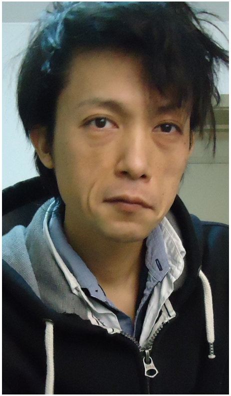
Vafa-Witten invariants of projective surfaces

Joint with Yuuji Tanaka



The Vafa-Witten equations

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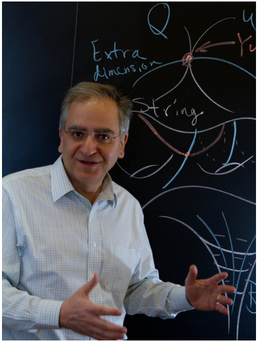

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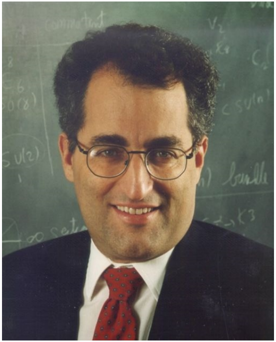

Ed Witten

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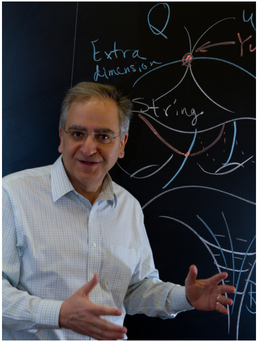

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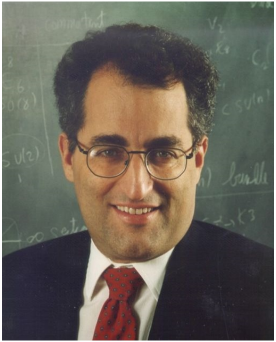

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Riemannian 4-manifold M, $SU(r)$ bundle $E \to M$, connection A, fields $B \in \Omega^+(\mathfrak{su}(E))$, $\Gamma \in \Omega^0(\mathfrak{su}(E))$,

$$\begin{aligned} F_A^+ + [B.B] + [B,\Gamma] &= 0, \\ d_A\Gamma + d_A^* B &= 0. \end{aligned}$$

Their prediction

VW invariants

Vafa and Witten told us to "count" (in an appropriate sense) solutions of these VW equations.

For $c_2 = n$, let VW_n ($\in \mathbb{Z}$? $\in \mathbb{Q}$?) denote the resulting Vafa-Witten invariants of M.

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"S-duality" voodoo should imply their generating series

$$q^{-\frac{e(S)}{12}} \sum_{n=0}^{\infty} \mathrm{VW}_n(M)\, q^n$$

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In particular, the infinite collection of numbers $\mathrm{VW}_n(M)$ should be determined by only finitely many of them.

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They were able to check their conjecture in many cases when M has positive curvature, due to a *vanishing theorem* $B = 0 = \Gamma$.

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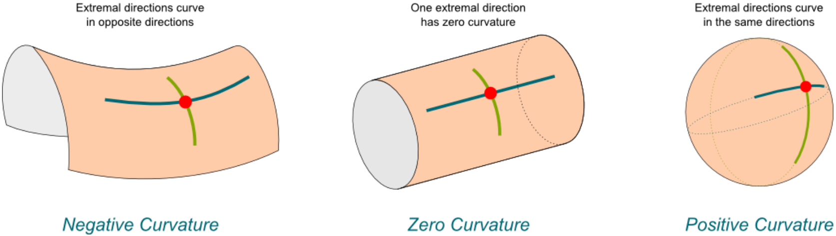

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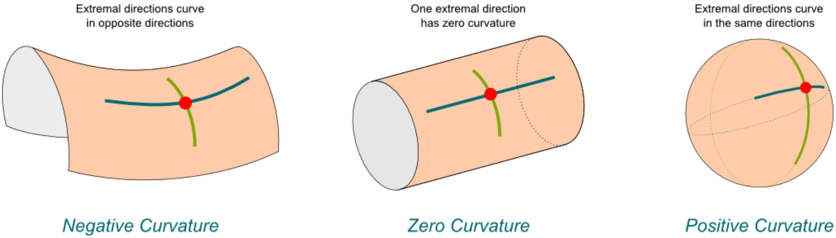

The equations then reduce to the anti-self-dual equations. These have a compact moduli space $\mathcal{M}_n^{\mathrm{asd}}$.

When there are *no reducible solutions*, the obstruction bundle is $T^{(*)}\mathcal{M}_n^{\mathrm{asd}}$ so we should have

$$\mathrm{VW}_n = \pm e(\mathcal{M}_n^{\mathrm{asd}}).$$

Kähler case

For general M no one can yet define Vafa-Witten invariants since the moduli space is inherently noncompact. ($|B|$, $|\Gamma|$ can $\to \infty$.)

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When $M = S$ is a Kähler surface we can rewrite B, Γ in terms of an $\mathrm{End}_0\, E$-valued $(2,0)$-form ϕ and an $\mathrm{End}_0\, E$-valued multiple of the Kähler form ω, giving

$$\begin{aligned} F_A^{0,2} &= 0, \\ F_A^{1,1} \wedge \omega + [\phi, \overline{\phi}] &= 0, \\ \overline{\partial}_A \phi &= 0. \end{aligned}$$

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So $\overline{\partial}_A$ makes E into a holomorphic bundle with a holomorphic Higgs field

$$\phi \ \in \ H^0(\mathrm{End}_0\, E \otimes K_S)$$

satisfying a moment map equation $F_A^{1,1} \wedge \omega + [\phi, \overline{\phi}\,] = 0$.

Hitchin-Kobayashi correspondence

At least when S is projective, Álvarez-Cónsul–García-Prada and Tanaka have proved an infinite dimensional Kempf-Ness theorem.

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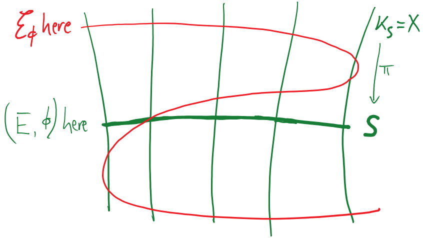
When $K_S < 0$ stability forces $\phi = 0$.
Similarly when $K_S \leq 0$ and stability = semistability.

Then we get the moduli space of (semi)stable sheaves E with $\det E = \mathcal{O}_S$ on S, and VW_n is some kind of Euler characteristic thereof.

Spectral construction

Put eigenspaces of $\phi : E \to E \otimes K_S$ over the corresponding eigenvalues in K_S. Defines a torsion sheaf $\mathcal{E}_\phi$ on $X = K_S$.



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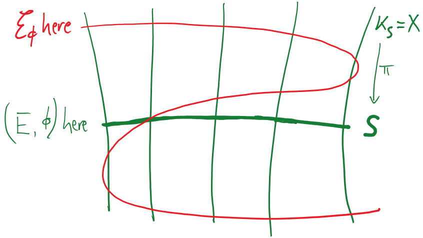
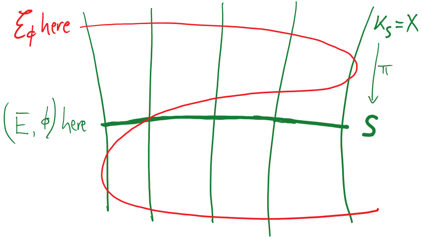


Over a point of S we have a vector space V and a endomorphism ϕ. This makes V into a finite-dimensional $\mathbb{C}[x]$-module (and so a torsion sheaf) by letting x act through ϕ.

$\mathrm{Higgs}_{K_S}(S) \longleftrightarrow \mathrm{Coh}_c(X)$



Globally over S, we make E into a $\pi_* \mathcal{O}_S = \bigoplus_i K_S^{-i}$-module by

$$E \otimes K_S^{-i} \xrightarrow{\phi^i} E.$$

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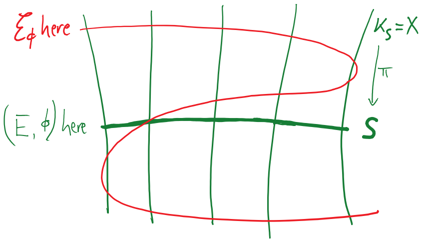

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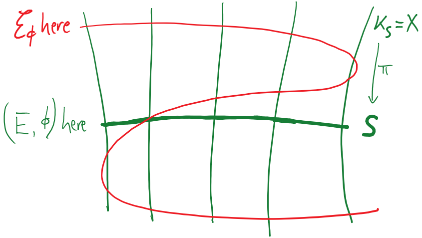

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$$\det E = \mathcal{O}_S, \ \mathrm{tr}\, \phi = 0 \iff$$
$\mathcal{E}$ has centre of mass 0 on each fibre, and $\det \pi_* \mathcal{E} = \mathcal{O}_S$.

Virtual cycle

When stability = semistability, deformation-obstruction theory of sheaves $\mathcal{E}$ on Calabi-Yau 3-fold X is perfect, symmetric:

$$\begin{aligned} \text{Deformations} \quad & \operatorname{Ext}^1_X(\mathcal{E},\mathcal{E}) \\ \text{Obstructions} \quad & \operatorname{Ext}^2_X(\mathcal{E},\mathcal{E}) \;\cong\; \operatorname{Ext}^1_X(\mathcal{E},\mathcal{E})^* \\ \text{Higher obstructions} \quad & \operatorname{Ext}^{\geq 3}_X(\mathcal{E},\mathcal{E})_0 \;=\; 0 \end{aligned}$$

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$\mathbb{C}^*$-fixed locus compact, so can define an invariant by virtual $\mathbb{C}^*$-localisation. Local DT invariant.

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$\mathbb{C}^*$-localisation then defines an invariant

$$\mathrm{VW}_n \in \mathbb{Q}$$

when stability $=$ semistability.

$\mathbb{C}^*$-fixed loci

Invariant computed from two types of $\mathbb{C}^*$-fixed locus:

1. $\phi = 0$. We get the moduli space $\mathcal{M}^{asd}$ of stable sheaves on S.
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Unstudied. Our (limited) computations give more modular forms predicted by Vafa-Witten 100 years ago by "cosmic strings".

The Vafa-Witten prediction

For general type surfaces with a smooth connected canonical divisor, [VW] predicts

The formula we propose is then

$$
\begin{aligned}
Z_x = {} & \left(\frac{1}{4}G(q^2)\right)^{\nu/2}\left(\delta_{x,0}(-1)^{\nu}\left(\frac{\theta_0}{\eta^2}\right)^{1-g} + \delta_{x,x_0}\left(\frac{\theta_1}{\eta^2}\right)^{1-g}\right) \\
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In particular, this convinces us that our virtual localisation definition is the right one. An alternative definition via Behrend-weighted Euler characteristic has various advantages (integers in stable case, natural generalisation to semistable case, natural refinement and categorification) and also gives modular forms, but the wrong ones.

Semistable case

Motivated by Mochizuki and Joyce-Song, we consider pairs

$$(\mathcal{E}, s)$$

of a torsion sheaf $\mathcal{E}$ on X and a section $s \in H^0(\mathcal{E}(n)),\ n \gg 0,$
$\big(\mathcal{E}$ has centre of mass 0 on the fibres of $X \to S$, and $\det \pi_* \mathcal{E} \cong \mathcal{O}_S.\big)$
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which are stable

- $\mathcal{E}$ is semistable,
- if $\mathcal{F} \subsetneq \mathcal{E}$ has the same Giesker slope then s does not factor through $\mathcal{F}$.

These also admit a perfect symmetric obstruction theory governed by

$$\operatorname{Ext}^*_X(I^\bullet, I^\bullet)_\perp \qquad \text{where} \quad I^\bullet \ = \ \big\{\mathcal{O}_X(-N) \to \mathcal{E}\big\}.$$

Invariants in the semistable case

Again we use virtual $\mathbb{C}^*$-localisation to define pairs invariants $P^{\perp}_{\alpha}(n)$, where $\alpha = (\mathrm{rank}(E), c_1(E), c_2(E))$.

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From these we define VW invariants by the conjectural formula

$$P^{\perp}_{\alpha}(n) \;=\; \sum_{\substack{\ell\geq 1,\,(\alpha_i=\delta_i\alpha)_{i=1}^{\ell}:\\ \sum_{i=1}^{\ell}\delta_i=1}} \frac{(-1)^{\ell}}{\ell!}\prod_{i=1}^{\ell}(-1)^{\chi(\alpha_i(n))}\chi(\alpha_i(n))\mathsf{VW}_{\alpha_i}(S)$$

when $H^{0,1}(S) = 0 = H^{0,2}(S)$. If either is $\neq 0$ we instead use only the first term

$$P^{\perp}_{\alpha}(n) \;=\; (-1)^{\chi(\alpha(n))}\chi(\alpha(n))\mathsf{VW}_{\alpha}(S).$$

Results

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- When $\deg K_S < 0$ the same is true. (Here we prove the pairs moduli space is smooth, and the invariants equal those defined by (Behrend-weighted) Euler characteristic. To these we can apply Joyce-Song's work.)
- When S is a K3 surface the same is true. (Joint work with Davesh Maulik. We work on compact $S \times E$, where Behrend-weighted Euler characteristic invariants equal virtual invariants. We then degenerate E to a nodal rational curve to access $S \times \mathbb{C}$. This introduces an exponential, which accounts for the difference between our simplified pairs formula and Joyce-Song's.)

Refinement

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For invariants defined by virtual localisation we need to do something different. Maulik proposes taking the virtual K-theoretic invariant (in the stable = semistable case)

$$\chi\Big(\big(K^{\mathrm{vir}}_{\mathcal{M}_{VW}}\big)^{\frac{1}{2}}\Big)$$

defined by K-theoretic localisation, and refining it to a polynomial in $t^{1/2}$ by using the $\mathbb{C}^*$ action.

Refinement

(Behrend-weighted) Euler characteristics e have natural refinements H^* (of the perverse sheaf of vanishing cycles).

For invariants defined by virtual localisation we need to do something different. Maulik proposes taking the virtual K-theoretic invariant (in the stable = semistable case)

$$\chi\Big(\big(K^{\mathrm{vir}}_{\mathcal{M}_{VW}}\big)^{\frac{1}{2}}\Big)$$

defined by K-theoretic localisation, and refining it to a polynomial in $t^{1/2}$ by using the $\mathbb{C}^*$ action.

Refinement

By (virtual) Riemann-Roch this amounts to replacing the virtual localisation definition

$$\mathsf{VW} \;=\; \int_{\left[\mathcal{M}_{VW}^{\mathbb{C}^*}\right]^{\mathsf{vir}}} \frac{1}{e(N^{\mathsf{vir}})}$$

by

$$\int_{[\mathcal{M}_{VW}^{\mathbb{C}^*}]} \frac{\mathrm{ch}\big((K_{\mathcal{M}_{VW}}^{\mathsf{vir}})^{\frac{1}{2}}\big)\,\mathsf{Td}\left(T_{\mathcal{M}_{VW}}^{\mathsf{vir}}\right)}{e(N^{\mathsf{vir}})}\,.$$

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Both are integrals in equivariant cohomology taking values in $H^*(B\mathbb{C}^*) \cong \mathbb{Z}[t]$ (localised and extended to $\mathbb{Q}[t^{\pm\frac{1}{2}}]$). The first is a constant, whereas the second can be a more general Laurent polynomial in $t^{1/2}$.

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On $\mathcal{M}_2$ computations are work in progress, trying K-theoretic cosection localisation, and refined Carlsson-Okounkov operators.

Nested Hilbert schemes

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We get the nested Hilbert scheme

$$S^{[n_1,n_2]} \;=\; \left\{\mathcal{I}_1 \subseteq \mathcal{I}_2 \subset \mathcal{O}_S \colon\ \mathrm{length}(\mathcal{O}_S/\mathcal{I}_i) = n_i\right\}$$

embedded in $S^{[n_1]} \times S^{[n_2]}$ as the locus of ideals $(\mathcal{I}_1, \mathcal{I}_2)$ where

$$\mathrm{Hom}_S(\mathcal{I}_1, \mathcal{I}_2) \;\neq\; 0.$$

Carlsson-Okounkov operators

In this way we can see $S^{[n_1,n_2]} \overset{\iota}{\hookrightarrow} S^{[n_1]} \times S^{[n_2]}$ as the degeneracy locus of the complex of vector bundles $R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)$.

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By the Thom-Porteous formula for the degeneracy locus, this gives

$$\iota_*\big[S^{[n_1,n_2]}\big]^{\mathrm{vir}} \;=\; c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big)$$

on $S^{[n_1]} \times S^{[n_2]}$. The latter has been computed by Carlsson-Okounkov in terms of Grojnowski-Nakajima operators on $H^*(S^{[*]})$.