Diffusion of characteristic polynomials and edge universality in non-hermitian random matrix models

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[Burda, Grela, MAN, Tarnowski and Warchoł, Phys. Rev. Lett. 113 (2014) 104102;
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Blaizot, Grela, MAN,Tarnowski and Warchoł - *On Ornstein-Uhlenbeck diffusion of hermitian and non-hermitian matrices-unexpected links*, math-th/1512.06599, to be published in JSTAT, 2016]
Yizhuang Liu and Ismail Zahed (Stony Brook)
[Liu, MAN, Zahed; hep-lat/1602.02578]

Outline

- Hermitian overture: diffusion of hermitian matrices - "Dysonian way" vs "Burgulent way"
- Unraveling the diffusion of nonhermitian matrices - Ginibre ensemble
- Non-trivial example - application to finite density QCD

"Dysonian way" ([Dyson; 1962])

- *After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler......*
$d\lambda_i(\tau) = \frac{1}{\sqrt{N}} dB_i(\tau) + \frac{1}{N} \sum_{i \neq j}^{N} \frac{1}{\lambda_i - \lambda_j} d\tau - a\lambda_i d\tau$
- *The word "time" in this paper will always refer to a fictitious time which is a property of mathematical model.....*
In our case, "time" may be real time, area of the string, temperature, length of the mesoscopic wire etc.
- *This term* [Coulombic] *is mainly sensitive to the local (microscopic) configurations of the gas particles... at the microscopic scale...After local equilibrium is established..the gas must adjust itself by a macroscopic motion on the time scale* [which is N times larger comparing to the microscopic one]*... "a rigorous proof that this picture is accurate would require a much deeper mathematical analysis...*
We give support to this picture.

Diffusion of N by N hermitian matrices $H = H^\dagger$

- Gaussian Unitary Ensemble (GUE)

$$H_{ij} = \begin{cases} x_{ii} & \text{if } i = j \\ \frac{x_{ij}+iy_{ij}}{\sqrt{2}} & \text{if } i < j \end{cases}$$

where all x_{ij}, y_{ij} drawn from standard Gaussians, so

- $< dH_{ij} >= -aH_{ij}d\tau, \quad < (dH_{ij})^2 >= \frac{1}{N}d\tau$
- Probability distribution $\partial_\tau P(H,\tau) = L_{OU}P(H,\tau)$, with $P(H,\tau) = Cexp\left(-\frac{Na}{1-e^{-2a\tau}}\mathrm{tr}(H - H_0 e^{-a\tau})^2\right)$
- $< F(H) >_\tau = \int [dH]P(H,\tau)F(H)$

"Burgulent way"

- We define $D_N(z,\tau) =< \det(z\mathbf{1}_N - H) >_\tau$
- Integrable, exact eq. (for any N and for any initial conditions)
  $\partial_\tau D_N(z,\tau) = -\frac{1}{2N}\partial_{zz} D_N(z,\tau) + az\partial_z D_N(z,\tau) - aND_N(z,\tau)$
  [Blaizot,MAN,Warchoł; 2008-2013]
- Inverse Cole-Hopf transform $f_N = \frac{1}{N}\partial_z \ln D_N$
- Burgers equation $\partial_\tau f_N + f_N \partial_z f_N - a\partial_z(zf_N) = \nu_s \partial_{zz} f_N$
- Spectral viscosity $\nu_s = -\frac{1}{2N}$

Burgers equation trivia

- Navier-Stokes eq. in $d = 1$ without pressure term.
- Toy model for turbulence ($f(x,t)$ - height of the wave at position x and time t)
  $\partial_t f + f\partial_x f = \nu \partial_{xx} f$, where ν is a viscosity
  [Burgers; 1939]
- Exactly solvable by [Hopf-Cole;1950-51] transformation
  $f = -2\nu\partial_x \ln d$, so $\partial_t d = \nu\partial_{xx} d$
- Inviscid limit ($\nu \to 0$): Euler equation $\partial_t f + f\partial_x f = 0$, solvable by the method of characteristics with implicit solution: $f = f_0(x - tf)$, where $f_0 = f(x, 0)$.
- Inviscid equation develops singularities (shocks) at $t^\star = 1/f_0'$.

Naive large N, i.e. inviscid limit

- Green's function $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z\mathbf{1}_N - H}\right\rangle_\tau = \frac{1}{N}\left\langle \sum_{k=1}^N \frac{1}{z-\lambda_k}\right\rangle$
- $G(z,\tau) = \lim_{N\to\infty} f_N = \lim_{N\to\infty} \frac{1}{N}\partial_z \ln < \det(z-H) >$
  $= \lim_{N\to\infty} \frac{1}{N}\partial_z < \mathrm{Tr} \ln(z-H) >$
- inviscid complex Burgers equation
  $\partial_\tau G + G\partial_z G - a\partial_z(zG) = 0$
- Stationary limit $\tau \to \infty$ yields $\partial_z(\frac{G^2}{2} - azG) = 0$
- Spectrum from Sochocki Plemelj eq.
  $\frac{1}{\lambda-\lambda' \pm i\epsilon} = \mathrm{P.V.}\frac{1}{\lambda-\lambda'} \mp i\pi\delta(\lambda-\lambda')$
- $\rho(\lambda) = \frac{a}{\pi}\sqrt{2/a - \lambda^2}$
- Warning: Shock phenomena at the edges of the spectrum

Microscopic limit

- Burgers equation is exactly integrable
- Lamperti Transformation [Lamperti; 1962]
  $D(z,\tau) = (1+2a\tau')^{-N/2} D'(z',\tau')$
  $z' = e^{a\tau} z \quad \tau' = \frac{1}{2}(e^{2a\tau}-1)$
- Diffusion equation in "primed" variables $\partial_{\tau'} D' = -\frac{1}{2N} \partial_{z'z'} D'$
- Similar behavior (modulo the sign) for the inverse characteristic determinant.
- Solution at the vicinity of the shock leads to either Airy-like oscillations (trivial initial conditions) or Pearcey-like oscillations (non-trivial initial conditions)

Generic picture: Surfing the shock waves in QCD

- Tracing the singularities of the flow allows to understand the pattern of the evolution of the complex system without explicit solutions of the complicated hydrodynamic equations...
- Zooming at singularities allows to infer the universal scaling (critical) exponents, since viscous equations are exact for arbitrary number of colors.
- Example 1. [Durhuus-Olesen; 1981] transition for the Wilson loop spectra in large N Yang-Mills [Narayanan, Neuberger; 2008], [Blaizot, MAN; 2008]
- Example 2. Critical chiral transitions for the Dirac operator (e.g. Bessoid class) [Janik, MAN, Papp, Zahed; 1997], [Blaizot, MAN, Warchoł; 2013]

"Great chiral waves at Kanazawa" *collage based on Hokusai woodcut*

Do we have similar phenomena in nonhermitian RMM?

- [Ginibre ensemble, 1964] - academic exercise
- Random walk for $\beta = 2$ - [Osada; 2012] ???
- Random walk for $\beta = 1$ - [Mihail Poplavskyi, Roger Tribe, Oleg Zaboronski, 2012-2015]

Why to bother about nonhermitian operators...

Nonhermitian operators 52 years later...

- Nonhermitian quantum mechanics (resonances, complex potentials,...)
- Statistics (lagged correlators) $C_{i,j}(\Delta) = \frac{1}{T}\sum_{t=1}^{T} X_{i,t}X_{j,t+\Delta}$
- Complexity (directed graphs/networks, non-backtracking (Hashimoto) operators for sparse systems)
- "Pathological" Euclidean Dirac operators
- ....

Non-hermitian case - large N - electrostatic analogy

Analytic methods break down, since spectra are complex $\rho(z) = \frac{1}{N} \left\langle \sum_i \delta^{(2)}(z - \lambda_i) \right\rangle$.

- Electrostatic potential [Girko;1984],[Brown;1986],[Sommers et al.;1988]
  $\phi(z, \bar{z}) \equiv \lim_{\epsilon \to 0} \lim_{N \to \infty} \left\langle \frac{1}{N} \mathrm{tr} \ln[|z - X|^2 + \epsilon^2] \right\rangle$
- Green's function (electric field)
  $g = \partial_z \phi = \lim_{\epsilon \to 0} \lim_{N \to \infty} \left\langle \frac{1}{N} \mathrm{tr} \frac{\bar{z} - X^\dagger}{|z - X|^2 + \epsilon^2} \right\rangle$
- Gauss law $\rho(z, \tau) = \frac{1}{\pi} \partial_{\bar{z}} g|_{\epsilon=0} = \frac{1}{\pi} \frac{\partial^2 \phi}{\partial z \partial \bar{z}}|_{\epsilon=0}$
  Proof: $\delta^{(2)}(z) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2}$

"Linearization trick" [Janik, MAN, Papp, Zahed; 1997]

- $\phi(z,\bar{z}) \equiv \lim_{\epsilon\to 0} \lim_{N\to\infty} \left\langle \frac{1}{N} \mathrm{tr} \ln[|z-X|^2+\epsilon^2] \right\rangle$
  $= \lim_{\epsilon\to 0} \lim_{N\to\infty} \left\langle \frac{1}{N} \ln D_N \right\rangle$ where
  $D_N(z,\bar{z},\epsilon) = \det(Z \otimes \mathbf{1}_N - \mathcal{X})$ with
  $$Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$
- $\mathcal{G}(z,\bar{z}) = \frac{1}{N} \left\langle \mathrm{btr} \frac{1}{(Z-\mathcal{X})} \right\rangle = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix}$
  $$\mathrm{btr} \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2N\times 2N} \equiv \begin{pmatrix} \mathrm{tr} A & \mathrm{tr} B \\ \mathrm{tr} C & \mathrm{tr} D \end{pmatrix}_{2\times 2}$$
- $\mathcal{G}_{11} = g(z,\bar{z})$ yields spectral function
- $\mathcal{G}_{1\bar{1}} \cdot \mathcal{G}_{\bar{1}1}$ yields elements of a certain eigenvector correlator [Savin,Sokolov; 1997],[Chalker,Mehlig;1998].

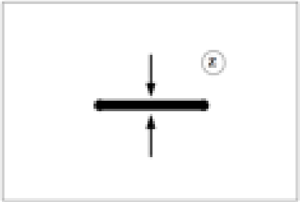
Loophole in the standard arguments

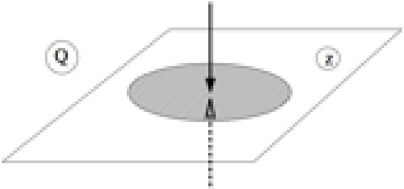
- For non-hermitian matrices X, we have left and right eigenvectors $X = \sum_k \lambda_k |R_k >< L_k|$ where $X|R_k >= \lambda_k |R_k >$ and $< L_k|X = \lambda_k < L_k|$
- $< L_j|R_k >= \delta_{jk}$, but $< L_i|L_j > \neq 0$ and $< R_i|R_j > \neq 0$.
- $D_N = \det(Z - \mathcal{X}) = \det[U^{-1}(Z - \mathcal{X})U] =$ $\det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -i\epsilon < L|L > \\ i\epsilon < R|R > & \bar{z}\mathbf{1}_N - \bar{\Lambda} \end{pmatrix}$
- Spectrum (Λ) entangled with diagonal part of the overlap of eigenvectors $O_{ij} \equiv < L_i|L_j >< R_j|R_i >$.
- Naive limit $\epsilon \to 0$ kills the entanglement leading to incomplete description of the non-hermitian RM

Cure: Hidden variable

We promote $i\epsilon$ to full, complex-valued dynamical variable.
Then, "orthogonal direction" w unravels the eigenvector correlator $O(z,t) = \frac{1}{N^2}\left\langle \sum_k O_{kk}\delta^{(2)}(z-\lambda_k(t))\right\rangle$, where $O_{ij} =< L_i|L_j >< R_j|R_i >$ and $|L_i>$ ($|R_i>$) are left (right) eigenvectors of X. [Janik, MAN, Noerenberg, Papp, Zahed; 1999]

Replacing $Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix}$ by **quaternion** $Q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$ provides algebraic generalization of free random variables calculus for nonhermitian RMM. [Janik, MAN, Papp, Zahed, 1997], [Feinberg, Zee; 1997],[Jarosz, MAN;2006], [Belinschi, Sniady, Speicher; 2015].





Approach to nonhermitian variables

- We replace $D_N(z,\tau) =< \det(z\mathbf{1}_N - H) >_\tau$ by the determinant $\mathcal{D}_N(z,\bar{z},w,\bar{w},\tau) =< \mathrm{det}(Q \otimes \mathbf{1}_N - \mathcal{X}) >_\tau$
- Using the evolution equation, we arrive at exactly integrable equation for any N and any initial conditions $\partial_\tau \mathcal{D}_N = \frac{1}{N}\partial_{w\bar{w}}\mathcal{D}_N - 2Na\mathcal{D}_N + ad\mathcal{D}_N$, where operator $d = z\partial_z + w\partial_w + \bar{z}\partial_{\bar{z}} + \bar{w}\partial_{\bar{w}}$
- Switching to "primed" variables (Lamperti transformation) removes the O-U drift, yielding exact 2d diffusion equation $\partial_{\tau'}\mathcal{D}'_N(z',w',\tau') = \frac{1}{N}\partial_{w'\bar{w}'}\mathcal{D}'_N(z',w',\tau')$

"Burgulent way" - nonhermitian case, $N = \infty$ limit

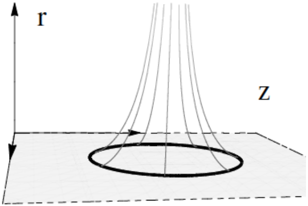
- The hermitian-case Burgers equation $\partial_{\tau'} g' + g' \partial_{z'} g' = 0$ is now superimposed by the system (two Cole-Hopf transforms)

$$\begin{aligned} \partial_{\tau'} v' &= v' \partial_{r'} v' \\ \partial_{\tau'} g' &= \partial_{z'} v'^2 \end{aligned}$$

  where v'^2 controls eigenvectors, g' controls the complex spectrum and $|w'| = r'$
- Evolution of overlaps (v') **prior** to the evolution of spectra
- Shock phenomenon in eigenvector sector
- "Missed " complex plane (w') is relevant - quaternion (Q') description.

Historical example

- For trivial initial conditions $X_0 = 0$
  $O(z) = \frac{1}{\pi}(1-|z|^2)\Theta(1-|z|)$
  [Chalker-Mehlig;1998],[Janik et al.;1998]
- $\rho(z) = \frac{1}{\pi}\Theta(1-|z|)$
  [Ginibre; 1964]
- Despite the fact, that in the large N limit, overlap of eigenvectors is **prior** to eigenvalues, this correlator was calculated 34 years after the spectral density calculation (*sic!*).
- "Duality" helps!



Microscopic universality

- $\mathcal{D}(z,r,\tau) = \frac{2N}{\tau}\int_0^\infty r' exp(-N\frac{r^2+r'^2}{\tau}) I_0(\frac{2Nrr'}{\tau})\mathcal{D}_0(z,r')dr'$
  where $\mathcal{D}_0(z,r') = (|z|^2 + r'^2)^N$
- Three saddle points $r'_0 = 0, r'_\pm = \pm\sqrt{\tau - |z|^2}$
- Unfolding $r' = \theta N^{-1/4}$, $|z| = \sqrt{\tau} + \eta N^{-1/2}$
- $\lim_{N\to\infty} \mathcal{D}(z = \sqrt{\tau} + \eta N^{-1/2}, r = 0, \tau) \sim \frac{1}{2\pi\tau}\mathrm{erfc}(\sqrt{2/\tau}\eta)$

Unexpected links

	GUE	GE
Spectrum	real	complex
Green's f.	complex-valued $G(z) = \frac{1}{N} \left\langle \mathrm{Tr}(z-H)^{-1} \right\rangle$	quaternion-valued $\mathcal{G}(Q) = \frac{1}{N} \left\langle \mathrm{bTr}(Q-\mathcal{X})^{-1} \right\rangle$
Det	$D(z,\tau) =< \det(z-H) >$	$\mathcal{D}(Q,\tau) =< \det(Q-\mathcal{X}) >$
Diffusion eq.	$\partial_\tau D = -\frac{1}{2N}\partial_{zz} D$	$\partial_\tau \mathcal{D} = +\frac{1}{N}\partial_{w\bar{w}}\mathcal{D}$
Viscosity	negative	positive
Universality	oscillatory (Airy)	smooth (Erfc)
R-transform	$R_{GUE}(G) = G$	$\mathcal{R}_{GG}(\mathcal{G}) = \begin{pmatrix} 0 & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & 0 \end{pmatrix}$
Voiculescu eq.	$\frac{\partial G}{\partial \tau} + R(G)\frac{\partial G}{\partial z} = 0$	$\frac{\partial \mathcal{G}_{ab}}{\partial \tau} + \sum_{c,d=1}^{2} \mathcal{R}[\mathcal{G}]_{cd} \frac{\partial \mathcal{G}_{ab}}{\partial Q_{cd}} = 0$
Shocks	eigenvalues	eigenvectors

Non-trivial example: Euclidean QCD at finite density

$Z_{N_f}(\tau, z = -im_f, \mu) =$

$\int dT dT^\dagger P(\tau, T) \det \begin{pmatrix} z & T - i\mu \\ T^\dagger - i\mu & z \end{pmatrix}^{N_f}$ where

$P(\tau, T) = e^{-\frac{N}{\tau}\mathrm{Tr}\, T^\dagger T}$

- For $\mu = 0$: Universality in the bulk (sine kernel) [MAN,Verbaarschot,Zahed;1989]
- For $\mu = 0$: Universality at the chiral point (Bessel kernel) [Shuryak,Verbaarschot,Zahed;1993]
- For $\mu \neq 0$: [Stephanov;1994], suggested the solution of the "mystery of the baryonic pion".

"Deformed Wishart model" [Liu,MAN,Zahed;2016]

- $\mathbf{Z}_{N_f}(\mathbf{z}, w, \mu) = \left\langle (\det(|\mathbf{z} - W|^2 + w\bar{w})^{N_f/4} \right\rangle$, where
  $W = T^\dagger T - i\mu(T + T^\dagger)$, $\mathbf{z} = z^2 + \mu^2$
- $\mathbf{Z}_{N_f}(\mathbf{z}, w, \mu) \equiv \left\langle e^{F+G} \right\rangle$ where
  $F = q^\dagger(\mathbf{z} - W)q + Q^\dagger(\bar{\mathbf{z}} - W^\dagger)Q$ and $G = \bar{w}q^\dagger Q + wQ^\dagger q$
- $\mathbf{z}, \bar{\mathbf{z}}$ act as complex masses for $q^\dagger q, Q^\dagger Q$, whereas $w, \bar{w}$ act as mixing masses for $q^\dagger Q$, $Q^\dagger q$.
- For $N_f = 4$, we managed to write exact for finite N "diffusive-like" evolution equation.

"Deformed Wishart model" [Liu,MAN,Zahed;2016] cont.

- WKB analysis reproduces (via known conformal mapping) expanding droplet boundary in agreement with the Stephanov result
- Model reproduces known Airy universalities for $\mu = 0$ and chiral universalities (after nontrivial unfolding of the type $\mu^2 N$ fixed for large N) of the 1-matrix model Osborn, Splittorff,Verbaarschot [2006-2008] and 2-matrix model of Akemann and Osborn [2003-2007]
- Model yields novel microscopic edge profile of the erfc type, depending on the value of the quark condensate, providing *a priori* a way of extracting the physical condensate from current and quenched Dirac lattice data, without having to solve the sign problem
- More work to be done to include temperature, realistic masses, numerical identification of zero mode zone... (work in progress).

Conclusions and open problems

- Formalism of Dysonian dynamics for non-hermitian RMM ($\beta = 2$), involving **coevolution of eigenvalues and eigenvectors**, based on **hidden variable**
- Unexpected **similarity** between hermitian and non-hermitian RMM based on **"Burgulence"** concepts
- Verification in various applications of hermitian and non-hermitian random matrix models
- Unexplored mathematics