On global fluctuations of non-colliding processes

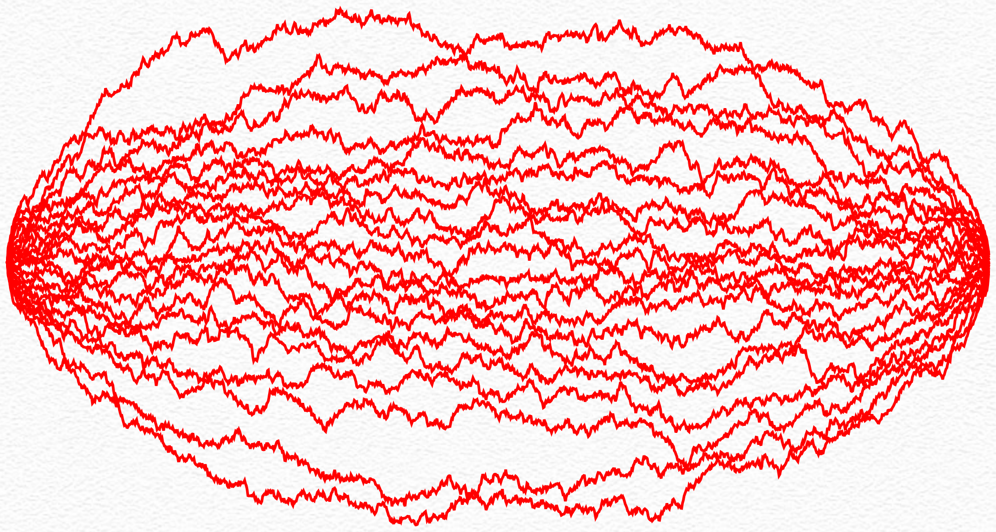
Maurice Duits

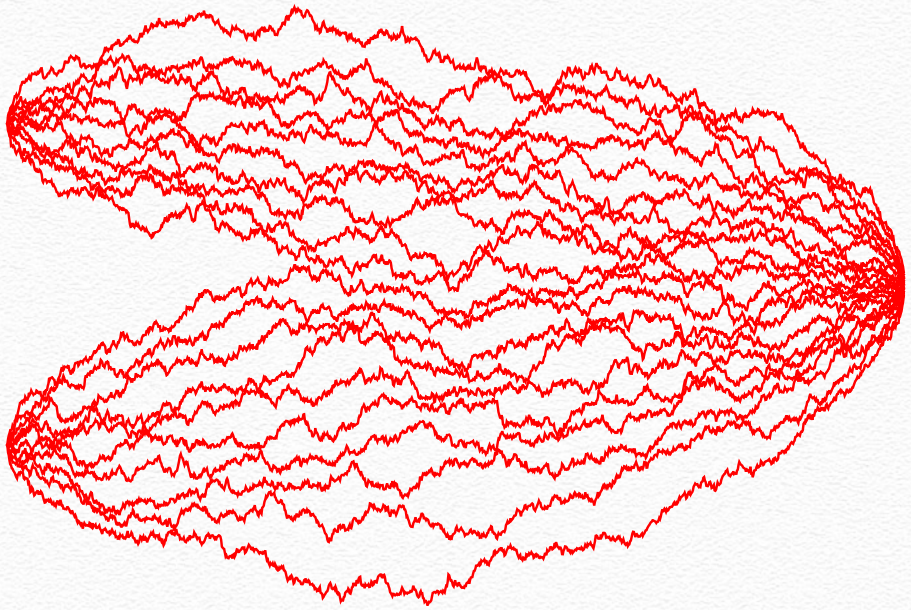
March 21, Warwick

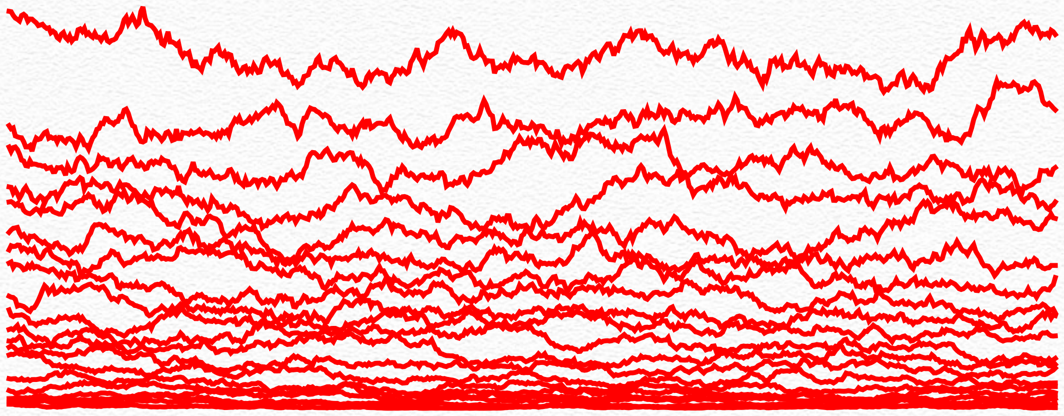
KTH VETENSKAP OCH KONST
ROYAL INSTITUTE OF TECHNOLOGY

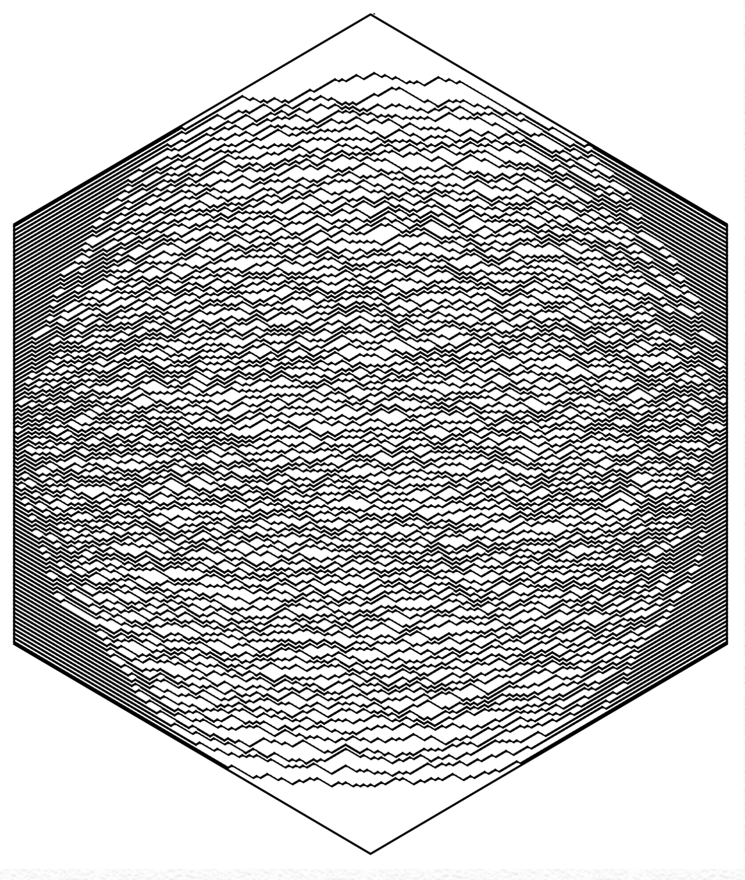
1.Motivation

Some examples of non-colliding processes or non-intersecting paths in the class of determinantal point processes









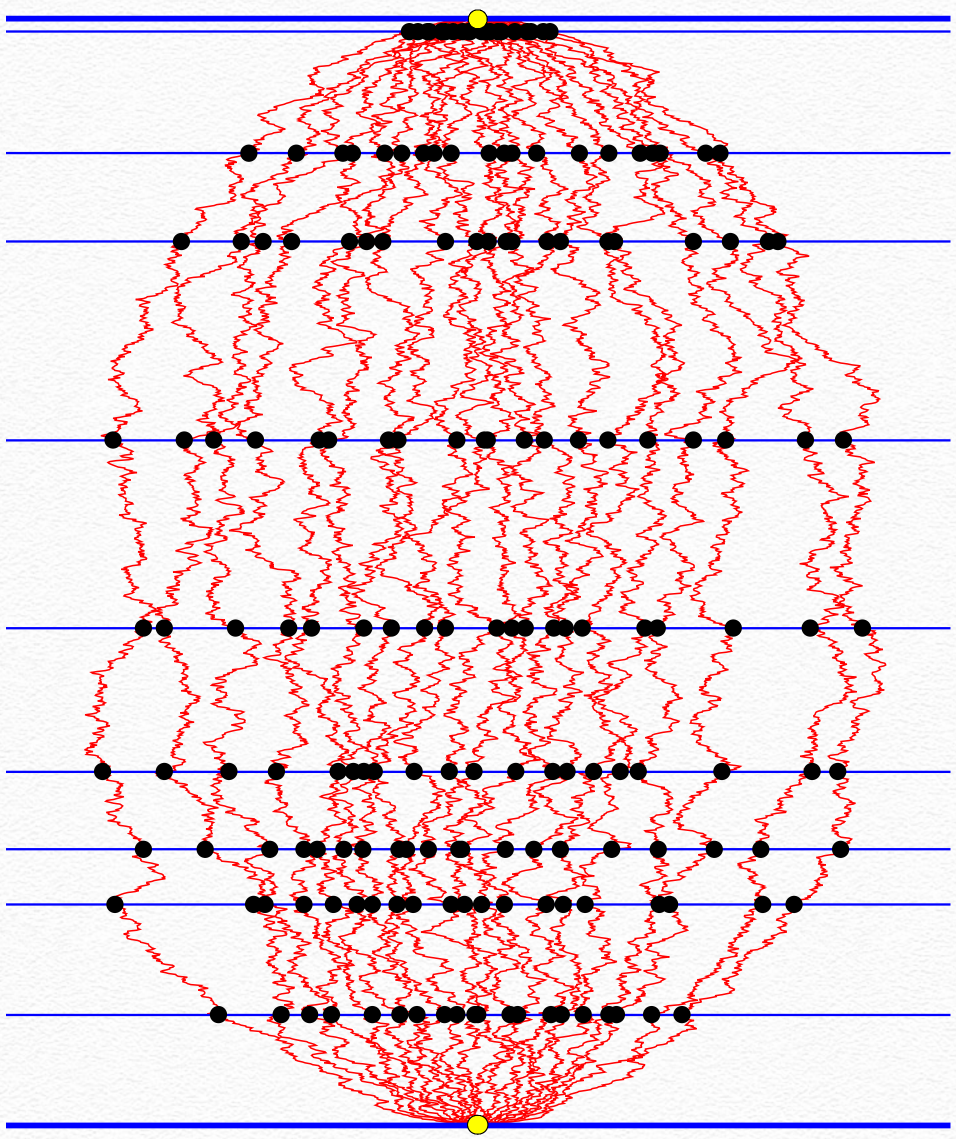
Global fluctuations via multi-time linear statistics

- Choose a number of horizontal sections τ_m for $m = 1, \ldots, N$ and consider the linear statistic

$$X_n(f) = \sum_{m=1}^{N} \sum_{j=1}^{n} f(\tau_m, x_j(\tau_m))$$

- In this talk we will be interested in the behavior of the fluctuations of $X_n(f)$ as $n \to \infty$

- We will discuss N=1 separately.



2. Some example of results for linear statistics

Linear statistics and the GUE

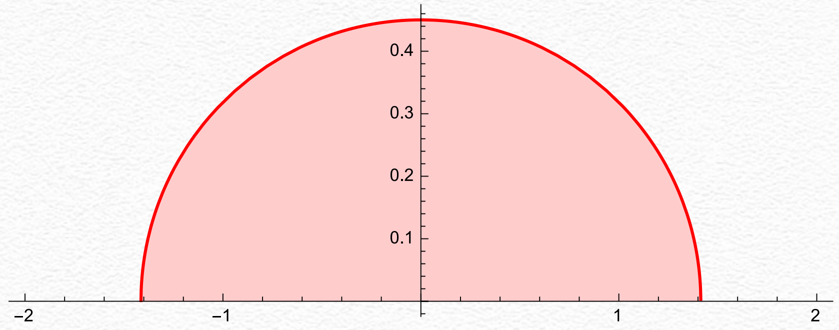
- For a smooth function f consider the linear statistic

$$X_n(f) = \sum_{j=1}^{n} f(x_j)$$

where, for now, the points $x_1, \ldots, x_n$ are the eigenvalues of a (properly scaled) GUE matrix.

- Then, as $n \to \infty$, the global distribution is given by

$$\frac{1}{n}\mathbb{E}X_n(f) \to \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} f(x)\sqrt{2-x^2}\mathrm{d}x$$



- We will be mostly interested in the limiting behavior of the fluctuations

$$X_n(f) - \mathbb{E}X_n(f)$$

Linear statistics and the GUE

- **Central Limit Theorem**:
  For the fluctuations we have as $n \to \infty$, for sufficiently smooth functions f,

  $$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2)$$

  where

  $$\sigma_f^2 = \sum_{k=1}^{\infty} k|f_k|^2$$

  and

  $$f_k = \frac{1}{\pi}\int_0^{\pi} f(\sqrt{2}\cos\theta)\cos k\theta \mathrm{d}\theta$$

  Johansson '98,….

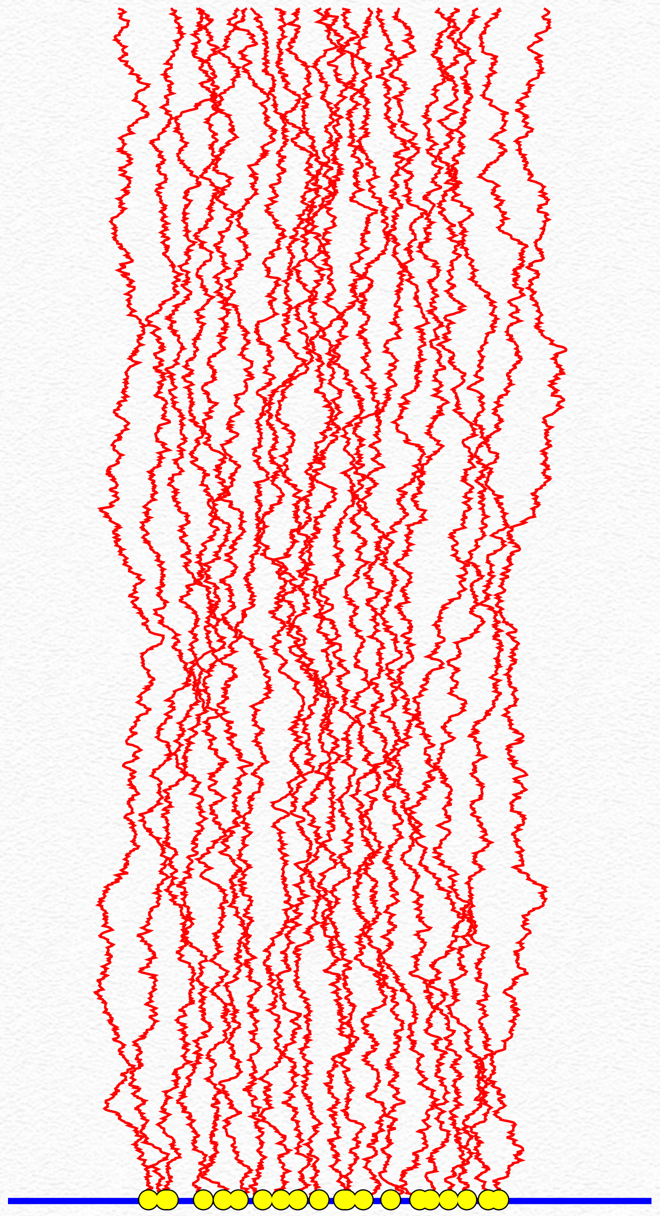
- No normalization! Only for sufficiently smooth functions. One dimensional section of a 2d Gaussian Free Field.

Linear statistics and the GUE

- Dyson (1962). Consider a Hermitian matrix for which the entries (up to symmetry) evolve according to independent Ornstein-Uhlenbeck processes. Then the ordered eigenvalues form a non-colliding process

- The invariant measure is the eigenvalue distribution for the GUE.

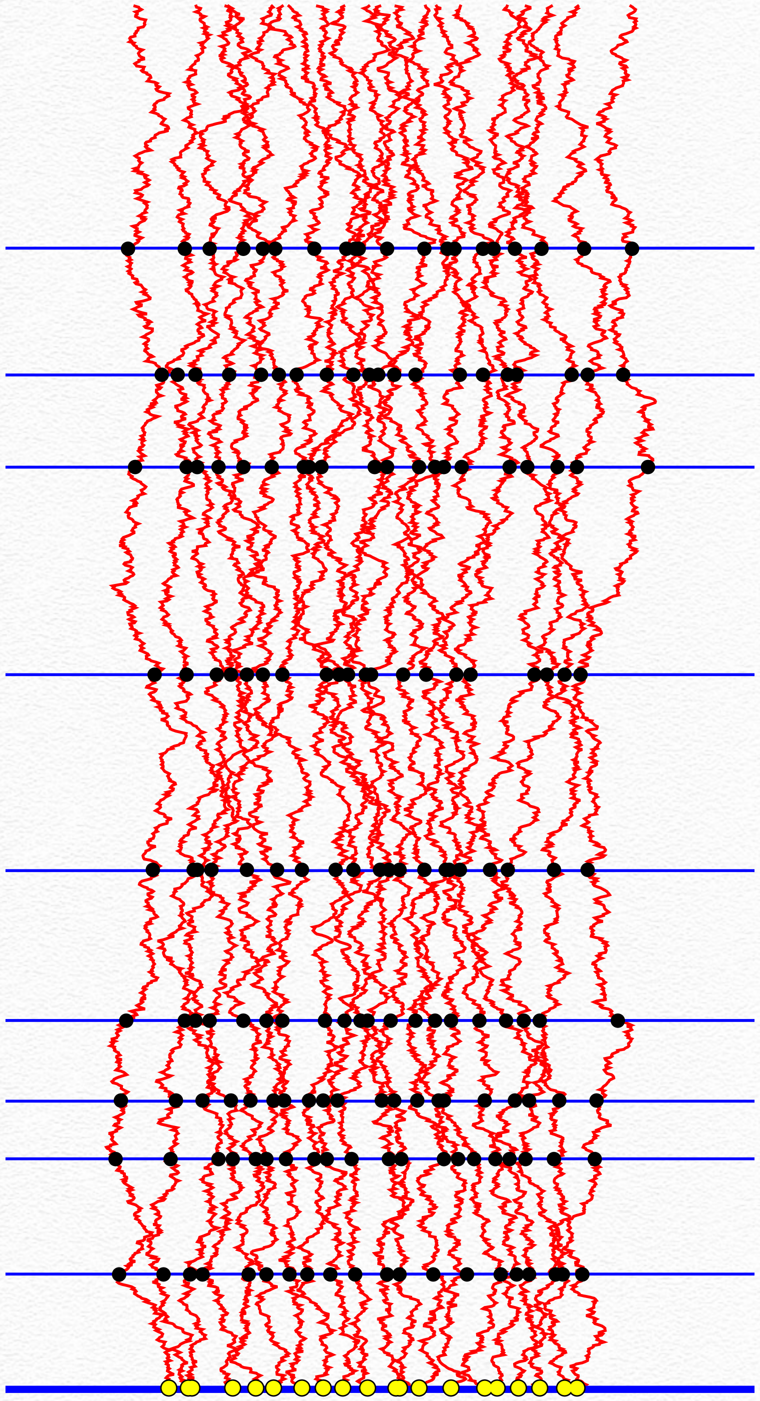
- What about multi-time fluctuations in the stationary situation?



Stationary DBM

- Choose a number of horizontal sections τ_m for $m = 1, \ldots, N$

$$X_n(f) = \sum_{m=1}^{N} \sum_{j=1}^{n} f(\tau_m, x_j(\tau_m))$$



- **Multi-time Global CLT**: As $n \to \infty$, for sufficiently smooth functions f,

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2)$$

where

$$\sigma(f)^2 = \sum_{m_1, m_2 = 1}^{N} \sum_{k=1}^{\infty} \mathrm{e}^{-|\tau_{m_1} - \tau_{m_2}|k} k f_k^{(m_1)} f_k^{(m_2)},$$

and

$$f_k^{(m)} = \frac{1}{\pi} \int_0^{\pi} f(m, \sqrt{2}\cos\theta) \cos k\theta \mathrm{d}\theta$$

Borodin, D.

3. Biorthogonal Ensembles

Single time fluctuations

Biorthogonal Ensembles

- Consider a probability measure on $\mathbb{R}^n$ defined by

$$\frac{1}{Z_n} \det\left(\phi_j(x_k)\right)_{j,k=1}^{n} \det\left(\psi_j(x_k)\right)_{j,k=1}^{n} \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$

where μ is a Borel measure ϕ_j and ψ_j are some square integrable functions with respect to μ

- By taking linear combination of the the rows in the determinants and the fact that the above is probability measure, we can assume, without loss of generality, that

$$\int \phi_j(x)\psi_k(x)\mathrm{d}\mu(x) = \delta_{jk}$$

This also implies that $Z_n = n!$

Recurrence Matrices

- Our main assumption will be that the functions ϕ_j and ψ_j are part of large system of biorthogonal functions $\{\phi_j\}_j$ and $\{\psi_j\}_j$,

$$\int \psi_j(x)\psi_k(x)\mathrm{d}\mu(x) = \delta_{jk}$$

for which there exists a banded matrix $\mathcal{J}$ such that

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \end{pmatrix} = \mathcal{J} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \end{pmatrix}$$

- This means that the ϕ_j satisfy a finite term recurrence

$$x\phi_j(x) = \sum_{|k|\leq\rho} \mathcal{J}_{j,j+k}\phi_{j+k}(x)$$

Orthogonal polynomials and Jacobi matrices

- An example of such a system are the *Orthogonal Polynomial Ensembles*. In that case $\phi_j(x) = \psi_j(x) = p_j(x)$ are the orthonormal polynomials. That is, the polynomials of degree *j* with positive leading coefficients satisfying

$$\int p_j(x) p_k(x) \mathrm{d}\mu(x) = \delta_{jk}$$

- There exists $\{a_k\}_k \subset (0,\infty)$ and $\{b_k\}_k \subset \mathbb{R}$ such that

$$x p_k(x) = a_{k+1} p_{k+1}(x) + b_{k+1} p_k(x) + a_k p_{k-1}(x)$$

- The recurrence matrix, or Jacobi operator, is symmetric tridiagonal

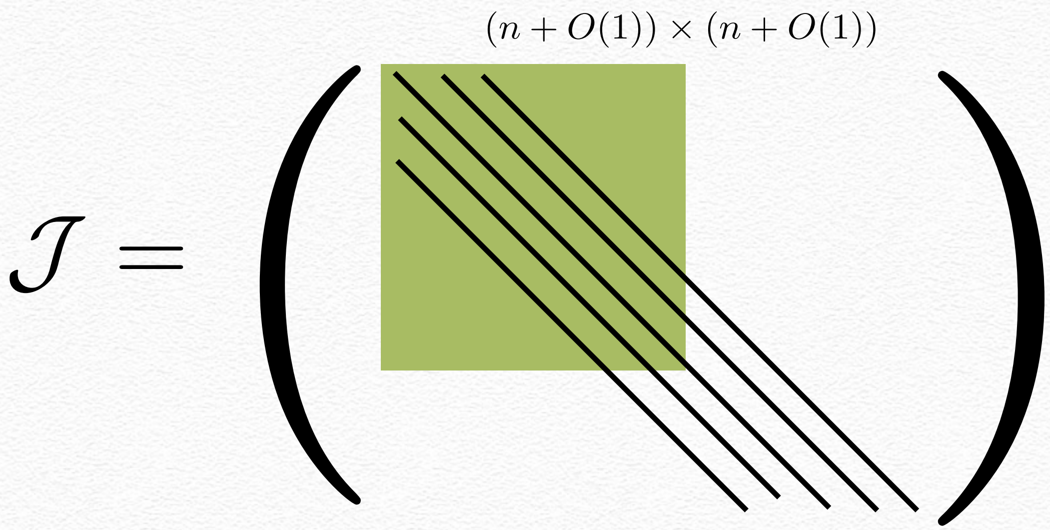
$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \ldots \\ a_1 & b_2 & a_2 & \ldots \\ 0 & a_2 & b_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Relevant part of the recurrence matrix (1)

- *Limit shape:* If is a polynomial then

$$\lim_{n\to\infty} \frac{1}{n}\left(\mathbb{E}X_n(f) - \operatorname{Tr} P_n f(\mathcal{J}) P_n\right) = 0$$

This means that the expected value of the moments of the empirical measure only depend on coefficients in an upper right block of size.



Relevant parts of the recurrence matrix (2)

- *Fluctuations:* The moments of the fluctuations

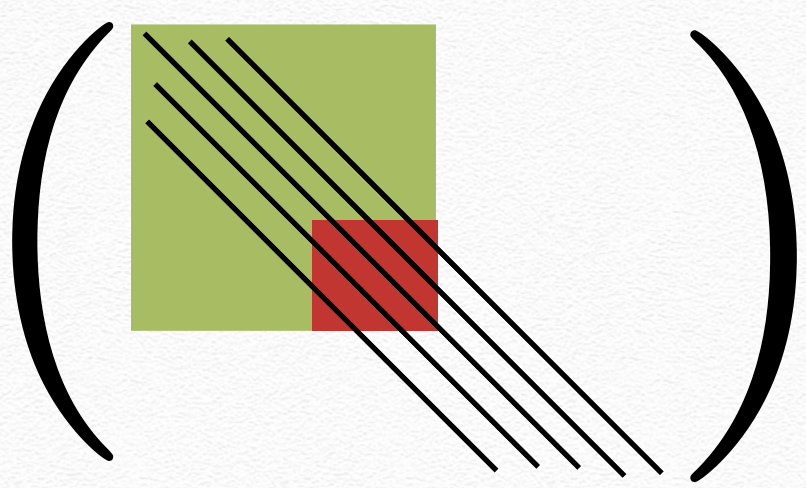
$$\mathbb{E}\left[(X_n(f) - \mathbb{E}X_n(f))^m\right]$$

  of the moments of the empirical measure only depend on O(1) size block around n,n entry

- More precisely, *The m-th moment only depends on the entries*

$$\mathcal{J}_{n+j,n+k}, \qquad |j|, |k| \leq M.$$

  where M *depends on* m *and the degree* f *of but not on* n.

$\mathcal{J} =$ 

Universality of fluctuations

Theorem *(Breuer-D, to appear in JAMS)*

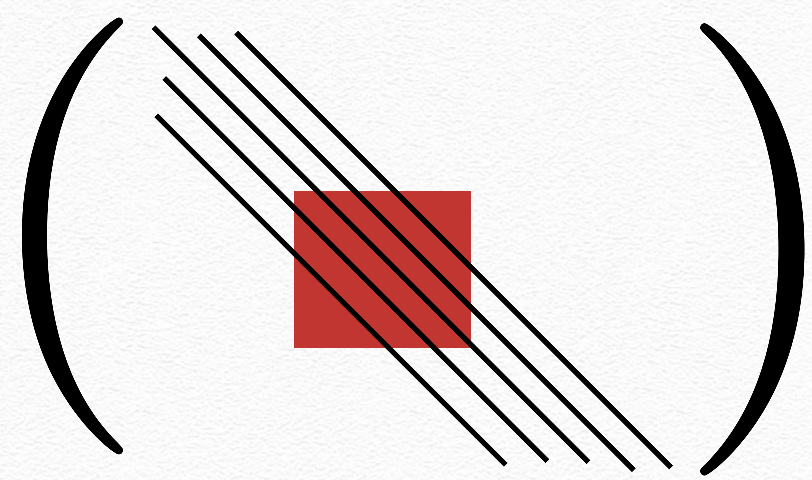
Given two biorthogonal ensembles, with recurrence matrices $\mathcal{J}$ and $\tilde{\mathcal{J}}$ such that for all k and ℓ we have

$$\lim_{n\to\infty}\left(\mathcal{J}_{n+k,n+\ell}-\tilde{\mathcal{J}}_{n+k,n+\ell}\right)=0,$$

Then, for any polynomial $f(x)$ and for each m, as $n\to\infty$,

$$\mathbb{E}\left[(X_n(f)-\mathbb{E}X_n(f))^m\right]-\tilde{\mathbb{E}}\left[\left(X_n(f)-\tilde{\mathbb{E}}X_n(f)\right)^m\right]\to 0$$

- Question: *What is the typical behavior in the red blocks?*

$\mathcal{J} =$ 

Central Limit Theorem

Theorem *(Breuer-D, to appear in JAMS)*
Suppose that we have a biorthogonal ensemble with a recurrence matrix satisfying

$$\lim_{n\to\infty} J_{n+k,n+\ell} = a_{k-\ell}$$

Then for any polynomial $f(x)$ as $n \to \infty$,

$$X_n(f) - \mathbb{E}X_n(f) \to N\left(0, \sum_{k=1}^{\infty} k f_k f_{-k}\right)$$

where

$$f_k = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} f\left(\sum_j a_j z^j\right) \frac{\mathrm{d}z}{z^{k+1}}$$

´In the symmetric case $\phi_j = \psi_j$, we also formulate general and weak conditions to allow C^1 functions of polynomial growth.

Example GUE

- The eigenvalues of a GUE matrix have the

$$\frac{1}{n!}\left(\det\left(p_{j-1}(x_k)\right)_{j,k=1}^{n}\right)^2 \prod_{j=1}^{n} \mathrm{e}^{-nx_j^2}\mathrm{d}x_j$$

where $p_j(x)$ are rescaled and normalized Hermite polynomials, i.e.

$$\int p_j(x)p_k(x)\mathrm{e}^{-nx^2}\mathrm{d}x = \delta_{jk}$$

- The (rescaled) Hermite polynomials have there recurrence

$$xp_k(x) = \sqrt{\frac{k+1}{2n}}p_{k+1}(x) + \sqrt{\frac{k}{2n}}p_{k-1}(x)$$

- Hence we find the following recurrence matrix and the relevant limit

$$\mathcal{J} = \frac{1}{\sqrt{2n}}\begin{pmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & \ddots & \ddots \\ & & & \ddots & \end{pmatrix} \qquad (\mathcal{J})_{n+k,n+l} = \begin{cases} \frac{1}{\sqrt{2}} & |k-l| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Remarks on the generality

- This theorem gives a *criterium* for which the model has Gaussian limiting fluctuation for the linear statistics. Work is still needed to show that the criteria is practical.

- For models involving classical functions / orthogonal polynomials, the criteria is often satisfied and can be simply looked up on any standard reference on special functions.

- Often, the question of finding leading term in the asymptotics for the recurrence coefficients is an intrinsically easier question that finding the full asymptotic description for the correlation kernel.

- In certain cases there are general result ensuring that the criterium is fulfilled for a remarkably wide class of models.

- The criterium fails in multi-cut cases since the recurrence coefficients have quasi-periodic behavior. The fluctuations are not always gaussian.
  Pastur '06, Borot-Guionnet '13, Shcherbina '13

Example: OPRL

- As the first consider $\phi_j = \psi_j = p_{j-1}$ the OP's for μ

$$\frac{1}{n!}\left(\det p_j(x_k)\right)^2 \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n) \sim \prod_{1\leq i<j\leq n} (x_i - x_j)^2 \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$

- Recurrence

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_{k+1}p_k(x) + a_k p_{k-1}(x)$$

- Jacobi operator

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- We allow varying measures so all objects may depend on n

Example: OPRL

- For non-varying measures we have the:

  Denissov-Rakhmanov Theorem: If μ is a compactly supported measure for which the essential part of the support is a single interval $[b-2a, b+2a]$ and the absolutely continuous part has a density that is strictly positive a.e. on the essential part. Then

  $$a_n \to a \qquad\qquad b_n \to b$$

- For varying measure related to Unitary Ensembles, we have limits of the recurrence coefficients for one-cut situations.

- Also many discrete examples from tiling models fall in this class.

DBM with random initial condition

- Take as the initial configuration the eigenvalues of a matrix take randomly from a unitary ensemble with a convex potential V

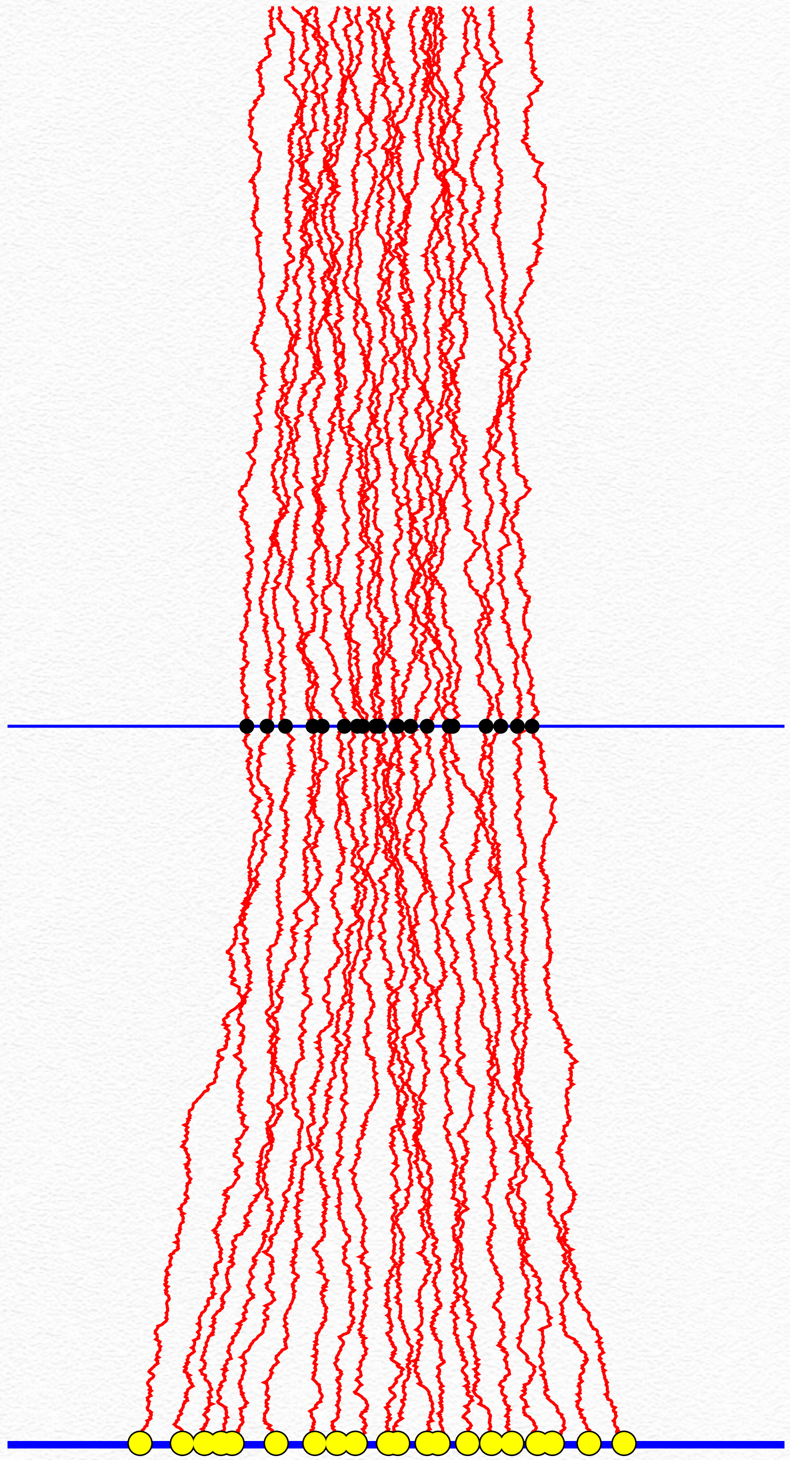
$$\frac{1}{n}\mathrm{e}^{-n\,\mathrm{Tr}\,V(M)}\mathrm{d}M$$

- Then run DBM and consider the distribution of the points at time t. These have the same distribution as the eigenvalues of

$$\mathrm{e}^{-t}M_V + \sqrt{1-\mathrm{e}^{-2t}}M_{\mathrm{GUE}}$$

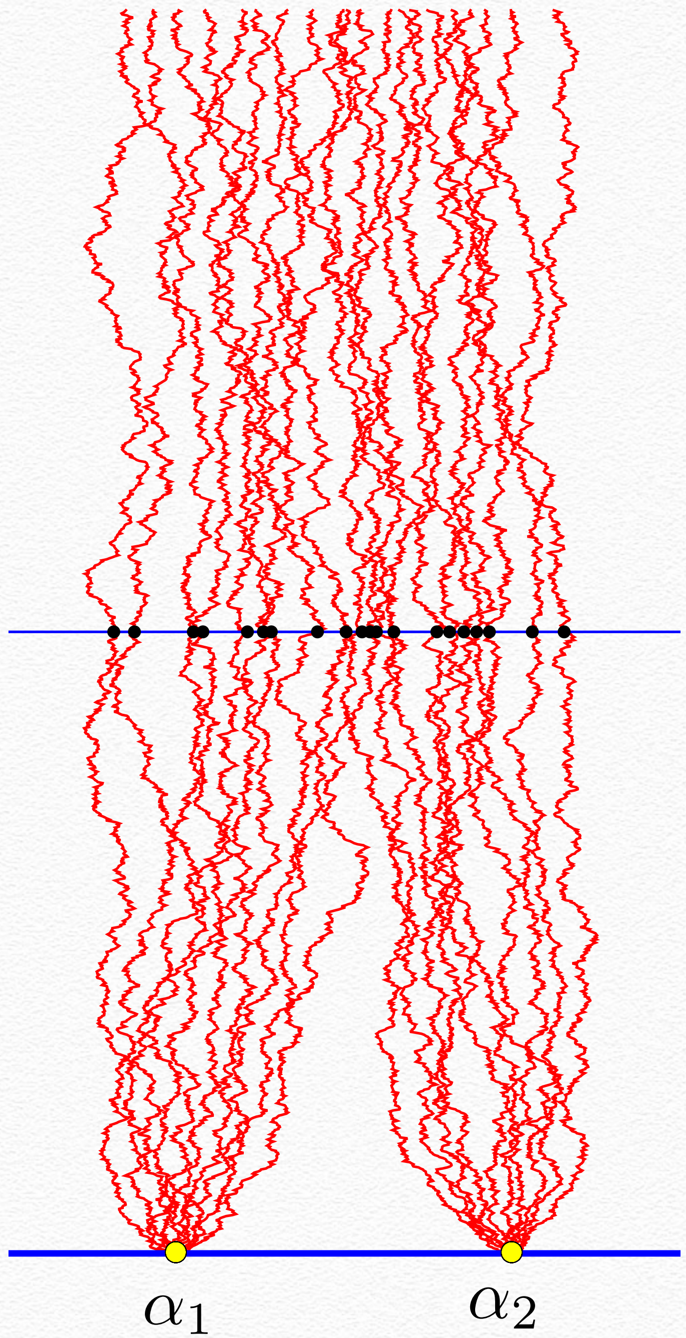
(Recall talk by Kuijlaars on Monday)

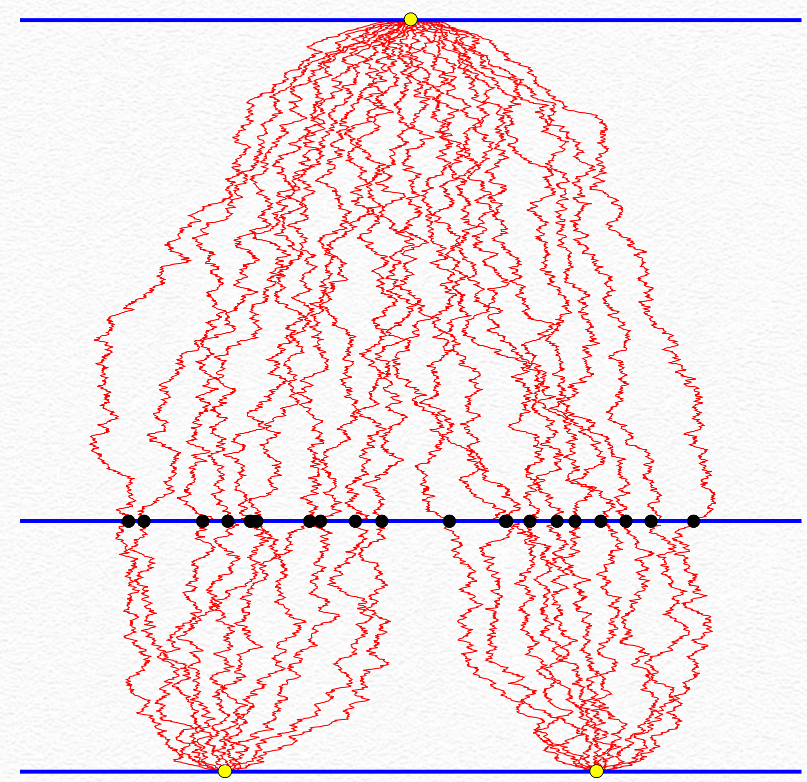
- In this case, we have multiple orthogonal polynomials with wider band recurrence matrices but we still find a Toeplitz structure in the limit with relative ease, see *Breuer-D*.



DBM with two initial points

- Now consider DBM with two starting points α_1 and α_2. We assume that the number of paths n is even and that half of the points start at each point.





(Equivalent picture of brownian bridges)

Multiple Hermite polynomials

- The locations at time t form a biorthogonal ensembles constructed by Multiple Hermite Polynomials *(Johansson '02, Bleher-Kuijlaars '04)*

$$P_{k_1,k_2}(x) = \frac{1}{\sqrt{2\pi}\mathrm{i}} \int_{\mathrm{iR}} (z-a_1)^{k_1}(z-a_2)^{k_2} \mathrm{e}^{\frac{1}{2}(z-x)^2} \mathrm{d}z.$$

$$Q_{k_1,k_2}(x) = \frac{1}{\sqrt{2\pi}2\pi\mathrm{i}} \oint_{\gamma_{a_1,a_2}} (w-a_1)^{-k_1}(w-a_2)^{-k_2} \mathrm{e}^{-\frac{1}{2}(w-x)^2} \mathrm{d}z.$$

The P_{k_1,k_2} is a polynomial of degree k_1+k_2

- Then, up to a rescaling and re-parametrization, the points at time t form a biorthogonal ensemble with

$$\{\phi_j\}_{j=1}^n = \{P_{0,0}, P_{1,0}, P_{1,1}, P_{2,1}, \ldots\}$$

$$\{\psi_j\}_{j=1}^n = \{Q_{1,0}, Q_{1,1}, Q_{2,1}, Q_{2,2}, \ldots\}$$

Recurrence relations for Multiple Hermite Polynomials

- One can show that

$$xP_{k_1,k_2}(x) = P_{k_1+1,k_2}(x) + a_1 P_{k_1,k_2}(x) + k_1 P_{k_1-1,k_2}(x) + k_2 P_{k_1,k_2-1}(x)$$

- We can exchange indices according to the rule

$$P_{k_1+1,k_2}(x) = P_{k_1,k_2+1}(x) + (a_2 - a_1) P_{k_1,k_2}(x).$$

- Then we have the following recurrence relation

$$xP_{k_1,k_2}(x) = P_{k_1+1,k_2}(x) + a_1 P_{k_1,k_2}(x) + (k_1 + k_2) P_{k_1,k_2-1}(x) - (a_2 - a_1) k_1 P_{k_1-1,k_2-1}(x).$$

$$xP_{k_1,k_2}(x) = P_{k_1,k_2+1}(x) + a_2 P_{k_1,k_2}(x) + (k_1 + k_2) P_{k_1-1,k_2}(x) - (a_1 - a_2) k_1 P_{k_1-1,k_2-1}(x).$$

Limit of the recurrence matrix

- After some computation by plugging in the recurrences we find

$$\lim_{n\to\infty}\left((\mathcal{J}^{(n)})_{n+k,n+l} - (\mathbb{J})_{n+k,n+l}\right) = 0.$$

where

$$\mathbb{J} = \begin{pmatrix} A_0 & A_{-1} & & & \\ A_1 & A_0 & A_{-1} & & \\ & A_1 & A_0 & A_{-1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$A_1 = (\mathrm{e}^t - \mathrm{e}^{-t}) \begin{pmatrix} \frac{1}{2}(a_1 - a_2) & 1 \\ 0 & \frac{1}{2}(a_2 - a_1) \end{pmatrix}$$

$$A_0 = \mathrm{e}^{-t} \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} + (\mathrm{e}^t - \mathrm{e}^{-t}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A_{-1} = \mathrm{e}^{-t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

A more general Central Limit Theorem

- The CLT of Breuer-D. can be generalized into

Theorem

Suppose there exists and such that

$$\lim_{n\to\infty}\left((\mathcal{J}^{(n)})_{n+k,n+l} - (\mathbb{J})_{n+k,n+l}\right) = 0.$$

and that for a polynomial we can decompose $p(\mathbb{J}) = p(\mathbb{J})_+ + p(\mathbb{J})_-$ such that

1) $p(\mathbb{J})_+$ is lower triangular,
2) $p(\mathbb{J})_-$ is upper triangular,
3) $[p(\mathbb{J})_+, p(\mathbb{J})_-]$ is trace class

Then, as $n \to \infty$, we have

$$X_n(p) - \mathbb{E}X_n(p) \xrightarrow{\mathcal{D}} N\left(0, \mathrm{Tr}[p(\mathbb{J})_+, p(\mathbb{J})_-]\right),$$

Some comments

- If $\mathbb{J}$ is a Toeplitz matrix, the conditions are satisfied for every polynomial and therefore the aforementioned theorem in Breuer-D is a direct consequence of this theorem. This follows by simple identities for Toeplitz matrices.

- If $\mathbb{J}$ is a block Toeplitz matrix, then this generically fails, although it may be true for some polynomials (e.g. in the symmetric two-cut case for unitary ensemble, even polynomials still satisfy a CLT)

- However, in the case of multiple Hermite polynomials, the condition is true for every polynomial! This is because of a very particular structure in the blocks.

- It is an interesting problem for the theory of multiple orthogonal polynomials to characterize conditions under which this happens.

CLT for DBM with two starting points

Theorem [D. upcoming]
Let f be a polynomial. Then

$$X_n(f) - \mathbb{E}X_n(f) \to N\left(0, \sum_{k=1}^{\infty} k\,\mathrm{Tr}\,(f_k f_{-k})\right)$$

where

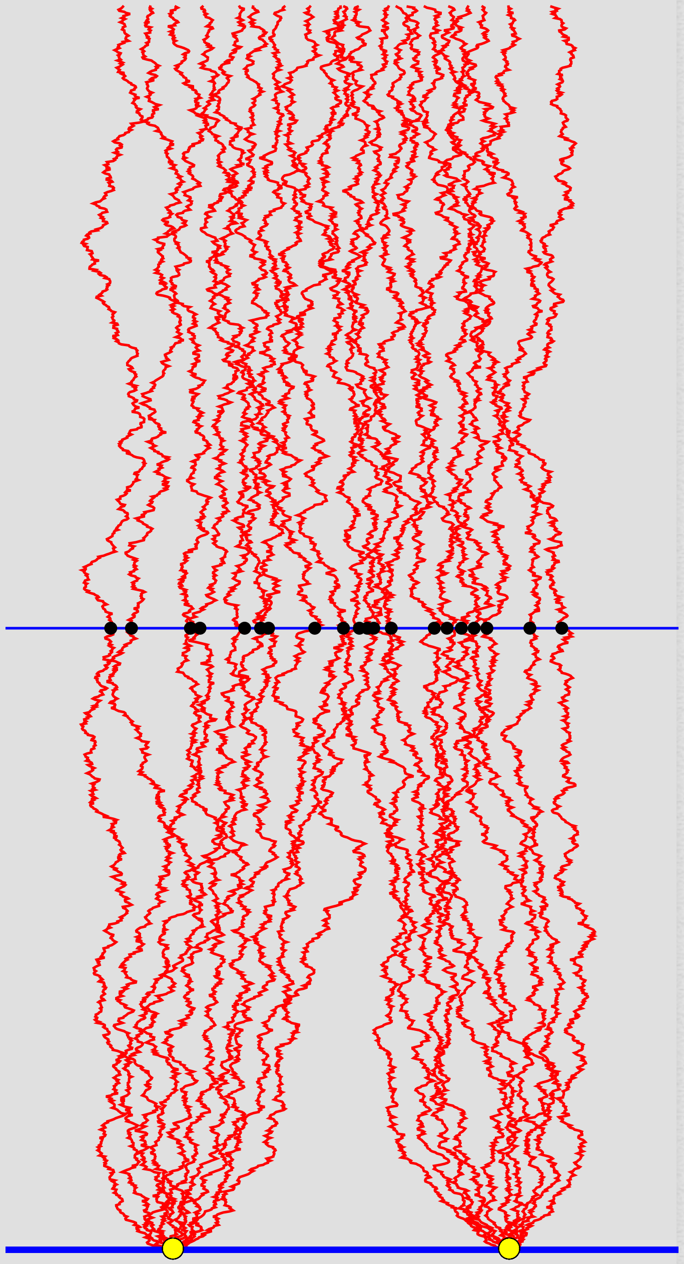
$$f_k = \frac{1}{2\pi \mathrm{i}} \oint p\,(zA_1 + A_0 + A_2/z)\,\frac{\mathrm{d}z}{z^{k+1}}$$

and

$$A_1 = (\mathrm{e}^t - \mathrm{e}^{-t}) \begin{pmatrix} \frac{1}{2}(a_1 - a_2) & 1 \\ 0 & \frac{1}{2}(a_2 - a_1) \end{pmatrix}$$

$$A_0 = \mathrm{e}^{-t} \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} + (\mathrm{e}^t - \mathrm{e}^{-t}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A_{-1} = \mathrm{e}^{-t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



4. Extended Biorthogonal Ensembles

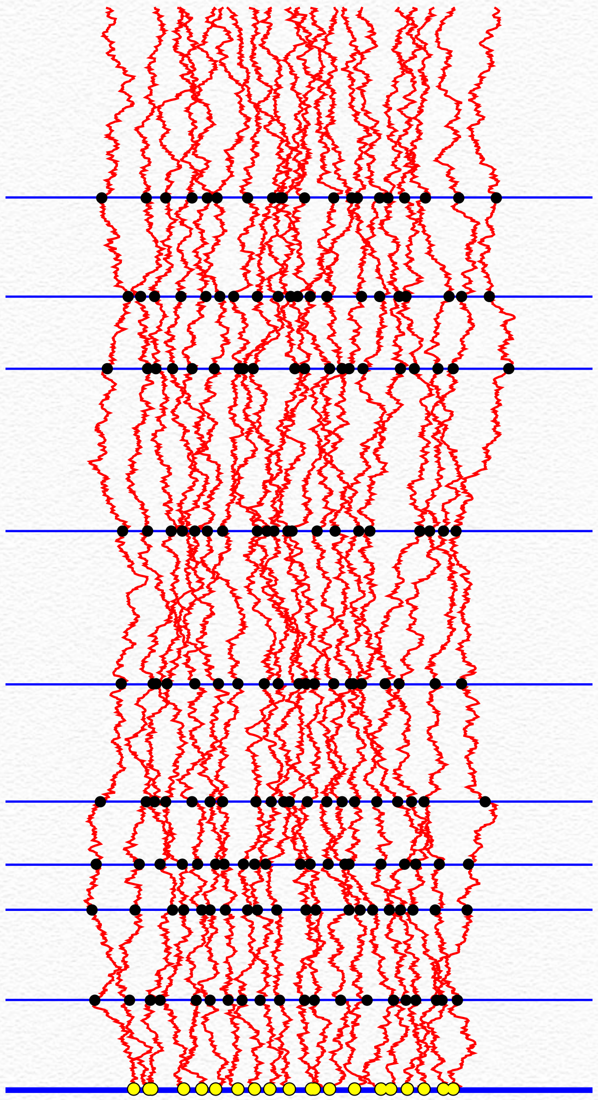
Multi-time fluctuations

Extended Biorthogonal Ensembles

- Now consider a probability measure on $(\mathbb{R}^n)^N$ of the form

$$\frac{1}{Z_n}\det\left(\phi_{j,1}(x_{1,k})\right)_{j,k=1}^n \prod_{m=1}^{N-1}\det\left(T_m(x_{m,i},x_{m+1,j})\right)_{i,j=1}^n$$

$$\times\det\left(\psi_{j,N}(x_{N,k})\right)_{j,k=1}^n \prod_{m=1}^{N}\prod_{j=1}^{n}\mathrm{d}\mu_m(x_{m,j}),$$



- We assume that the operators

$$\mathcal{T}_m : \mathbb{L}_2(\mu_m)\to\mathbb{L}_2(\mu_{m+1})$$

defined by

$$\mathcal{T}_m f(y)=\int f(x)T_m(x,y)\mathrm{d}\mu_m(x),$$

are bounded.

Recurrence matrices

- We assume that the functions are biorthogonal in the sense

$$\int \psi_{j,N}(x)\mathcal{T}_{N-1}\mathcal{T}_{N-2}\cdots\mathcal{T}_1\phi_{k,1}(x)\mathrm{d}\mu_N(x) = \delta_{jk},$$

- Then define the functions $\phi_{j,m}$ and $\psi_{j,m}$

$$\phi_{j,m} = \mathcal{T}_{m-1}\cdots\mathcal{T}_1\phi_{j,1}, \qquad \psi_{j,m} = \mathcal{T}_m^*\cdots\mathcal{T}_{N-1}^*\psi_{j,N},$$

which are biorthogonal as well

$$\int \psi_{j,m}(x)\phi_{k,m}(x)\mathrm{d}\mu(x) = \delta_{jk}$$

- The marginals for fixed m gives a biorthogonal ensemble with respect to the functions $\phi_{j,m}$ and $\psi_{j,m}$.

Recurrence matrices

- We assume that $\phi_{j,1}$ and $\psi_{k,N}$ can be embedded into families such that

$$\int \psi_{j,N}(x)\mathcal{T}_{N-1}\mathcal{T}_{N-2}\cdots\mathcal{T}_1\phi_{k,1}(x)\mathrm{d}\mu_N(x) = \delta_{jk},$$

and that there exists band matrices $\mathcal{J}_m$

$$x\begin{pmatrix}\phi_{0,m}(x)\\ \phi_{1,m}(x)\\ \phi_{2,m}(x)\\ \vdots\end{pmatrix} = \mathcal{J}_m\begin{pmatrix}\phi_{0,m}(x)\\ \phi_{1,m}(x)\\ \phi_{2,m}(x)\\ \vdots\end{pmatrix}.$$

- The matrices may depend on n, except for the bandwidths.

Universality

Consider linear statistics

$$X_n(f) = \sum_{m=1}^{N} \sum_{j=1}^{n} f(m, x_{j,m})$$

***Theorem** (D. arXiv)* Suppose we have two different extended biorthogonal ensembles such that for each m

$$\lim_{n\to\infty} \Big((\mathcal{J}_m)_{n+k,n+l} - (\tilde{\mathcal{J}}_m)_{n+k,n+l} \Big) = 0$$

Then for any $f(x,m)$ such that it is a polynomial in x and $k \in \mathbb{N}$

$$\mathbb{E}\left[(X_n(f) - \mathbb{E}X_n(f))^k \right] - \widetilde{\mathbb{E}}\left[\Big(X_n(f) - \widetilde{\mathbb{E}}X_n(f) \Big)^k \right] \to 0,$$

as $n \to \infty$

Central Limit Theorem

Theorem *(D. arXiv)* Suppose that for each m, k, ℓ

$$\lim_{n\to\infty} (\mathbb{J}_m)_{n+k,n+l} = a_{k-l}^{(m)},$$

Then for any $f(m,x)$ that is a polynomial in x, as $n \to \infty$,

$$X_n(f) - \mathbb{E}X_n(f) \to$$

$$N\left(0, 2\sum_{m_1=1}^{N}\sum_{m_2=m_1+1}^{N}\sum_{k=1}^{\infty} k f_k^{(m_1)} f_{-k}^{(m_2)} + \sum_{m=1}^{N}\sum_{k=1}^{\infty} k f_k^{(m)} f_{-k}^{(m)}\right),$$

where

$$f_k^{(m)} = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} f(m, \sum_j a_j^{(m)} z^j) \frac{\mathrm{d}z}{z^{k+1}}.$$

Example: Stationary DBM

In stationary DBM we have

$$\frac{1}{Z_n}\det\left(\phi_{j,1}(x_{1,k})\right)_{j,k=1}^{n}\prod_{m=1}^{N-1}\det\left(P_{t_{m+1}-t_m}(x_{m,i},x_{m+1,j})\right)_{i,j=1}^{n}$$

$$\times\det\left(\psi_{j,N}(x_{N,k})\right)_{j,k=1}^{n}\prod_{m=1}^{N}\prod_{j=1}^{n}\,\mathrm{d}x_{m,j}$$

where

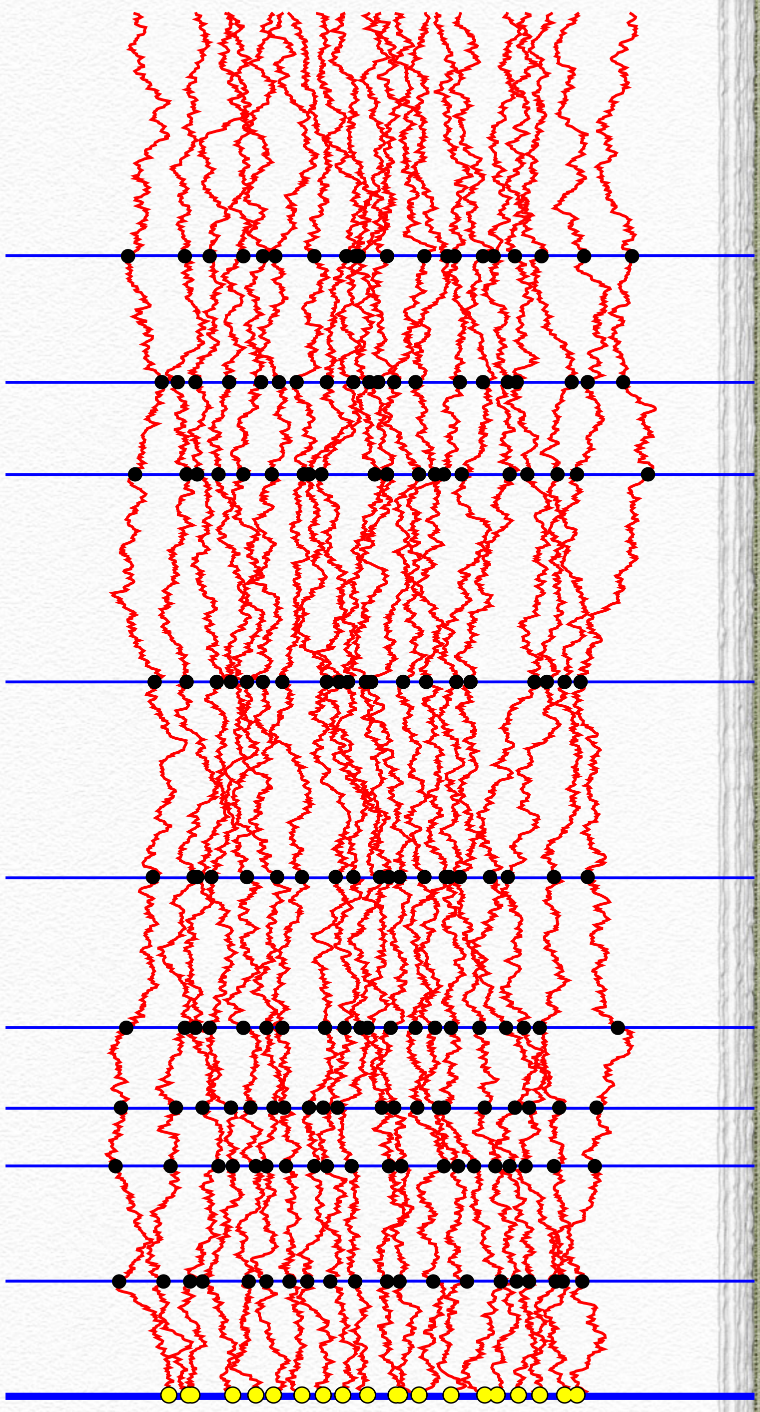
$$\phi_{j,1}=\mathrm{e}^{-\tau_1(j-1)}H_{j-1}(\sqrt{n}x)$$

$$\psi_{j,N}=\mathrm{e}^{\tau_N(j-1)}H_{j-1}(\sqrt{n}x)\,\mathrm{e}^{-nx^2}$$

where H_j stands for the j-th Hermite polynomial and

$$P_t(x,y)=\frac{1}{\sqrt{2\pi n}}\mathrm{e}^{-\frac{n}{1-\mathrm{e}^{-2t}}(\mathrm{e}^{-t}x-y)^2}$$

is the transition probability for the OU process.



Example: Dyson's Brownian motion (2)

- Then

$$\phi_{j,m}(x) = \mathcal{T}_{m-1} \cdots \mathcal{T}_1 \phi_{j,1}(x) = \mathrm{e}^{-\tau_m (j-1)} H_{j-1}(\sqrt{n} x)$$

and hence we obtain the recurrence matrices

$$(\mathcal{J}_m)_{jk} = \mathrm{e}^{-\tau_m (j-k)} (\mathcal{J}_{Hermite})_{jk}$$

- **Multi-time Global CLT**: As $n \to \infty$, for sufficiently smooth functions f

$$X_n(f) - \mathbb{E} X_n(f) \to N(0, \sigma(f)^2)$$

where

$$\sigma(f)^2 = \sum_{m_1, m_2 = 1}^{N} \sum_{k=1}^{\infty} \mathrm{e}^{-|\tau_{m_1} - \tau_{m_2}| k} k f_k^{(m_1)} f_k^{(m_2)},$$

and

$$f_k^{(m)} = \frac{1}{\pi} \int_0^{\pi} f(m, \sqrt{2} \cos \theta) \cos k\theta \mathrm{d}\theta$$

Further comments

- This construction can be generalized in a straightforward way. The Hermite functions are eigenfunctions for the generator of the Ornstein-Uhlenbeck processes. Similar statements are true other process for which the generator has the classical orthogonal polynomials as eigenfunctions. Such as Laguerre and Jacobi polynomials in the continuous case and in the discrete case one can turn to birth/death processes, Meixner, Charlier, Krawtchouk.

- It can also be applied to tiling models, such as lozenge tilings of a regular hexagon.

- One can rigorously show that the CLT explains a convergence of the fluctuation of the associated random height function to the *Gaussian Free Field*.

- By an extension we can also handle certain cases involving multiple orthogonal polynomials:

Multi-time CLT for DBM with two starting points

Theorem [D. upcoming]
Let $f(\tau, x)$ be a polynomial in x . Then

$$X_n(f) - \mathbb{E}X_n(f) \to$$

$$N\left(0, 2\sum_{m_1=1}^{N}\sum_{m_2=m_1+1}^{N}\sum_{k=1}^{\infty} k\,\mathrm{Tr}\left(f_k^{(m_1)} f_{-k}^{(m_2)}\right) + \sum_{m=1}^{N}\sum_{k=1}^{\infty} k\,\mathrm{Tr}\left(f_k^{(m)} f_{-k}^{(m)}\right)\right)$$

where

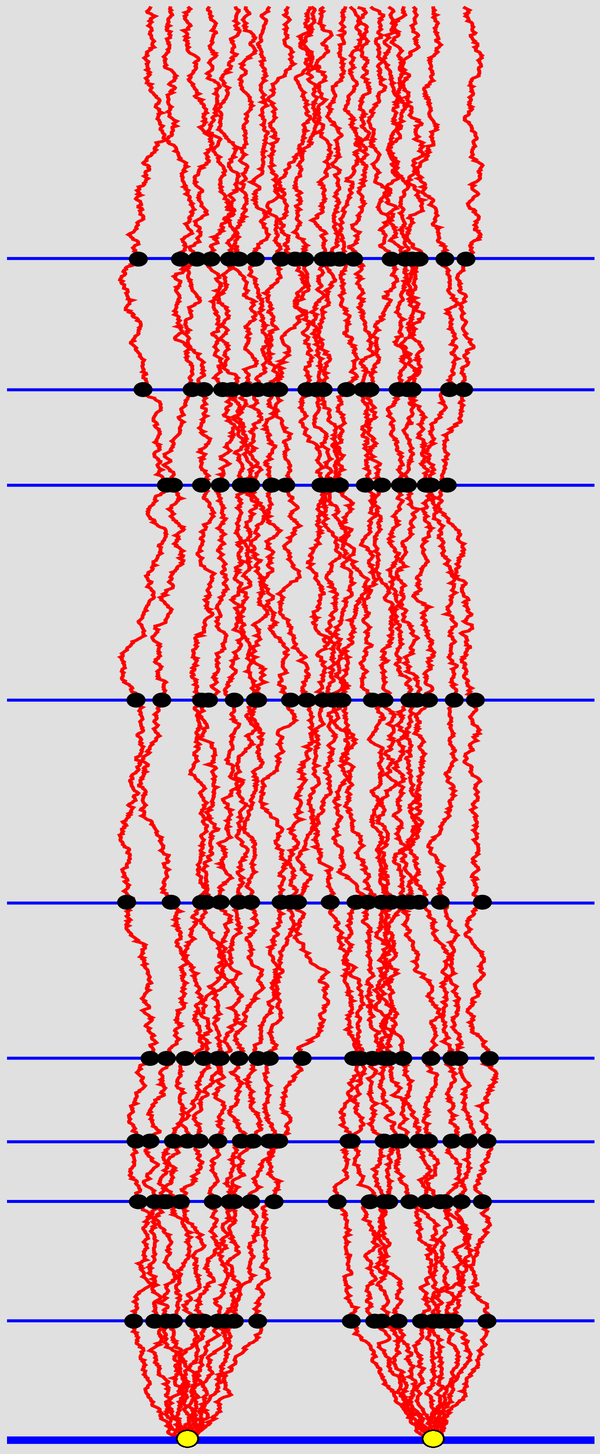
$$f_k^{(m)} = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} f\left(\tau_m, A_1^{(m)} z + A_0^{(m)} + A_{-1}^{(m)} z^{-1}\right) \frac{\mathrm{d}z}{z^{k+1}}$$

and

$$A_1^{(m)} = (\mathrm{e}^{\tau_m} - \mathrm{e}^{-\tau_m}) \begin{pmatrix} \frac{1}{2}(a_1 - a_2) & 1 \\ 0 & \frac{1}{2}(a_2 - a_1) \end{pmatrix}$$

$$A_0^{(m)} = \mathrm{e}^{-\tau_m} \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} + (\mathrm{e}^{\tau_m} - \mathrm{e}^{-\tau_m}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A_{-1}^{(m)} = \mathrm{e}^{-\tau_m} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



Thank you for you attention!