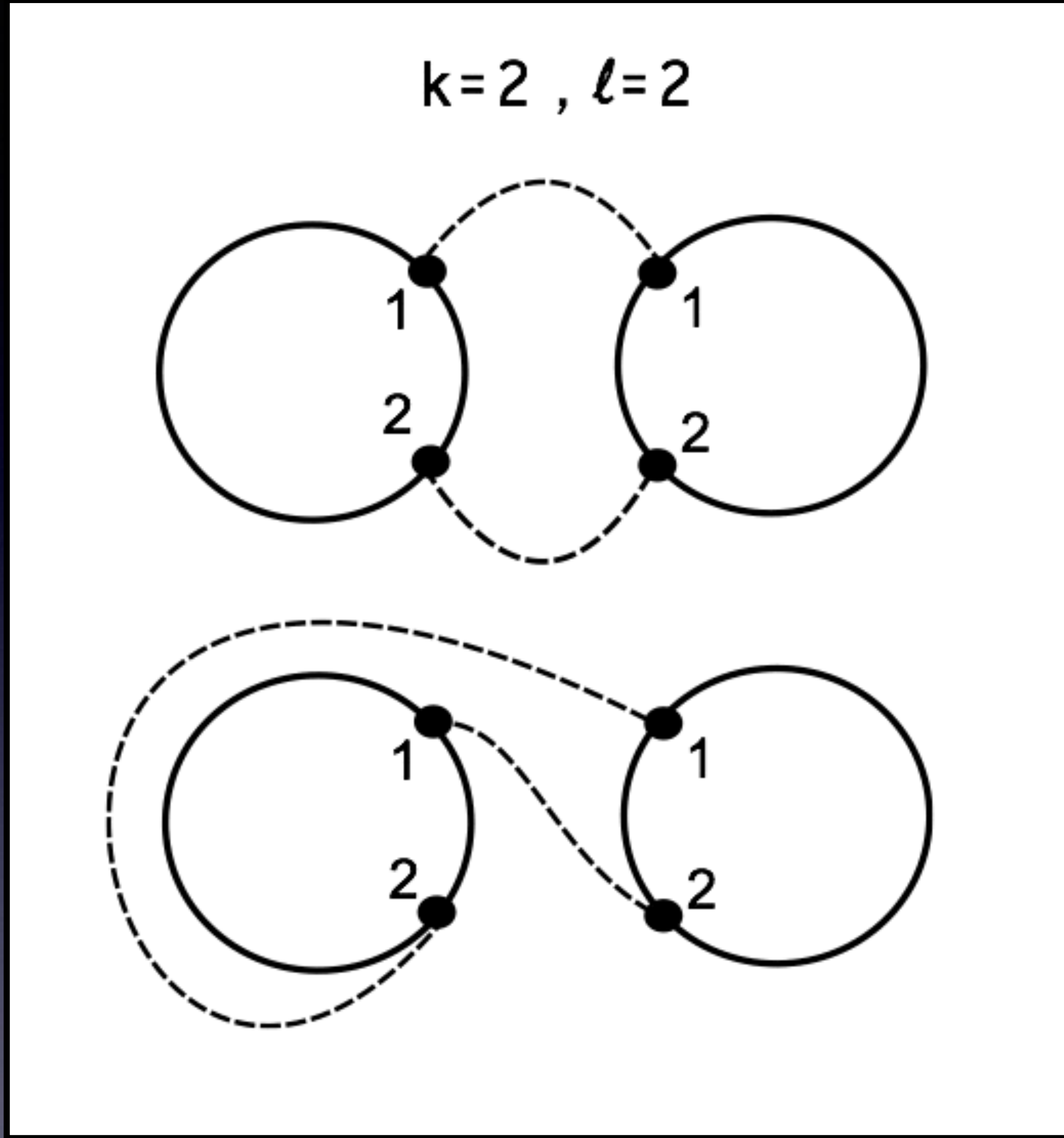
Pierpaolo Vivo

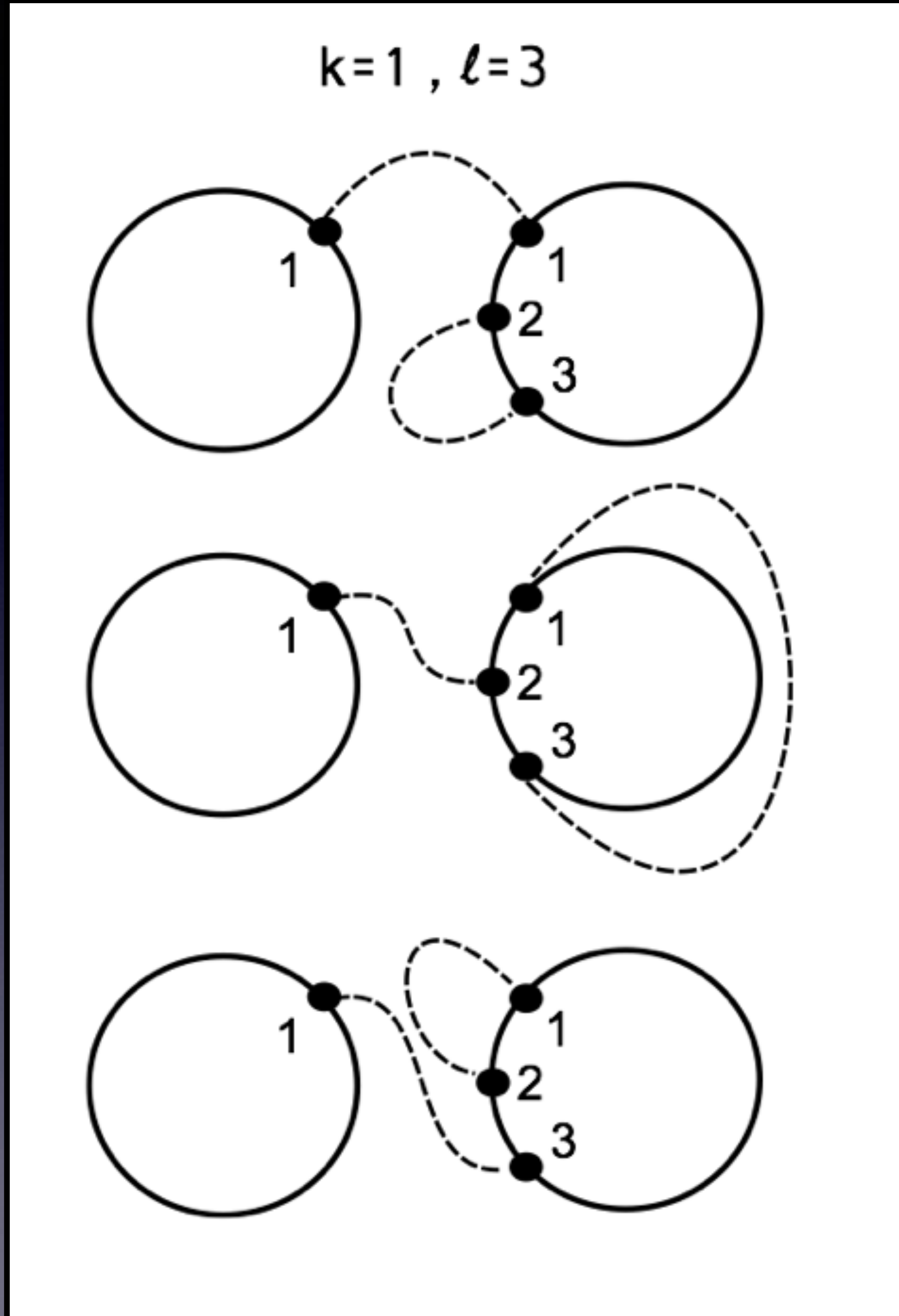
KING'S College LONDON

with F. D. Cunden   Phys. Rev. Lett. **113**, 070202 (2014)

with F.D. Cunden and F. Mezzadri   J. Phys. A **48**, 315204 (2015)


k = 1 , ℓ = 1
1
1


k=2 , ℓ=2
1
2
1
2
1
2
1
2


k=1 , ℓ=3
1
1
2
3
1
1
2
3
1
1
2
3

Non-crossing pairings of two circles

$$\begin{pmatrix}
1 & 0 & 3 & 0 & 10 & 0 & 35 & 0 \\
0 & 2 & 0 & 8 & 0 & 30 & 0 & 112 \\
3 & 0 & 12 & 0 & 45 & 0 & 168 & 0 \\
0 & 8 & 0 & 36 & 0 & 144 & 0 & 560 \\
10 & 0 & 45 & 0 & 180 & 0 & 700 & 0 \\
0 & 30 & 0 & 144 & 0 & 600 & 0 & 2400 \\
35 & 0 & 168 & 0 & 700 & 0 & 2800 & 0 \\
0 & 112 & 0 & 560 & 0 & 2400 & 0 & 9800
\end{pmatrix}$$

No pairings are possible if ‘k’ and ‘l’ have different parity

- Formula for these numbers?

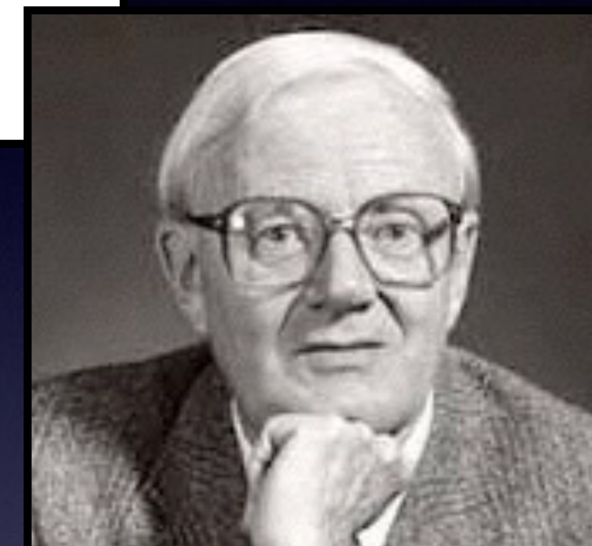
- Connection to random matrices?

- Formula for these numbers?

A NEW BRANCH OF ENUMERATIVE GRAPH THEORY

BY W. T. TUTTE

Communicated by Walter Rudin, April 6, 1962



A CENSUS OF SLICINGS

W. T. TUTTE

$$\begin{cases} \frac{2\kappa\ell}{\kappa+\ell}\binom{\kappa-1}{\lfloor\frac{\kappa}{2}\rfloor}\binom{\ell-1}{\lfloor\frac{\ell}{2}\rfloor} & \text{if } \kappa=\ell \mod 2 \\ 0 & \text{if } \kappa\neq\ell \mod 2, \end{cases}$$

- Formula for these numbers?

A NEW BRANCH OF ENUMERATIVE GRAPH THEORY

```
In[30]:= T = Table[If[(EvenQ[k] && EvenQ[l]) || (OddQ[k] && OddQ[l]),
           (2 k l)/(k + l) Binomial[k - 1, Floor[k / 2]] Binomial[l - 1, Floor[l / 2]], 0], {k, 1, 8}, {l, 1, 8}];
         T // MatrixForm
Out[31]//MatrixForm=
```

1	0	3	0	10	0	35	0
0	2	0	8	0	30	0	112
3	0	12	0	45	0	168	0
0	8	0	36	0	144	0	560
10	0	45	0	180	0	700	0
0	30	0	144	0	600	0	2400
35	0	168	0	700	0	2800	0
0	112	0	560	0	2400	0	9800

$$\begin{cases} \dfrac{2\kappa\ell}{\kappa+\ell}\dbinom{\kappa-1}{\lfloor\frac{\kappa}{2}\rfloor}\dbinom{\ell-1}{\lfloor\frac{\ell}{2}\rfloor} & \text{if } \kappa = \ell \mod 2 \\ 0 & \text{if } \kappa \neq \ell \mod 2, \end{cases}$$

- Connection to random matrices?

$$G_{ij} = \frac{1}{\sqrt{N}} x_{ij} \qquad x_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ for } i < j \text{ and } x_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,1)$$

GUE

- Connection to random matrices?

- $$G_{ij} = \frac{1}{\sqrt{N}} x_{ij} \qquad x_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ for } i < j \text{ and } x_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,1)$$

GUE

- $$G_{N,\kappa} = \frac{1}{N} \mathrm{Tr} G^{\kappa}$$

- Connection to random matrices?

- $$G_{ij} = \frac{1}{\sqrt{N}} x_{ij} \qquad x_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ for } i<j \text{ and } x_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,1)$$

GUE

- $$G_{N,\kappa} = \frac{1}{N}\mathrm{Tr} G^{\kappa}$$

- $$\alpha^{\mathcal{G}}_{\kappa,\ell} = \lim_{N\to\infty} \beta N^2 \mathrm{Cov}\left(G_{N,\kappa}, G_{N,\ell}\right)$$

- Connection to random matrices?

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 10 & 0 & 35 & 0 \\ 0 & 2 & 0 & 8 & 0 & 30 & 0 & 112 \\ 3 & 0 & 12 & 0 & 45 & 0 & 168 & 0 \\ 0 & 8 & 0 & 36 & 0 & 144 & 0 & 560 \\ 10 & 0 & 45 & 0 & 180 & 0 & 700 & 0 \\ 0 & 30 & 0 & 144 & 0 & 600 & 0 & 2400 \\ 35 & 0 & 168 & 0 & 700 & 0 & 2800 & 0 \\ 0 & 112 & 0 & 560 & 0 & 2400 & 0 & 9800 \end{pmatrix}$$

- G … $(0,1)$

$(1/2)\alpha^{\mathcal{G}}_{\kappa,\ell} = \#\{$Non-crossing pairings of two circles with κ points on the 1st circle and ℓ points on the 2nd circle such that the two circles are connected$\}$.

- $$\alpha^{\mathcal{G}}_{\kappa,\ell} = \lim_{N\to\infty} \beta N^2 \mathrm{Cov}\left(G_{N,\kappa}, G_{N,\ell}\right)$$

- Communities interested in this?

Combinatorics

Free Probability

RMT

- Communities interested in this?

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Combinatorics

Real Second-Order Freeness and the Asymptotic Real Second-Order Freeness of Several Real Matrix Models

Catherine Emily Iska Redelmeier

Free Probability

RMT

JOURNAL OF MATHEMATICAL PHYSICS **47**, 063302 (2006)

Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models

Ioana Dumitriu[a] and Alan Edelman

University of California, Berkeley, Berkeley, California 94720

GLOBAL FLUCTUATIONS FOR LINEAR STATISTICS OF β-JACOBI ENSEMBLES

IOANA DUMITRIU* and ELLIOT PAQUETTE†

- Communities interested in this?

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Free

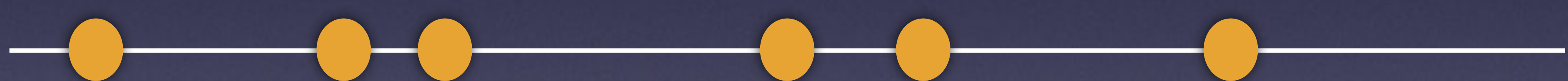
satisfy this definition. Second-order freeness then has the role for fluctuations that first-order freeness has for moments. In particular, if random matrices A and B are asymptotically second-order free, the asymptotic fluctuations of $A+B$ and AB can be calculated from the asymptotic moments and fluctuations of A and B.

Asymptotic second-order freeness as defined in [18] is not generally satisfied by real ensembles of random matrices. If random matrices $A_{k,N}$ and $B_{l,N}$, $k,l = 1,\ldots,p$ are elements of the algebra generated by a model studied in this paper, then the relation satisfied instead is

$$\lim_{N\to\infty} \operatorname{cov}\left(\operatorname{Tr}\left(\mathring{A}_{1,N}\cdots\mathring{A}_{p,N}\right),\operatorname{Tr}\left(\mathring{B}_{1,N}\cdots\mathring{B}_{p,N}\right)\right)$$
$$=\sum_{k=0}^{p-1}\prod_{i=1}^{p}\lim_{N\to\infty}\mathbb{E}\left(\operatorname{tr}\left(\mathring{A}_{i,N}\mathring{B}_{k-i,N}\right)\right)+\sum_{k=0}^{p-1}\prod_{i=0}^{p}\lim_{N\to\infty}\mathbb{E}\left(\operatorname{tr}\left(\mathring{A}_{i,N}\mathring{B}^{T}_{k+i,N}\right)\right)$$

- General problem

$$\mathrm{d}\mathbb{P}_{N,\beta}(\{\lambda_k\}) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\beta\left(-\frac{1}{2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j|+N\sum_i V(\lambda_i)\right)} \prod_{i=1}^{N} \mathrm{d}\lambda_i.$$



Can be interpreted as the joint probability density of eigenvalues of a β-ensemble of random matrices

- General setting

$$\mathrm{d}\mathbb{P}_{N,\beta}(\{\lambda_k\}) = \frac{1}{Z_{N,\beta}} e^{-\beta\left(-\frac{1}{2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j|+N\sum_i V(\lambda_i)\right)} \prod_{i=1}^{N} \mathrm{d}\lambda_i .$$

"Inverse Temperature"

"External potential"

We further assume a “one-cut” property: the external potential is such that the spectral density in the large-N limit is supported on a single interval of the real line

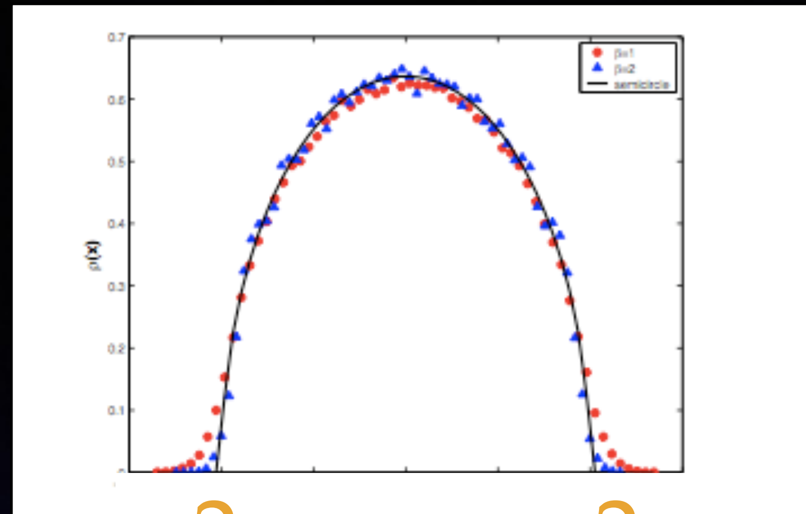
Examples:

β-Gaussian

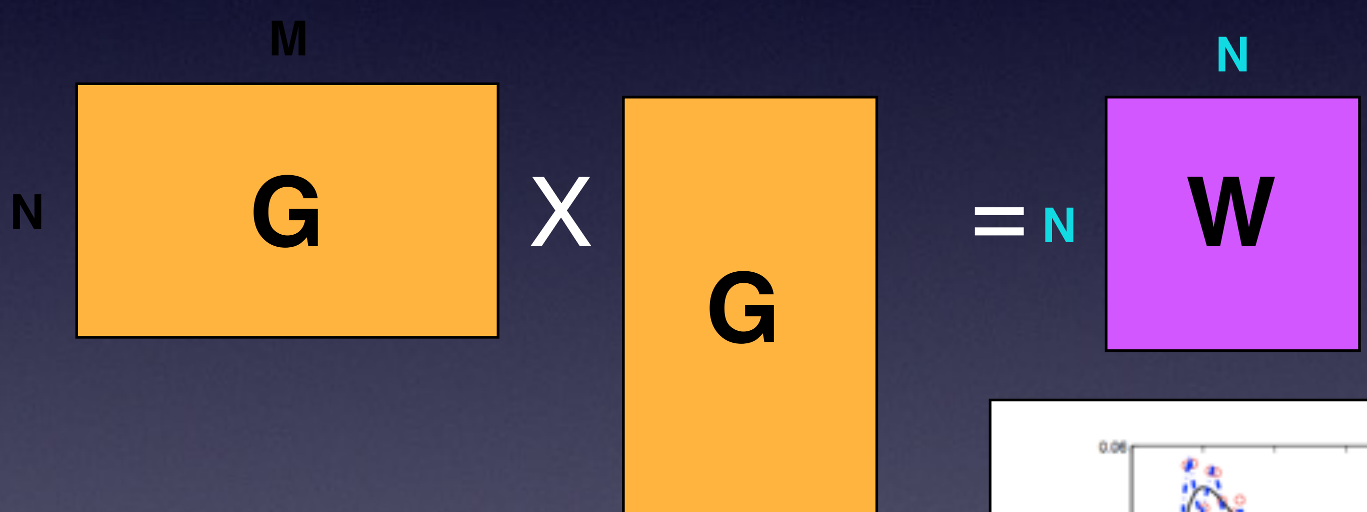
β-Wishart

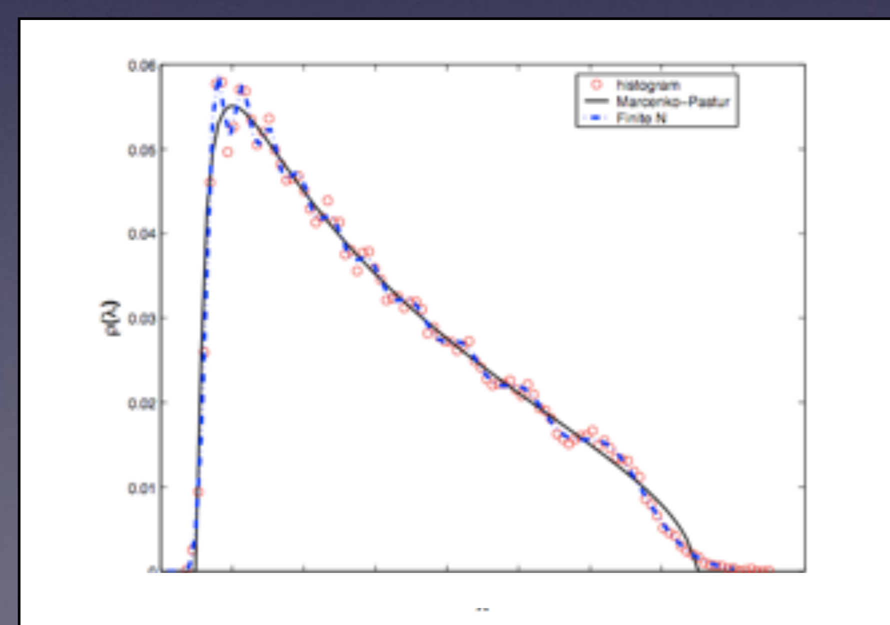
β-Jacobi

Gaussian



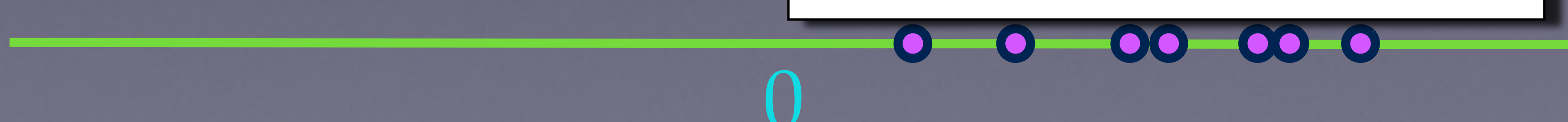
$$\mathbf{G}_{ij} \sim \mathcal{N}(0, 2/N)$$





Wishart

$$\mathbf{W} = \mathbf{G}\mathbf{G}^{\dagger}$$



$$X_{N,\kappa} = N^{-1}\mathrm{Tr}\mathcal{X}_N^{\kappa}$$

$$\lim_{N\to\infty} N^2\mathrm{Cov}(X_{N,\kappa}, X_{N,\ell}) = \frac{1}{\beta}\alpha_{\kappa,\ell}.$$

$$X_{N,\kappa} = N^{-1}\mathrm{Tr}\mathcal{X}_N^\kappa$$

$$\lim_{N\to\infty} N^2 \mathrm{Cov}(X_{N,\kappa}, X_{N,\ell}) = \frac{1}{\beta}\alpha_{\kappa,\ell}.$$

Our main result:

$$F(z,\zeta) = \sum_{\kappa,\ell=0}^{\infty} \alpha_{\kappa,\ell} z^\kappa \zeta^\ell = \frac{z\zeta}{(z-\zeta)^2}\left[\frac{2\lambda_-\lambda_+ z\zeta - (\lambda_- + \lambda_+)(z+\zeta) + 2}{2\sqrt{(1-\lambda_- z)(1-\lambda_+ z)(1-\lambda_-\zeta)(1-\lambda_+\zeta)}} - 1\right]$$

Generating function of the covariances of *any* one-cut β-ensemble

UNIVERSAL!

AIP | Journal of Mathematical Physics

Global spectrum fluctuations for the β -Hermite and β -Laguerre ensembles via matrix models

Ioana Dumitriu and Alan Edelman

Claim 3.15.2: For any fixed integers k and l,

$$\lim_{n\to\infty} \mathrm{Cov}(n_{i,\gamma}(n), \eta_{j,\gamma}(n)) = \frac{2}{\beta}(\mathrm{Sum}_1(i,j,\gamma) + \mathrm{Sum}_2(i,j,\gamma)),$$

where

$$\mathrm{Sum}_1(i,j,\gamma) = \sum_{q=1}^{i+j-1} (-1)^{q+1}\gamma^{i+j-q}\frac{\binom{i+j}{q}}{i+j}\sum_{j=q+1}^{i+j}\frac{(-1)^j}{\binom{i+j-1}{j-1}}\sum_{\substack{r+s=j\\ 1\le r\le i\\ 1\le s\le j}} rs\binom{i}{r}^2\binom{j}{s}^2,$$

$$\mathrm{Sum}_2(i,j,\gamma) = \sum_{q=0}^{i+j-2} (-1)^{q}\gamma^{i+j-q}\frac{\binom{i+j}{q}}{i+j}\sum_{j=q}^{i+j-2}\frac{(-1)^j}{\binom{i+j-1}{j}}\sum_{\substack{r+s=j\\ 0\le r\le i-1\\ 0\le s\le j-1}} (i-r)(j-s)\binom{i}{r}^2\binom{j}{s}^2.$$

$$\alpha^{W}_{\kappa,\ell} = \lim_{N\to\infty}\beta N^2 \mathrm{Cov}\left(W_{N,\kappa}, W_{N,\ell}\right) = 4(1+c)^{\kappa+\ell}\sum_{\substack{0\le p\le\kappa\\ 0\le q\le\ell\\ p=q \bmod 2}}\left(\frac{\sqrt{c}}{1+c}\right)^{p+q}\frac{pq}{p+q}\binom{\kappa}{p}\binom{\ell}{q}\binom{p-1}{\lfloor\frac{p}{2}\rfloor}\binom{q-1}{\lfloor\frac{q}{2}\rfloor}.$$

AIP | Journal of Mathematical Physics

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Claim 3.15.2: For any fixed integers k and l,

$$\lim_{N\to\infty} N^2 \mathrm{Cov}\left(W_{N,\kappa}, W_{N,\ell}\right) = \frac{1}{\beta}\begin{bmatrix} 2c & 4\left(c+c^2\right) & 6\left(c+3c^2+c^3\right) \\ 4\left(c+c^2\right) & 4\left(2c+5c^2+2c^3\right) & 12\left(c+5c^2+5c^3+c^4\right) \\ 6\left(c+3c^2+c^3\right) & 12\left(c+5c^2+5c^3+c^4\right) & 6\left(3c+24c^2+46c^3+24c^4+3c^5\right) \end{bmatrix}$$

$$\mathrm{Sum}_1(i,j,\gamma) = \sum_{q=1} (-1)^{q+1}\gamma^{i+j-q} \frac{\ldots}{i+j} \sum_{j=q+1} \frac{\ldots}{\binom{i+j-1}{j-1}} \sum_{\substack{r+s=j\\ 1\le r\le i\\ 1\le s\le j}} rs\binom{i}{r}\binom{j}{s},$$

$$\mathrm{Sum}_2(i,j,\gamma) = \sum_{q=0}^{i+j-2} (-1)^q \gamma^{i+j-q} \frac{\binom{i+j}{q}}{i+j} \sum_{j=q}^{i+j-2} \frac{(-1)^j}{\binom{i+j-1}{j}} \sum_{\substack{r+s=j\\ 0\le r\le i-1\\ 0\le s\le j-1}} (i-r)(j-s)\binom{i}{r}^2\binom{j}{s}^2.$$

$$\alpha^{W}_{\kappa,\ell} = \lim_{N\to\infty} \beta N^2 \mathrm{Cov}\left(W_{N,\kappa}, W_{N,\ell}\right) = 4(1+c)^{\kappa+\ell} \sum_{\substack{0\le p\le\kappa\\ 0\le q\le\ell\\ p=q \bmod 2}} \left(\frac{\sqrt{c}}{1+c}\right)^{p+q} \frac{pq}{p+q}\binom{\kappa}{p}\binom{\ell}{q}\binom{p-1}{\lfloor\frac{p}{2}\rfloor}\binom{q-1}{\lfloor\frac{q}{2}\rfloor}.$$

Random Matrices: Theory and Applications
Vol. 1, No. 4 (2012) 1250013 (60 pages)

DOI: 10.1142/S201032631250013X

GLOBAL FLUCTUATIONS FOR LINEAR STATISTICS OF β-JACOBI ENSEMBLES

IOANA DUMITRIU* and ELLIOT PAQUETTE†

The matrix $C_{k,l}$ for $k, l \geq 1$ can now be defined by

$$C_{k,l} := \frac{\alpha}{4}\int_{-a}^{0} \frac{1}{1+2\sigma}[(\partial_x p_k \partial_x p_m + \partial_y p_k \partial_y p_m)(1 - x^2 - y^2) - (\partial_x p_k \partial_y p_m + \partial_y p_k \partial_x p_m)(2xy)]d\sigma. \tag{2.14}$$

$$\alpha^{\mathfrak{J}}_{\kappa,\ell} = \left[\frac{\gamma_1^2 + \gamma_1\gamma_2 + 2(\gamma_1+\gamma_2+1)}{(\gamma_1+\gamma_2+2)^2}\right]^{\kappa+\ell} \sum_{\substack{0\leq p\leq\kappa \\ 0\leq q\leq\ell}} \left[\frac{\sqrt{(\gamma_1+1)(\gamma_2+1)(\gamma_1+\gamma_2+1)}}{\gamma_1^2+\gamma_1\gamma_2+2(\gamma_1+\gamma_2+1)}\right]^{p+q} \binom{\kappa}{p}\binom{\ell}{q}\alpha^{\mathfrak{G}}_{p,q},$$

$$\lim_{N\to\infty} N^2 \mathrm{Cov}\left(J_{N,\kappa}, J_{N,\ell}\right) = \frac{1}{\beta}\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{15}{128} & \frac{7}{64} & \frac{105}{1024} \\ \frac{1}{8} & \frac{9}{64} & \frac{9}{64} & \frac{35}{256} & \frac{135}{1024} \\ \frac{15}{128} & \frac{9}{64} & \frac{75}{512} & \frac{75}{512} & \frac{4725}{32768} \\ \frac{7}{64} & \frac{35}{256} & \frac{75}{512} & \frac{1225}{8192} & \frac{1225}{8192} \\ \frac{105}{1024} & \frac{135}{1024} & \frac{4725}{32768} & \frac{1225}{8192} & \frac{19845}{131072} \end{bmatrix}$$

$$\alpha^{\mathcal{J}}_{\kappa,\ell} = \left[\frac{\gamma_1^2+\gamma_1\gamma_2+2(\gamma_1+\gamma_2+1)}{(\gamma_1+\gamma_2+2)^2}\right]^{\kappa+\ell} \sum_{\substack{0\le p\le\kappa \\ 0\le q\le\ell}} \left[\frac{\sqrt{(\gamma_1+1)(\gamma_2+1)(\gamma_1+\gamma_2+1)}}{\gamma_1^2+\gamma_1\gamma_2+2(\gamma_1+\gamma_2+1)}\right]^{p+q} \binom{\kappa}{p}\binom{\ell}{q}\alpha^{\mathcal{G}}_{p,q},$$

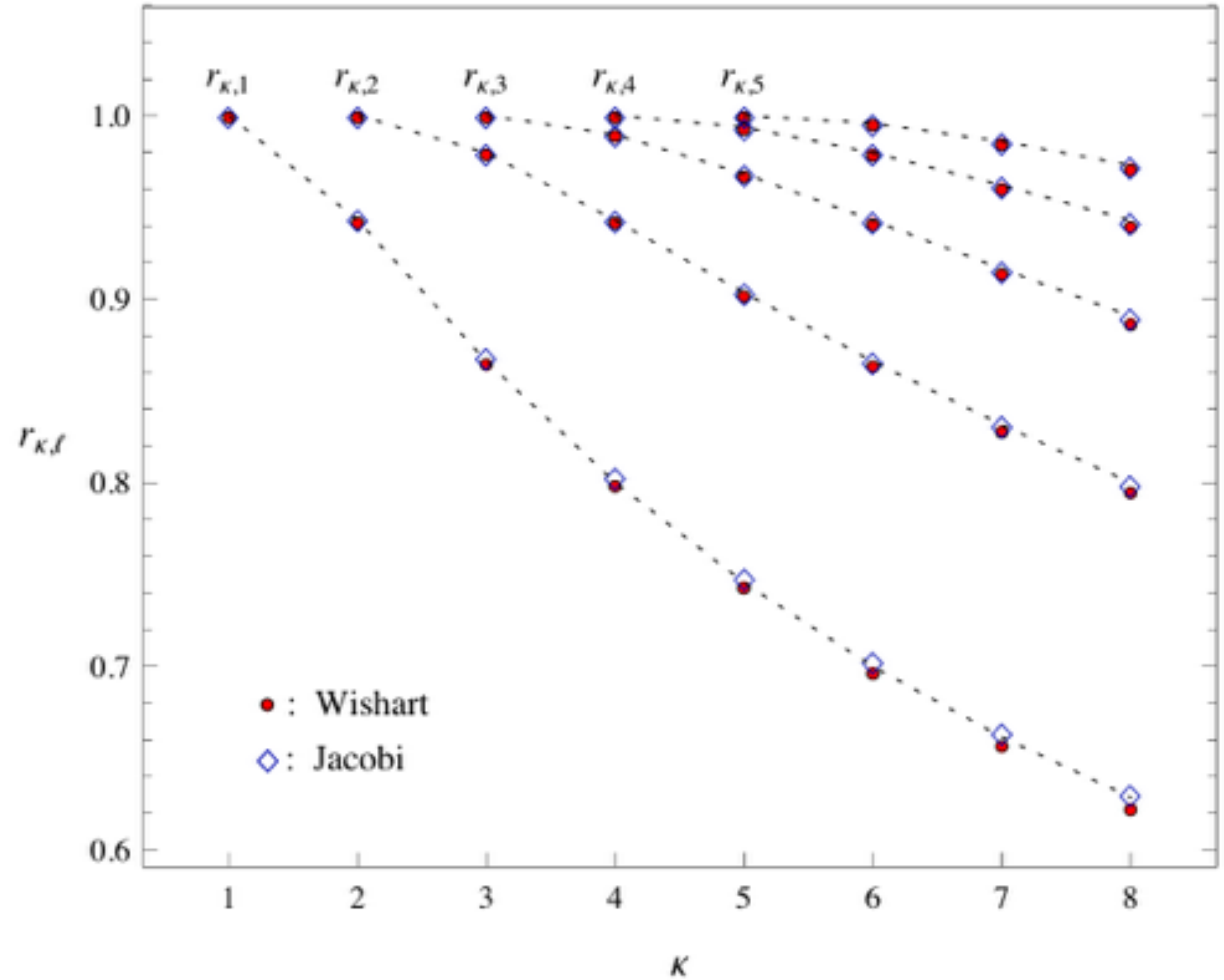

FIG. 2: Limiting correlations $r_{\kappa,\ell} = \alpha_{\kappa,\ell}/\sqrt{\alpha_{\kappa,\kappa}\alpha_{\ell,\ell}}$ of the moments for the Wishart ensemble $\mathcal{W}_N$ with $c = 1$ and the Jacobi ensemble $\mathcal{J}_N$ with $\gamma_1 = \gamma_2 = 0$. The densities of states $\rho_{\mathcal{W}}$ and $\rho_{\mathcal{J}}$ are supported on $[a, b] = [0, 4]$ and $[a, b] = [0, 1]$, respectively. According to Remark III, the correlation coefficients $r_{\kappa,\ell}$ are the same for these ensembles. Here we show a comparison of our prediction (50) with numerical simulations.

- More general setting

2 linear statistics, A and B (differentiable)

$A = \sum_{i=1}^{N} a(\lambda_i)$

$B = \sum_{i=1}^{N} b(\lambda_i)$

on eigenvalues of random matrices distributed as

$$\mathcal{P}_\beta(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}} \mathrm{e}^{-\beta E(\boldsymbol{\lambda})} \,,$$

$$E(\boldsymbol{\lambda}) = -\sum_{i<j} \log|\lambda_i - \lambda_j| + N \sum_i V(\lambda_i) \,.$$

such that the spectral density has single support

Problem

Behaviour of Covariance
of A and B for N>>1

Constant value (dependent on β)...
dependence on potential V ?

Solution

$$\mathrm{Cov}(A,B) \to \frac{1}{\beta\pi^2}\int_0^{+\infty} \mathrm{d}k\,\varphi(k)\mathrm{Re}\left[\tilde{a}(k)\tilde{b}^\star(k)\right] + \mathcal{O}(N^{-1})$$

where

$$\varphi(k) = \begin{cases} k & \text{for } (\lambda_-,\lambda_+) = (-\infty,\infty) \\ k\tanh(\pi k) & \text{otherwise} \end{cases}$$

$$\tilde{f}(k) = \begin{cases} \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx} f(x) & \text{for } (\lambda_-,\lambda_+) = (-\infty,\infty) \\ \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx} f(T_{\lambda_\pm}(\mathrm{e}^x)) & \text{otherwise} \end{cases}$$

... depends only on the **edges** of the support of the spectral density

Usually, interest is for a *single* linear statistics...

- Politzer (1989).
- Basor and Tracy (1993).
- Chen and Manning (1994).
- Costin and Lebowitz (1995).
- Baker and Forrester (1997).
- Pastur and Shcherbina (1997).
- Johansson (1998).
- Soshnikov (2000).
- Wieand (2002).
- Pastur (2006).
- Lytova and Pastur (2009).
- + ...

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 4, NUMBER 5 MAY 1963

Statistical Theory of the Energy Levels of Complex Systems. IV

FREEMAN J. DYSON AND MADAN LAL MEHTA*
Institute for Advanced Study, Princeton, New Jersey
(Received 21 January 1963)

PHYSICAL REVIEW B VOLUME 47, NUMBER 23 15 JUNE 1993-I

Random-matrix theory of mesoscopic fluctuations in conductors and superconductors

C W J Beenakker
Instituut Lorentz, University of Leiden, P O Box 9506, 2300 RA Leiden, The Netherlands
(Received 25 January 1993)

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Nuclear Physics B 422 [FS] (1994) 515–520
NUCLEAR PHYSICS B [FS]

Universality of Brézin and Zee's spectral correlator

C.W.J. Beenakker
Instituut Lorentz, University of Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands
Received 13 October 1993; revised 21 January 1994; accepted 25 January 1994

$$A = \sum_{i=1}^{N} a(\lambda_i)$$

1. Dyson-Mehta (DM) [1]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2}\int_0^\infty \mathrm{d}k\, k|\hat{a}(k)|^2 + \mathcal{O}(N^{-1})$$

$$\hat{a}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{ikx} a(x)$$

2. Beenakker (B1) [2]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2}\int_0^\infty \mathrm{d}k\, k \tanh(\pi k)|\hat{F}(k)|^2 + \mathcal{O}(N^{-1})$$

$$\hat{F}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{ikx} a\left(\frac{1}{1+\mathrm{e}^x}\right)$$

3. Beenakker (B2) [3]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2}\mathrm{Pr}\iint_{\lambda_-}^{\lambda_+} \mathrm{d}\lambda\mathrm{d}\mu\, \Theta(\lambda,\mu)\frac{a(\lambda)}{\lambda-\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}a(\mu) + \mathcal{O}(N^{-1})$$

$$\Theta(\lambda,\mu) = \sqrt{\frac{(\mu-\lambda_-)(\lambda_+-\mu)}{(\lambda-\lambda_-)(\lambda_+-\lambda)}}$$

Goals

- Generalize these formulae to the covariance case (2 linear statistics)
- Clarify when and where the classical variance formulae should/can be used

Ingredients:

(i) Conformal change of variables

ii) Definition of covariance in terms of 2-point kernel

iii) Functional derivative identity + electrostatic equation

iv) Universality of the smoothed kernel on $(0, \infty)$

(i)

$$\mathcal{P}_\beta(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}} \mathrm{e}^{-\beta\left[-\sum_{i<j} \ln|\lambda_i - \lambda_j| + N\sum_i V(\lambda_i)\right]} \equiv \frac{\mathrm{e}^{-\beta E(\boldsymbol{\lambda})}}{\mathcal{Z}} \, .$$

$$\mathrm{Cov}(A,B) = \int_{\Lambda^N} \mathrm{d}\boldsymbol{\lambda} \mathcal{P}_\beta(\boldsymbol{\lambda}) \left(A(\boldsymbol{\lambda}) - \langle A \rangle\right) \left(B(\boldsymbol{\lambda}) - \langle B \rangle\right) \, .$$

Let us consider the conformal map

$$\boldsymbol{\lambda} = T\boldsymbol{x} \quad : \quad \lambda_i = \frac{ax_i + b}{cx_i + d} \tag{20}$$

with $ad - bc \neq 0$. Then

$$\tilde{\mathcal{P}}_\beta(\boldsymbol{x}) = \frac{1}{\tilde{\mathcal{Z}}} \mathrm{e}^{-\beta \tilde{E}(\boldsymbol{x})} \, , \tag{21}$$

$$\tilde{E}(\boldsymbol{x}) = -\sum_{i<j} \log|x_i - x_j| + N\sum_i \tilde{V}(x_i) + \mathcal{O}(N)$$

$$\mathrm{Cov}(A,B) = \int_{\tilde{\Lambda}^N} \mathrm{d}\boldsymbol{x} \frac{1}{\tilde{\mathcal{Z}}} \mathrm{e}^{-\beta \tilde{E}(\boldsymbol{x})} A(T(\boldsymbol{x}))B(T(\boldsymbol{x})) - \langle A \rangle \langle B \rangle$$

$$\tilde{\rho}_N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i) \qquad \text{(random measure)}$$

$$\tilde{\rho}(x) = \lim_{N\to\infty} \langle \tilde{\rho}_N(x) \rangle \qquad \textbf{(single support)}$$

$$\tilde{\mathcal{K}}_N(x,x') = -\langle \tilde{\rho}_N(x)\tilde{\rho}_N(x') \rangle + \langle \tilde{\rho}_N(x) \rangle \langle \tilde{\rho}_N(x') \rangle$$

$$\tilde{\mathcal{K}}(x,x') = \lim_{N\to\infty} N^2 \tilde{\mathcal{K}}_N(x,x')$$

(ii)

$$\mathrm{Cov}(A,B) = -N^2 \iint \mathrm{d}x\mathrm{d}x' \tilde{\mathcal{K}}_N(x,x') a(T(x)) b(T(x'))$$

$$\to - \iint_\sigma \mathrm{d}x\mathrm{d}x' \tilde{\mathcal{K}}(x,x') a(T(x)) b(T(x')) + \mathcal{O}(1/N)$$

We can choose a,b,c,d in a convenient way...

The support σ in the large-N limit is moved to (0,∞)

$$\begin{aligned}\mathrm{Cov}(A,B) &= -N^2 \iint \mathrm{d}x\mathrm{d}x' \mathcal{K}_N(x,x') a(T(x)) b(T(x')) \\ &\to -\iint_\sigma \mathrm{d}x\mathrm{d}x' \mathcal{K}(x,x') a(T(x)) b(T(x')) + \mathcal{O}(1/N)\end{aligned}$$

What is the kernel in this case?

iii) Functional derivative identity...

$$\tilde{\mathcal{K}}_N(x,x') = \frac{1}{\beta N^2}\frac{\delta\langle\tilde{\rho}_N(x)\rangle}{\delta\tilde{V}(x')}\left(1+\mathcal{O}(N^{-1})\right)$$

[Beenakker 1993]

$$\frac{\delta\langle\tilde{\rho}_N(x)\rangle}{\delta\tilde{V}(x')} = \frac{\delta}{\delta\tilde{V}(x')}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x)\mathcal{P}_\beta(\boldsymbol{x})$$

$$= \frac{\delta}{\delta\tilde{V}(x')}\frac{1}{\tilde{\mathcal{Z}}}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x)\mathrm{e}^{-\beta\tilde{E}(\boldsymbol{x})}$$

$$= \frac{-\beta}{\tilde{\mathcal{Z}}}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x)\mathrm{e}^{-\beta\tilde{E}(\boldsymbol{x})}\frac{\delta\tilde{E}(\boldsymbol{x})}{\delta\tilde{V}(x')} +$$

$$+\frac{\beta}{\tilde{\mathcal{Z}}^2}\iint \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}''\,\tilde{\rho}_N(x)\mathrm{e}^{-\beta(\tilde{E}(\boldsymbol{x})+\tilde{E}(\boldsymbol{x}''))}\frac{\delta\tilde{E}(\boldsymbol{x}'')}{\delta\tilde{V}(x')}$$

$$= -\beta N^2\left[\frac{1}{\tilde{\mathcal{Z}}}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x)\tilde{\rho}_N(x')\mathrm{e}^{-\beta\tilde{E}(\boldsymbol{x})} - \right.$$

$$\left.\left(\frac{1}{\tilde{\mathcal{Z}}}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x)\mathrm{e}^{-\beta\tilde{E}(\boldsymbol{x})}\right)\left(\frac{1}{\tilde{\mathcal{Z}}}\int \mathrm{d}\boldsymbol{x}\,\tilde{\rho}_N(x')\mathrm{e}^{-\beta\tilde{E}(\boldsymbol{x})}\right)\right] + \mathcal{O}(N)\,.$$

... + electrostatic equation

for $\tilde{\sigma} = (0, +\infty)$ support

$$\int_0^\infty \mathrm{d}y \tilde{\rho}(y) \ln|x-y| = \tilde{V}(x), \quad x > 0$$

$$\tilde{\rho}(x) = \int_0^\infty \mathrm{d}y \, \Phi(x,y)\tilde{V}(y)$$

$$\tilde{\mathcal{K}}(x,x') = \frac{1}{\beta}\frac{\delta\tilde{\rho}(x)}{\delta\tilde{V}(x')} = \Phi(x,x')$$

iv) [Beenakker 1993]

$$\Phi(x,x') = \partial_x \partial_{x'} \ln\left|\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{x}-\sqrt{x'}}\right|$$

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Multiloop correlators for two-dimensional quantum gravity

J. Ambjørn and J. Jurkiewicz [1]
Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

and

Yu.M. Makeenko
Institute for Theoretical and Experimental Physics, SU-117 259 Moscow, USSR

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Universality of the correlations between eigenvalues of large random matrices

E. Brézin [a] and A. Zee [a,b]

[a] *Laboratoire de Physique Théorique* [1]*, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France*
[b] *Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

Summary of the main steps

- Covariance: double integral over finite-N kernel
- Conformal transformation maps the original support to $(0,\infty)$ [unless it was $(-\infty,\infty)$]
- Limit of the finite-N kernel is universal upon smoothing (it depends only on the edge points of the support of average density)
- Inserting the smoothed kernel for a support $(0,\infty)$ inside the double integral gives the result (after simple algebra in Fourier space)

$$\mathrm{Cov}(A,B) \to \frac{1}{\beta\pi^2}\int_0^{+\infty} \mathrm{d}k\,\varphi(k)\mathrm{Re}\left[\tilde{a}(k)\tilde{b}^\star(k)\right] + \mathcal{O}(N^{-1})$$

$$\tilde{f}(k) = \begin{cases} \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx} f(x) & \text{for } (\lambda_-,\lambda_+) = (-\infty,\infty) \\ \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx} f(T_{\lambda_\pm}(\mathrm{e}^x)) & \text{otherwise} \end{cases}$$

- Invariant under the exchange A $\longrightarrow$ B
- It can be zero (decorrelation to leading order)

Recovering classical variance formulas (1)

1. Dyson-Mehta (DM) [1]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2}\int_0^\infty \mathrm{d}k\, k|\hat{a}(k)|^2 + \mathcal{O}(N^{-1})$$

$$\hat{a}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} a(x)$$

Recovered when A=B for a matrix model whose spectral density has support over the full real axis [example: Cauchy ensemble]

Recovering classical variance formulas (II)

2. Beenakker (B1) [2]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2}\int_0^\infty \mathrm{d}k\, k\tanh(\pi k)|\hat{F}(k)|^2 + \mathcal{O}(N^{-1})$$

$$\hat{F}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} a\left(\frac{1}{1+\mathrm{e}^x}\right)$$

Recovered when A=B for a matrix model whose spectral density has support over [0,1]
[example: Jacobi ensemble]

Recovering classical variance formulas (III)

3. Beenakker (B2) [3]

$$\mathrm{Var}(A) = \frac{1}{\beta\pi^2} \mathrm{Pr} \iint_{\lambda_-}^{\lambda_+} \mathrm{d}\lambda \mathrm{d}\mu \, \Theta(\lambda, \mu) \frac{a(\lambda)}{\lambda - \mu} \frac{\mathrm{d}}{\mathrm{d}\mu} a(\mu) + \mathcal{O}(N^{-1})$$

$$\Theta(\lambda, \mu) = \sqrt{\frac{(\mu - \lambda_-)(\lambda_+ - \mu)}{(\lambda - \lambda_-)(\lambda_+ - \lambda)}}$$

Recovered when A=B for a matrix model whose spectral density has finite support

Conclusions

- Covariance formula for 2 (differentiable) linear statistics: it depends only on the edge points of the support of the average density
- Applicable to "Coulomb-gas" ensembles whose level density is supported on a single interval
- Conformal transformation maps the support to $(0,\infty)$
- Smoothed 2-point kernel for $(0,\infty)$ is known and completely universal

$$\mathrm{Cov}(A,B) \to \frac{1}{\beta\pi^2}\int_0^{+\infty} \mathrm{d}k\,\varphi(k)\mathrm{Re}\left[\tilde{a}(k)\tilde{b}^{\star}(k)\right] + \mathcal{O}(N^{-1})$$

Outlook

- Extensions to multi-cut ensembles?
- ...to non-invariant ensembles?
- ...to non-Hermitian ensembles?
- ...to biorthogonal ensembles?
- Non-differentiable linear statistics
- Central Limit Theorem for *joint* linear statistics?