A Variational Approach to the Regularity of the Fractional Obstacle Problem

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Abstract. This are lecture notes for the course *A variational approach to the regularity of the fractional obstacle problem* that I taught at the , held in Warwick, June 10th - 16th 2017.

Contents

1. Introduction

1.1. The fractional Laplacian and its obstacle problem. We define the fractional Laplacian as follows: let $s \in (0,1)$ be a fixed constant and, for every $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\,\mathrm{d}x < +\infty,$$

we set

$$\begin{aligned}(-\Delta)^s f(x) &:= c(n,s)\mathrm{PV}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|y|^{n+2s}}\,\mathrm{d}y \\ &= c(n,s)\mathrm{PV}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{f(x)-f(y)}{|y|^{n+2s}}\,\mathrm{d}y, \qquad (1.1)\end{aligned}$$

where the constant $c(n,s) > 0$ is given by

$$c(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos(y_1)}{|y|^{n+2s}}\,\mathrm{d}y\right)^{-1}.$$

The fractional obstacle problem is then the following: let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, we look at a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases}\min\left\{f-\phi, (-\Delta)^s f(x)\right\} = 0 & \text{in } \mathbb{R}^n,\\ \lim_{|x|\to+\infty} f(x) = 0.\end{cases} \qquad (1.2)$$

1.2. The local version: the lower dimensional obstacle problem. Although the fractional Laplacian is a non-local operator, one can use the so called *extension method* to write it a local operator in a space with one extra variable. More precisely, let us consider the half space $\mathbb{R}^{n+1}_+ := \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ and the (degenerate) elliptic boundary value problem

$$\begin{cases}\mathrm{div}\big(x_{n+1}^a \nabla u(x)\big) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ u(x',0) = f(x') & \forall\, x' \in \mathbb{R}^n \times \{0\},\end{cases} \qquad (1.3)$$

where $a := 1-2s \in (-1,1)$.

Lemma 1.2.1 (Caffarelli–Silvestre [5]). *There exists a dimensional constant $C > 0$ such that, for every $f \in$, we have*

$$(-\Delta)^s f(x') = C \lim_{x_{n+1}\downarrow 0^+} x_{n+1}^a \frac{\partial u}{\partial x_{n+1}}(x', x_{n+1}). \qquad (1.4)$$

Proof. □

Without loss of generality, we can consider the functions u extended to the whole $\mathbb{R}^{n+1}$ evenly:

$$u(x', x_{n+1}) = u(x', -x_{n+1}) \quad \forall\, x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}.$$

Eq. (1.3) is the Euler–Lagrange equation of the functional

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u(x)|^2 \, x_{n+1}^a \, \mathrm{d}x$$

In particular, the function u can be found as the the minimizer of the above energy with constraint $u(x',0) = f(x')$.

1.3. The scalar Signorini problem. As far as the local regularity of the solution to the fractional obstacle problem is concerned, we can look at the function u which is a minimizer (actually the unique minimizer) with respect to its own boundary conditions of the weighted Dirichlet energy

$$\min \int_{B_R} |\nabla u(x)|^2 \, |x_{n+1}|^a \, \mathrm{d}x \quad : \quad u(x',0) \geq \phi(x'). \tag{1.5}$$

Such problem is sometimes called the *lower dimensional obstacle problem*, because the constraint $u(x',0) \geq \phi(x')$ is given on a low dimensional submanifold.

The above problem also arises in elasticity theory and in the case of $s = \frac{1}{2}$ (*i.e.* $a = 0$) it is called the *scalar Signorini problem*.

In this perspective, one is also led to consider the simplest version of such a problem, namely the case of zero obstacle $\phi \equiv 0$ and boundary value $u|_{\partial B_r} = g$ with $g \geq 0$ on $\partial B_R \cap \{x_{n+1} = 0\}$. This is the problem we consider: given any boundary value $g \in H^1(B)$ even symmetric with respect to x_{n+1} and with $g|_{B_1'} > 0$, we consider the minimization problem

$$\min_{u \in \mathcal{A}_g} \int_B |\nabla u(x)|^2 \mathrm{d}x, \tag{1.6}$$

where

$$\mathcal{A}_g := \left\{ v \in g + H^1_0(B_1) : v|_{B_1'} \geq 0, \; v(x', x_{n+1}) = v(x', -x_{n+1}) \right\}.$$

It follows from the direct method in the calculus of variations and from the convexity of the energy that there exists a unique minimizer to the Signorini problem (1.6) and that it is even symmetric with respect to x_{n+1}.

The main questions concerning the solutions to the thin obstacle problem we would like to address are those regarding the regularity. In this regard, we need to distinguish between two kind of regularity:

(a) the regularity of the solution u itself; namely, whether for smooth obstacles is the solutions also smooth, or if not which the best regularity we can hope for;
(b) the regularity of the *free boundary* $\Gamma(u)$, *i.e.* of the relative boundary of the coincide set $\Lambda(u)$:

$$\Lambda(u) := \left\{ x' \in B_R' \; : \; u(x',0) = \phi(x') \right\}$$

and

$$\Gamma(u) := \partial_{\{x_{n+1}=0\}}\big\{x' \in B_R' \,:\, u(x',0) = \phi(x')\big\}.$$

2. Optimal regularity of the solutions

In this chapter we prove the following result.

THEOREM 2.0.1. *Let $u \in H^1(B_1)$ be a solution to the lower dimensional obstacle problem. Then, $u \in C^{1,1/2}_{\mathrm{loc}}(B_1^+ \cup B_1')$ and*

$$\|u\|_{C^{1,1/2}(B^+_{1/2}\cup B'_{1/2})} \leq C(n)\,\|u\|_{L^2(B_1)},$$

where $C(n) > 0$ is a dimensional constant.

In particular, this gives a precise meaning to the Signorini "ambiguous boundary conditions".

COROLLARY 2.0.2. *Every solution u to the lower dimensional obstacle problem is characterized by the following set of equalities and inequalities:*

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+, \\ u \geq 0, \quad -\partial_{n+1}u \geq 0, \quad u\,\partial_{n+1}u = 0 & \text{on } B_1', \end{cases} \tag{2.1}$$

where the value of $\partial_{n+1}u$ on B_1' is well-defined according to the regularity of Theorem 2.0.1.

REMARK 2.0.3. The regularity in Theorem 2.0.1 is optimal: in the sense that there exists solutions u_0 which are not $C^{1,\alpha}$ for any $\alpha > \frac{1}{2}$: e.g.,

$$u_0(x) = \left(2\,x_1 - \sqrt{x_1^2 + x_{n+1}^2}\right)\sqrt{\sqrt{x_1^2 + x_{n+1}^2} + x_1}.$$

2.1. $W^{2,2}$-theory: penalization method.

We consider a smooth function $\beta : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\beta(t) = 0 \quad \forall\, t \geq 0, \qquad \beta'(t) \geq 0 \quad \forall\, t \in \mathbb{R}, \qquad \beta''(t) \leq 0 \quad \forall\, t \in \mathbb{R},$$

and

$$\beta(t) = 2t + 1 \quad \forall\, t \leq -2 \quad \text{and} \quad |\beta(t)| \leq 2\,|t| \quad \forall t \in \mathbb{R}.$$

We set

$$\beta_\varepsilon(t) := \varepsilon^{-1}\beta\big({}^{t}\!/\!_{\varepsilon}\big).$$

Then, one can consider the solutions of following boundary value problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } B_1^+, \\ -\partial_{n+1}u_\varepsilon + \beta_\varepsilon\big(u_\varepsilon\big) = 0 & \text{in } B_1', \\ u_\varepsilon = g & \text{in } (\partial B_1)^+. \end{cases} \tag{2.2}$$

The weak solutions to (2.2)

$$\int_{B_1^+} \nabla u_\varepsilon \cdot \nabla \eta \,\mathrm{d}x = -\int_{B_1'} \beta_\varepsilon(u_\varepsilon)\,\eta\,\mathrm{d}x' \quad \forall\, \eta \in H^1_0(B_1) \tag{2.3}$$

are the unique minimizer of the following variational problem:

$$\min_{v\in H^1(B_1)\, v|_{\partial B_1}=g} \quad \frac{1}{2}\int_{B_1^+} |\nabla v(x)|^2 \mathrm{d}x + \int_{B_1'} F_\varepsilon(v(x',0))\,\mathrm{d}x', \tag{2.4}$$

where F_ε is a primitive of the function β_ε:

$$F_\varepsilon(t) := \begin{cases} 0 & t\geq 0,\\ -\int_t^0 \beta_\varepsilon(s)\,\mathrm{d}s & t<0. \end{cases}$$

Note that $F_\varepsilon \geq 0$ and $F_\varepsilon(t)\leq C\varepsilon^{-1}|t|^2$ for a dimensional constant $C>0$. It is then simple to verify that the energies in (2.4) are coercive, lower semicontinuous and convex, and therefore there exists a unique minimizer $u_\varepsilon \in H^1(B_1^+)$. Without loss of generality, we can extend it to the whole B_1 by even reflection.

Moreover, we have the following.

LEMMA 2.1.1. *Let g, u, u_ε be as above. Then, u_ε converges to u in $L^2(B_1)$ and there exists a constant $C>0$ such that*

$$\|u_\varepsilon\|_{H^1(B)} \leq C\,\|u\|_{H^1(B)} \quad \forall\, \varepsilon>0. \tag{2.5}$$

PROOF. We start noticing that

$$\begin{aligned}\frac{1}{2}\int_{B_1^+}|\nabla u_\varepsilon(x)|^2\mathrm{d}x &\leq \frac{1}{2}\int_{B_1^+}|\nabla u_\varepsilon(x)|^2\mathrm{d}x + \int_{B'} F_\varepsilon(u_\varepsilon(x',0))\,\mathrm{d}x'\\ &\leq \frac{1}{2}\int_{B_1^+}|\nabla u(x)|^2\mathrm{d}x + \int_{B'} F_\varepsilon(u(x',0))\,\mathrm{d}x'\\ &= \frac{1}{2}\int_{B_1^+}|\nabla u(x)|^2\mathrm{d}x.\end{aligned}$$

Therefore, we deduce the existence of a constant $C>0$ (depending on g) such that (2.5) holds.

We test now (2.3) with $\eta := u_\varepsilon\,\zeta^2$ where $\zeta\in C_c^1(B)$ is a cut-off function: it follows that

$$\begin{aligned}\int_{B_1'}\beta_\varepsilon(u_\varepsilon)\,u_\varepsilon\,\zeta^2\,\mathrm{d}x' &= -\int_{B_1^+}\nabla u_\varepsilon\cdot\nabla\big(u_\varepsilon\,\zeta^2\big)\,\mathrm{d}x\\ &\leq \int_{B_1^+}\Big(2\,|\nabla u_\varepsilon|^2\,\zeta^2 + 4\,u_\varepsilon^2\,|\nabla\zeta|^2\Big)\,\mathrm{d}x \overset{(2.5)}{\leq} C,\end{aligned}$$

where the constant $C>0$ depends on u and ζ. We therefore deduce that for every $\delta>0$

$$\mathcal{H}^n\Big(\{u_\varepsilon<-\delta\}\cap B_1'\cap\{\zeta=1\}\Big)|\beta_\varepsilon(-\delta)|\,\delta\leq C.$$

Since $|\beta_\varepsilon(t)|\geq |2\,\varepsilon^{-2}|t|+\varepsilon^{-1}|$ for $t<-2\varepsilon$, we deduce that for $2\varepsilon<\delta$

$$\mathcal{H}^n\Big(\{u_\varepsilon<-\delta\}\cap B_1'\cap\{\zeta=1\}\Big)\leq C\,\frac{\varepsilon^2}{\delta\,|\varepsilon-2\,\delta|}.$$

In particular, any weak limit w of u_ε satisfies $w|_{B_1'} \geq 0$ and

$$\int_{B_1^+} |\nabla w(x)|^2 \mathrm{d}x \leq \liminf_{\varepsilon \to 0^+} \int_{B_1^+} |\nabla u_\varepsilon(x)|^2 \mathrm{d}x \leq \int_{B_1^+} |\nabla u(x)|^2 \mathrm{d}x.$$

Since u is the unique solution to the Signorini problem, we infer that $w = u$. $\square$

Next we show that u_ε are actually uniformly $H^2_{\mathrm{loc}}(B_1)$. To this aim we introduce the notation: for $i = 1, \ldots, n$ and $h \in \mathbb{R}$,

$$\tau_{h,i} u(x) := \frac{u(x + h\, e_i) - u(x)}{h}.$$

Proposition 2.1.2. *The solution to the thin obstacle problem u are $H^2_{\mathrm{loc}}(B_1^+)$ and there exists a dimensional constant $C > 0$ such that for every $\varepsilon > 0$*

$$\int_{B_r^+(x_0)} \frac{|\nabla(\partial_i u)|^2}{|x - x_0|^{n-1}} \, \mathrm{d}x \leq \frac{C}{r^{n+1}} \int_{B_{2r}^+ \setminus B_r^+(x_0)} |\partial_i u|^2 \, \mathrm{d}x, \tag{2.6}$$

for every $x_0 \in B'_{1/2}$ and for every $r \in (0, 1/4)$.

In particular, the Signorini ambiguous boundary conditions in Corollary 2.0.2 are satisfied in the sense of traces.

Proof. Without loss of generality it suffices to consider the case $x_0 = 0$. Let $\zeta \in C^1_c(B)$ be a test function with

$$\zeta \equiv 1 \text{ in } B_r,\ \zeta \equiv 0 \text{ in } B_1 \setminus B_{2r} \text{ and } |\nabla \zeta| \leq C\, r^{-1},$$

and for a small parameter $\delta > 0$ let

$$\Psi(x) := \min\left\{ |x|^{1-n}, \delta^{1-n} \right\}.$$

We test (2.3) with $\eta := \tau_{-h,i}\big(\tau_{h,i} u_\varepsilon \Psi \zeta^2\big)$ and $i = 1, \ldots, n$. In the following we omit to write the index i and use the change of variables at the base of the partial integration for the discrete derivatives

$$\int (\tau_h f)\, g \, \mathrm{d}x = \int f\, (\tau_{-h} g) \, \mathrm{d}x,$$

and the fact that τ_h and ∇ commute: $\nabla(\tau_h) f = \tau_h(\nabla f)$. We have

$$\begin{aligned} \int_{B_1^+} \nabla u_\varepsilon \cdot \nabla \eta \, \mathrm{d}x &= \int_{B_1^+} \nabla(\tau_h u_\varepsilon) \cdot \nabla\big((\tau_h u_\varepsilon) \Psi \zeta^2\big) \, \mathrm{d}x \\ &= \int_{B_1^+} |\nabla(\tau_h u_\varepsilon)|^2 \Psi \zeta^2 \, \mathrm{d}x + I + II, \end{aligned} \tag{2.7}$$

with

$$I = \int_{B_1^+} (\tau_h u_\varepsilon) \zeta^2 \, \nabla(\tau_h u_\varepsilon) \cdot \nabla \Psi \, \mathrm{d}x,$$

$$II = 2 \int_{B^+} (\tau_h u_\varepsilon)\, \Psi\, \zeta\, \nabla(\tau_h u_\varepsilon) \cdot \nabla \zeta \, \mathrm{d}x.$$

II can be estimated via Hölder as follows:

$$|II| \le \frac{1}{2}\int_{B_1^+} |\nabla(\tau_h u_\varepsilon)|^2 \Psi \zeta^2 \,\mathrm{d}x + C\int_{B_1^+} (\tau_h u_\varepsilon)^2 |\nabla \zeta|^2\, \Psi \,\mathrm{d}x.$$

For what concerns I, we make an integration by parts (we used $\Delta\Psi = 0$ in $B-1 \setminus B_\delta$):

$$\begin{aligned}
I &= \frac{1}{2}\int_{B_1^+\setminus B_\delta} \zeta^2\, \nabla(\tau_h u_\varepsilon)^2 \cdot \nabla\Psi \,\mathrm{d}x\\
&= -\int_{B_1^+\setminus B_\delta} \zeta\, (\tau_h u_\varepsilon)^2 \nabla\Psi\cdot\nabla\zeta\,\mathrm{d}x - \frac{1}{2}\int_{B_1'\setminus B_\delta} \zeta^2\,(\tau_h u_\varepsilon)^2 \nabla\Psi\cdot e_n \,\mathrm{d}x'\\
&\quad - \frac{1}{2}\int_{(\partial B_\delta)^+} \zeta^2\,(\tau_h u_\varepsilon)^2 \nabla\Psi\cdot \frac{x}{|x|}\,\mathrm{d}\mathcal{H}^n(x)\\
&= -\int_{B_1^+\setminus B_\delta} \zeta\, (\tau_h u_\varepsilon)^2 \nabla\Psi\cdot\nabla\zeta\,\mathrm{d}x - \frac{(1-n)}{2\,\delta^n}\int_{(\partial B_\delta)^+} \zeta^2\,(\tau_h u_\varepsilon)^2 \,\mathrm{d}\mathcal{H}^n(x)\\
&\ge -\int_{B_1^+\setminus B_\delta} \zeta\, (\tau_h u_\varepsilon)^2 \nabla\Psi\cdot\nabla\zeta\,\mathrm{d}x \ge -\frac{C}{r^{n+1}}\int_{B_{2r}^+\setminus B_r^+} (\tau_h u_\varepsilon)^2\,\mathrm{d}x.
\end{aligned}$$

Next we note that

$$\begin{aligned}
&-\int_{B_1'} \beta_\varepsilon(u_\varepsilon)\tau_{-h}(\tau_h u_\varepsilon)\Psi\zeta^2\,\mathrm{d}x' = -\int_{B_1'} \tau_h\big(\beta_\varepsilon(u_\varepsilon)\big)(\tau_h u_\varepsilon)\Psi\zeta^2\,\mathrm{d}x'\\
&\quad = -\int_{B'} \frac{\beta_\varepsilon(u_\varepsilon(x+he_i)) - \beta_\varepsilon(u_\varepsilon(x))}{h}\,\frac{u_\varepsilon(x+he_i) - u(x)}{h}\Psi\zeta^2\,\mathrm{d}x' \le 0.
\end{aligned}$$

In particular, we have derived that

$$\begin{aligned}
\int_{B_1^+} |\nabla(\tau_h u_\varepsilon)|^2\Psi\zeta^2\,\mathrm{d}x &= -I - II - \int_{B_1'} \beta_\varepsilon(u_\varepsilon)\tau_{-h}(\tau_h u_\varepsilon)\Psi\zeta^2\,\mathrm{d}x'\\
&\le \frac{C}{r^{n+1}}\int_{B_{2r}^+\setminus B_r^+} (\tau_h u_\varepsilon)^2\,\mathrm{d}x + \frac{1}{2}\int_{B_1^+} |\nabla(\tau_h u_\varepsilon)|^2\Psi\zeta^2\,\mathrm{d}x,
\end{aligned}$$

from which

$$\int_{B_1^+} |\nabla(\tau_h u_\varepsilon)|^2\Psi\zeta^2\,\mathrm{d}x \le \frac{C}{r^{n+1}}\int_{B_{2r}^+\setminus B_r^+} (\tau_h u_\varepsilon)^2\,\mathrm{d}x.$$

In particular, it follows that $\partial_i\partial_j u_\varepsilon$ exists in $L^2_{\mathrm{loc}}(B_1^+)$ for every $1 = 1,\dots,n$ and for every $j = 1,\dots,n+1$, with uniform bounds. In particular, using $\Delta u_\varepsilon = 0$ in B_1^+, we also infer $\partial_{n+1}\partial_{n+1} u_\varepsilon \in L^2_{\mathrm{loc}}(B_1^+)$.

For what concerns the estimate relative to $\nabla(\partial_{n+1}u_\varepsilon)$, we test the equation (2.3) with $\eta := \partial_{n+1}\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)$:

$$
\begin{aligned}
-\int_{B_1'}\beta_\varepsilon(u_\varepsilon)\partial_{n+1}\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x' &= \int_{B_1'}\partial_{n+1}u_\varepsilon\,\partial_{n+1}\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x'\\
&= \int_{B_1^+}\nabla u_\varepsilon\cdot\nabla\Big(\partial_{n+1}\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\Big)\,\mathrm{d}x\\
&= -\int_{B_1^+}\nabla\big(\partial_{n+1}u_\varepsilon\big)\cdot\nabla\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x-\int_{B_1'}\nabla u_\varepsilon\cdot\nabla\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x'.
\end{aligned}
$$

Therefore we deduce that (we denote with ∇' the derivatives in the horizontal directions)

$$
\int_{B_1^+}\nabla\big(\partial_{n+1}u_\varepsilon\big)\cdot\nabla\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x = -\int_{B_1'}\nabla' u_\varepsilon\cdot\nabla'\big(\partial_{n+1}u_\varepsilon\,\Psi\,\zeta^2\big)\,\mathrm{d}x' \tag{2.8}
$$

$$
= -\int_{B_1'}\nabla' u_\varepsilon\cdot\nabla'\big(\beta_\varepsilon(u_\varepsilon)\,\Psi\,\zeta^2\big)\,\mathrm{d}x' \tag{2.9}
$$

$$
= -\int_{B_1'}|\nabla' u_\varepsilon|^2\,\beta_\varepsilon'(u_\varepsilon)\,\Psi\,\zeta^2\,\mathrm{d}x' - \int_{B'}\beta_\varepsilon(u_\varepsilon)\nabla' u_\varepsilon\cdot\nabla'(\Psi\,\zeta^2)\,\mathrm{d}x' \tag{2.10}
$$

$$
\leq -\int_{B'}\beta_\varepsilon(u_\varepsilon)\nabla' u_\varepsilon\cdot\nabla'(\Psi\,\zeta^2)\,\mathrm{d}x'. \tag{2.11}
$$

By the compact embeddings we have that $\beta_\varepsilon(u_\varepsilon) = \partial_n u_\varepsilon$ strongly converge in $L^2(B_1')$ to $\partial_{n+1}u$ and $\nabla' u_\varepsilon$ weakly converge in $L^2(B_1')$ to $\nabla' u$. Therefore, from the Signorini ambiguous boundary conditions we deduce that

$$
\lim_{\varepsilon\to 0}\int_{B_1'}\beta_\varepsilon(u_\varepsilon)\nabla' u_\varepsilon\cdot\nabla'(\Psi\,\zeta^2)\,\mathrm{d}x' = 0.
$$

We can then argue as above in (2.8) and deduce that

$$
\int_{B_1^+}|\nabla(\partial_{n+1}u)|^2\Psi\zeta^2\,\mathrm{d}x \leq \frac{C}{r^{n+1}}\int_{B_{2r}^+\setminus B_r^+}(\partial_{n+1}u)^2\,\mathrm{d}x.
$$

□

2.2. $C^{1,\alpha}$-regularity: hole-filling technique. The next step is to show the following intermediate regularity.

Theorem 2.2.1. *Let u be a solution to the Signorini problem. Then, there exists a constant $\alpha\in(0,1)$ such that $u\in C^{1,\alpha}_{\mathrm{loc}}(B_1^+\cup B_1')$ and*

$$
\|u\|_{C^{1,\alpha}(B_{1/2}^+\cup B_{1/2}')} \leq C\|u\|_{L^2(B_1^+)}. \tag{2.12}
$$

Proof. We consider the following integral quantities:

$$
\mathrm{I}(x_0,r) := \int_{B_r^+(x_0)}\sum_{i=1}^{n}\frac{|\nabla(\partial_i u)|^2}{|x-x_0|^{n-1}}\,\mathrm{d}x,
$$

and

$$\mathrm{II}(x_0,r) := \int_{B_r^+(x_0)} \frac{|\nabla(\partial_{n+1}u)|^2}{|x-x_0|^{n-1}}\,\mathrm{d}x.$$

By Proposition 2.1.2 we have that

$$\mathrm{I}(x_0,r) \le \frac{C}{r^{n+1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\nabla' u|^2\,\mathrm{d}x$$

and

$$\mathrm{II}(x_0,r) \le \frac{C}{r^{n+1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\partial_{n+1} u|^2\,\mathrm{d}x.$$

From the Signorini boundary conditions, for every $r \in (0, 1/4)$ we have to consider two possibilities: either

$$\mathcal{H}^n\big(\Lambda(u)\cap B'_{2r}\setminus B'_r(x_0)\big) \ge \frac{\mathcal{H}^n\big(B'_{2r}\setminus B'_r(x_0)\big)}{2},$$

or

$$\mathcal{H}^n\big(\{\partial_{n+1}u=0\}\cap B'_{2r}\setminus B'_r(x_0)\big) \ge \frac{\mathcal{H}^n\big(B'_{2r}\setminus B'_r(x_0)\big)}{2}.$$

Using a Poincarè-type inequality we have in the first case

$$\begin{aligned}\mathrm{I}(x_0,r) &\le \frac{C}{r^{n+1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\nabla' u|^2\,\mathrm{d}x \le \frac{C}{r^{n-1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\nabla(\nabla')u|^2\,\mathrm{d}x \\ &\le C\Big(\mathrm{I}(x_0,2r)-\mathrm{I}(x_0,r)\Big).\end{aligned}$$

In the second case we have that

$$\begin{aligned}\mathrm{II}(x_0,r) &\le \frac{C}{r^{n+1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\partial_n u|^2\,\mathrm{d}x \le \frac{C}{r^{n-1}} \int_{B_{2r}^+\setminus B_r^+(x_0)} |\nabla\partial_n u|^2\,\mathrm{d}x \\ &\le C\big(\mathrm{II}(x_0,2r)-\mathrm{II}(x_0,r)\big).\end{aligned}$$

In both cases we can add $C\mathrm{I}(x_0,r)$ or $C\mathrm{II}(x_0,r)$ to both sides and infer that for every $r \in (0, 1/4)$

$$\text{either}\quad \mathrm{I}(x_0,r) \le \theta\,\mathrm{I}(x_0,2r) \quad\text{or}\quad \mathrm{II}(x_0,r) \le \theta\,\mathrm{II}(x_0,2r) \tag{2.13}$$

with $\theta := \frac{C}{C+1} \in (0,1)$.

We claim that (2.13) leads to the following:

$$\mathrm{II}(x_0,r) \le C\,r^{2\alpha} \quad \forall\, r \in (0, 1/4),$$

for some constants $C, \alpha > 0$. Indeed, consider any $r \in (0, 1/4)$ and let $k \in \mathbb{N}$ be such that $r \in [4^{-k-1}, 4^{-k})$. Set for convenience $r_l := 2^{-l}$ for $l = 2, \ldots, 2k$. Then, there exists at least k radii r_l, say $\{r_{l_j}\}_{j=1}^M$ with $M \ge k$ and $l_j \le l_{j+1}$, such that at lest one between I and II decays. In particular, we deduce that

$$\mathrm{I}(x_0,r_{l_j}) \le \theta\,\mathrm{I}(x_0,2\,r_{l_j}) \le \theta\,\mathrm{I}(x_0,r_{l_{j-1}})$$

or

$$\mathrm{II}(x_0,r_{l_j}) \le \theta\,\mathrm{II}(x_0,2\,r_{l_j}) \le \theta\,\mathrm{II}(x_0,r_{l_{j-1}}).$$

Iterating this inequality we deduce that

$$\min\left\{\frac{\mathrm{I}(x_0,r)}{\mathrm{I}(x_0,1/4)},\frac{\mathrm{II}(x_0,r)}{\mathrm{II}(x_0,1/4)}\right\}\le\theta^k\le r^{2\alpha},$$

with $\alpha=-\frac{\log\theta}{4\log 4}$. In particular, considering that $\mathrm{I}(x_0,1/4)+\mathrm{II}(x_0,1/4)\le C\,\|Du\|_{L^2(B_1)}$, we deduce that

$$\min\big\{\mathrm{I}(x_0,r),\mathrm{II}(x_0,r)\big\}\le C\,\|Du\|_{L^2(B_1)}r^{2\alpha}.$$

Finally, note that since $\partial_{n+1}\partial_{n+1}u=-\sum_{i=1}^n\partial_{ii}u$ we also deduce that

$$\mathrm{II}(x_0,r)\le(n+1)\,\mathrm{I}(x_0,r)\le(n+1)\,\min\big\{\mathrm{I}(x_0,r),\mathrm{II}(x_0,r)\big\}\le C\,\|Du\|_{L^2(B_1)}r^{2\alpha}.$$

In particular,

$$\int_{B_r(x_0)^+}|\nabla\partial_{n+1}u|^2\le C\,r^{n-2+2\alpha},$$

and from Morrey's inequality it follows that $\partial_{n+1}u\in C^{0,\alpha}_{\mathrm{loc}}(B_1')$ and from Schauder estimates $u\in C^{1,\alpha}_{\mathrm{loc}}(B_1^+)$. □

2.3. Almgren's frequency function. We introduce the following integral quantities: for every $x_0\in B_1'$ and $r\in\big(0,\mathrm{dist}(x_0,\partial B_1)\big)$, we set

$$D(x_0,r):=\int_{B_r(x_0)}|\nabla u|^2\,\mathrm{d}x\quad\text{and}\quad H(x_0,r):=\int_{\partial B_r(x_0)}u^2\,\mathrm{d}\mathcal{H}^n,$$

and, if $H(x_0,r)>0$ (that is always the case for the solutions to the Signorini problem, unless $u\equiv 0$), we define

$$I(x_0,r):=\frac{r\,D(x_0,r)}{H(x_0,r)}.$$

The function I is called *Almgren's frequency function.*

Proposition 2.3.1. *Let u be a nonzero solution to the Signorini problem. Then,*

$$I(x_0,r_0)\le I(x_0,r_1)\quad\forall\,x_0\in B_1',\quad\forall\,0<r_0<r_1<\mathrm{dist}\big(x_0,\partial B_1\big).$$

Moreover, if there exist $x_0\in B_1'$, $k\in\mathbb{R}$ and $0<r_0<r_1<\mathrm{dist}(x_0,\partial B_1)$ such that $I(x_0,r)=k$ for all $r\in(r_0,r_1)$, then the solution u is k-homogeneous around x_0, i.e. there exists $w:\partial B_1\to\mathbb{R}$ such that

$$u(x)=|x-x_0|^k\,w\left(\frac{x-x_0}{|x-x_0|}\right)\quad\forall\,x\in B_1.$$

Proof. Without loss of generality we consider $x_0=0$. Note that the functions $D(r)$, $H(r)$ and $I(r)$ are absolutely continuous and we can compute as follows (we set $\nu(x):=x/|x|$ for the outward unit normal to B_r):

$$H'(r)=\frac{n}{r}\,H(r)+2\int_{\partial B_r}u\,u_\nu\mathrm{d}\mathcal{H}^n,\tag{2.14}$$

(use the change of variable $x = ry$, $y \in \partial B_1$);

$$D(r) = \int_{\partial B_r} u\, u_\nu \mathrm{d}\mathcal{H}^n, \tag{2.15}$$

(use the identity $\Delta(u^2) = 2|\nabla u|^2 + 2\,u\,\Delta u$ and the Signorini boundary conditions);

$$\begin{aligned}
D'(r) &= \int_{\partial B_r} |\nabla u|^2\,\mathrm{d}x = 2\,r^{-1}\int_{B_r^+} \mathrm{div}\big(|\nabla u|^2\,x\big)\,\mathrm{d}x \\
&= 2\,r^{-1}\int_{B_r^+}\Big((n+1)\,|\nabla u|^2 + 2\sum_{ij=1}^{n}\partial_{ij}u\partial_i u x_j\Big)\,\mathrm{d}x \\
&= 2\,r^{-1}\int_{B_r^+}\Big((n+1)\,|\nabla u|^2 - 2\sum_{ij=1}^{n}\partial_{ii}u\partial_j u x_j - 2\delta_{ij}\partial_i u\partial_j u\Big)\,\mathrm{d}x \\
&\quad + 4\,r^{-2}\sum_{ij=1}^{n}\int_{(\partial B_r)^+}\partial_i u\,x_i\,\partial_j u\,x_j\,\mathrm{d}\mathcal{H}^n - 4\,r^{-1}\int_{B_r'}\partial_{n+1}u\,(\nabla u\cdot x)\,\mathrm{d}x' \\
&= \frac{n-1}{r}\int_{B_r}|\nabla u|^2\,\mathrm{d}x + 2\int_{\partial B_r}(\partial_\nu u)^2\,\mathrm{d}\mathcal{H}^n,
\end{aligned}$$

where we used that $\Delta u = 0$ in B_r^+ and by the Signorini boundary conditions $\partial_n u(\nabla u\cdot x) = 0$ on B_r'. We can then derive that

$$\begin{aligned}
\frac{I'(r)}{I(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \\
&= 2\left(\frac{\int_{\partial B_r} u_\nu^2\,\mathrm{d}\mathcal{H}^n}{\int_{\partial B_r} u\,u_\nu \mathrm{d}\mathcal{H}^n} - \frac{\int_{\partial B_r} u\,u_\nu \mathrm{d}\mathcal{H}^n}{\int_{\partial B_r} u^2\,\mathrm{d}\mathcal{H}^n}\right) \geq 0,
\end{aligned}$$

the last inequality is due to the Cauchy–Schwarz inequality. In particular, if $I'(r) = 0$ for $r \in (r_0, r_1)$, then there exist a function $\lambda : (r_0, r_1) \to \mathbb{R}$ such that $u_\nu(x) = \lambda(|x|)u(x)$ for every $x \in B_{r_1} \setminus B_{r_0}$: in particular,

$$k = I(r) = \frac{rD(r)}{H(r)} = \frac{r\int_{\partial B_r} u\,u_\nu \mathrm{d}\mathcal{H}^n}{\int_{\partial B_r} u^2\,\mathrm{d}\mathcal{H}^n} = r\lambda(r) \quad \forall\, r \in (r_0, r_1).$$

We then deduce that $\nabla u(x)\cdot x = ku(x)$ for every $x \in B_{r_1} \setminus B_{r_0}$: by the Euler formula we get $u(x) = |x|^k w(x/|x|)$ for some function $w : \partial B_1 \to \mathbb{R}$ and by unique continuation for harmonic functions we conclude that this representation holds in all of B_1. □

A first consequence is the existence of homogeneous blowups at every free boundary point.

COROLLARY 2.3.2. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem and let $x_0 \in \Gamma(u)$. Then, for every infinitesimal sequence of decreasing radii $(r_i)_{i\in\mathbb{N}}$ with $r_0 \leq d_0 := \mathrm{dist}(x_0, \partial B_1)$, there exists a subsequence*

$(r_{i_k})_{k\in\mathbb{N}}$, *such that the rescaled functions* $u_{x_0,r_{i_k}} : B_{\frac{d_0}{r_{i_k}}} \to \mathbb{R}$ *defined by*

$$u_{x_0,r_{i_k}}(y) := \frac{r_{i_k}^{n/2} u(x_0 + r_{i_k} y)}{\int_{\partial B_{i_k}(x_0)} u^2 \,\mathrm{d}\mathcal{H}^n}$$

converge $C^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ *to a homogeneous function.*

Proof. It is immediate to verify that the functions u_{x_0,r_i} are solutions to the Signorini problem. Therefore, from Theorem 2.2.1 we have that the functions u_{x_0,r_i} are uniformly bounded in $C^{1,\alpha}$ for some $\alpha > 0$, from which the convergence up to subsequences follows.

In order to deduce the homogeneity of the limiting points u_{x_0}, we notice that

$$I_{u_{x_0}}(r) = \lim_{k\to+\infty} I_{u_{x_0,r_{i_k}}}(r) = \lim_{k\to+\infty} I_u(x_0, r_{i_k} r) = I_u(x_0, 0^+) \quad \forall\, r > 0.$$

In particular, $I_{u_{x_0}}(r)$ turns out to be constant and by Proposition 2.3.1 we deduce that u_{x_0} is homogeneous with respect to the origin and with homogeneity exponent $I_u(x_0, 0^+)$. □

Similarly, the following corollary will be useful later.

Corollary 2.3.3. *Let* $u \in H^1(B_1)$ *be a solution to the Signorini problem and let* $x_0 \in \Gamma(u) \cap B_{1/2}$ *and* $\lambda = I(x_0, 0^+)$. *Then,*

$$r^{-n-2\lambda} H(x_0, r) \le s^{-n-2\lambda} H(x_0, s) \tag{2.16}$$

for all $0 < r < s < 1/2$. *In particular, there exists a dimensional constant* $C > 0$ *such that*

$$\int_{B_r(x_0)} u^2 \,\mathrm{d}x \le C \|u\|_{L^2(B_1)} r^{n+1+2\lambda} \quad \forall\, r \in (0, 1/2). \tag{2.17}$$

Proof. Assume without loss of generality $x_0 = 0$. Using $r\,D(r) \ge \lambda\, H(r)$ and (2.14), we integrate the differential inequality

$$H'(r) = \frac{n}{r} H(r) + 2\, D(r) \ge \frac{n}{r} H(r) + \frac{\lambda}{r}$$

to obtain (2.16). Eq. (2.17) follows from (2.16), Theorem 2.0.1 and an integration in polar co-ordinates. □

2.4. Alt-Caffarelli-Friedman's monotonicity formula. Given any open set $S \subset \mathbb{S}^{d-1}$, let $\lambda(S)$ and v_λ be the first eigenvalue and the corresponding eigenfunction of the spherical Laplacian in S with Dirichlet boundary conditions:

$$\lambda(S) := \inf_{v\in H^1_0(S), v\neq 0} \frac{\int_S |\nabla_\tau v|^2 \,\mathrm{d}\mathcal{H}^{d-1}}{\int_S v^2 \,\mathrm{d}\mathcal{H}^{d-1}} = \frac{\int_S |\nabla_\tau v_\lambda|^2 \,\mathrm{d}\mathcal{H}^{d-1}}{\int_S v_\lambda^2 \,\mathrm{d}\mathcal{H}^{d-1}},$$

where $\nabla_\tau v$ denotes the (covariant) tangential derivative of $v : \mathbb{S}^{d-1} \to \mathbb{R}$. Corresponding to the eigenvalue $\lambda(S)$, one defines the *characteristic constant* $\alpha(S)$ given by the positive root of

$$\alpha^2 + \alpha\,(d-2) - \lambda = 0.$$

Note that $\alpha(S)$ is the homogeneity exponent of the harmonic extension of v_λ: writing in polar co-ordinates $u(r,\theta) := r^\alpha v_\lambda(\theta)$ we have that

$$\begin{aligned}\Delta u &= \frac{\mathrm{d}^2}{\mathrm{d}r^2}u + \frac{d-1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}u + \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}}u \\ &= \big(\alpha\,(\alpha-1) + (d-1)\,\alpha - \lambda\big)r^{\alpha-2}v_\lambda(\theta) = 0.\end{aligned}$$

We need the following result by Friedland and Hayman ([11]).

Theorem 2.4.1. *Let $S_1, S_2 \in \mathbb{S}^{d-1}$ be two disjoint open sets. Then,*

$$\alpha(S_1) + \alpha(S_2) \geq 2, \tag{2.18}$$

with equality if and only if S_1 and S_2 are two disjoint hemispheres.

The following is the monotonicity formula discovered by Alt, Caffarelli and Friedman [1].

Theorem 2.4.2. *Let u_1, u_2 be nonnegative continuous subharmonic functions in $B_1 \subset \mathbb{R}^d$. Assume that $u_1\,u_2 \equiv 0$ and $u_1(0) = u_1(0) = 0$. Then, the function*

$$J(r) := \frac{1}{r^4}\int_{B_r}\frac{|\nabla u_1(x)|^2}{|x|^{d-2}}\,\mathrm{d}x \int_{B_r}\frac{|\nabla u_2(x)|^2}{|x|^{d-2}}\,\mathrm{d}x$$

is monotone nondecreasing for $r \in (0,1)$ with

$$J(r) \leq C(n)\,\|u_1\|^2_{L^2(B_1)}\,\|u_2\|^2_{L^2(B_1)} \quad \forall\, r \in (0, 1/2).$$

Proof. We start establishing the following two inequalities: for $v = u_1$ or $v = u_2$, we have $v \in H^1_{\mathrm{loc}}(B_1)$ and for a.e. $r \in (0,1)$

$$\int_{B_r}\frac{|\nabla v|^2}{|x|^{d-2}}\,\mathrm{d}x \leq \frac{C}{r^d}\int_{B_{2r}\setminus B_r} v^2. \tag{2.19}$$

and

$$\int_{B_r}\frac{|\nabla v|^2}{|x|^{d-2}}\,\mathrm{d}x \leq \frac{d-2}{2r^{d-1}}\int_{\partial B_r} v^2\,\mathrm{d}\mathcal{H}^{d-1} + \frac{1}{r^{d-2}}\int_{\partial B_r} v\partial_\nu v\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.20}$$

where $\nu(x) := \frac{x}{|x|}$ is the outer unit normal and $C > 0$ is a dimensional constant. To this aim, we consider a regularization of v by convolution $v_\varepsilon := \varphi_\varepsilon \star v$, where φ_ε is a standard convolution kernel; and for every $\delta \in (0,r)$ we set

$$g_\delta = \min\big\{|x|^{2-d}, \delta^{2-d}\big\}.$$

From the subharmonicity and the positivity of v we get that

$$\Delta v_\varepsilon^2 = 2\,|\nabla v_\varepsilon|^2 + 2\,v_\varepsilon\,\Delta v_\varepsilon \geq 2\,|\nabla v_\varepsilon|^2.$$

In particular, if $\psi \in C_c^\infty(B_{2r})$ is such that

$$\psi \equiv 1 \quad \text{on} \quad B_r \quad \text{and} \quad r\,|D\psi| + r^2\,|D^2\psi| \leq C,$$

then we have that

$$\begin{aligned}
2\int_{B_{2r}} \psi\, g_\delta |\nabla v_\varepsilon|^2 &\leq \int_{B_{2r}} \psi\, g_\delta\, \Delta v_\varepsilon^2 \\
&= \int_{B_{2r}\setminus B_\delta} \Delta(\psi\, g_\delta) v_\varepsilon^2 \,\mathrm{d}x + \int_{\partial B_\delta} \psi\, \partial_\nu g_\delta\, v_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{d-1} \\
&\leq \frac{C}{r^d}\int_{B_{2r}\setminus B_r} v_\varepsilon^2 \,\mathrm{d}x + \frac{d-2}{r^{d-1}}\int_{\partial B_\delta} v_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

where we used that $\Delta g_\delta(x) = 0$ for $|x| > \delta$. Sending δ to zero, we infer that

$$\int_{B_r} \frac{|\nabla v_\varepsilon|^2}{|x|^{d-2}} \,\mathrm{d}x \leq \frac{C}{r^d}\int_{B_{2r}\setminus B_r} v_\varepsilon^2 \,\mathrm{d}x + C\, v_\varepsilon(0)^2.$$

In particular, the functions v_ε are uniformly in $H^1(B_r)$; and taking now the limit $\varepsilon \to 0$ (recall that v_ε converges to v uniformly and $v(0) = 0$), we conclude (2.19).

Similarly, for (2.20), we proceed as follows:

$$\begin{aligned}
2\int_{B_r} g_\delta |\nabla v_\varepsilon|^2 \,\mathrm{d}x &\leq \int_{B_r} \Delta v_\varepsilon^2\, g_\delta \,\mathrm{d}x \\
&= \int_{B_r\setminus B_\delta} \Delta g_\delta\, v_\varepsilon^2 \,\mathrm{d}x + 2\int_{\partial B_r} v_\varepsilon \nabla_\nu v_\varepsilon\, g_\delta \,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad - \int_{\partial B_r} \partial_\nu g_\delta\, v_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{d-1} + \frac{d-2}{r^{d-1}}\int_{\partial B_\delta} v_\varepsilon^2 \,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Considering that $\Delta g_\delta = 0$, we can take the limits $\delta \to 0$ and $\varepsilon \to 0$ in this order to infer (2.20).

Computing the derivative of J we have that

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}} \,\mathrm{d}x = r^{2-d}\int_{\partial B_r} |\nabla u_i|^2 \,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for a.e.} \quad r \in (0,1)$$

and therefore

$$\frac{J'(r)}{J(r)} = r^{2-d}\,\frac{\int_{\partial B_r} |\nabla u_1|^2 \,\mathrm{d}\mathcal{H}^{d-1}}{\int_{B_r} \frac{|\nabla u_1|^2}{|x|^{d-2}} \,\mathrm{d}x} + r^{2-d}\,\frac{\int_{\partial B_r} |\nabla u_2|^2 \,\mathrm{d}\mathcal{H}^{d-1}}{\int_{B_r} \frac{|\nabla u_2|^2}{|x|^{d-2}} \,\mathrm{d}x} - \frac{4}{r}.$$

We can then estimate as follows for $i = 1, 2$:

$$\begin{aligned}\int_{\partial B_r} |\nabla u_i|^2 \,\mathrm{d}\mathcal{H}^{d-1} &\geq \int_{\partial B_r} \left(|\partial_\nu u_i|^2 + |\nabla_\tau u_i|^2\right) \mathrm{d}\mathcal{H}^{d-1} \\ &\geq \int_{\partial B_r} \left(|\partial_\nu u_i|^2 + \lambda_i\, r^{-2} u_i^2\right) \mathrm{d}\mathcal{H}^{d-1} \\ &\geq 2\alpha_i\, r^{-1} \left(\int_{\partial B_r} (\partial_\nu u_i)^2 \,\mathrm{d}\mathcal{H}^{d-1}\right)^{1/2} \left(\int_{\partial B_1} u_i^2 \,\mathrm{d}\mathcal{H}^{d-1}\right)^{1/2} \\ &\quad + (\lambda_i - \alpha_i^2) r^{-2} \int_{\partial B_r} u_i^2 \,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

where λ_i denotes the lowest eigenvalue of the spherical Laplacian with Dirichlet boundary conditions in $S_i := (\{u_i > 0\} \cap \partial B_r)/r \subset \mathbb{S}^n$, and α_i is the corresponding characteristic number: in particular, $\alpha_i\,(d-2) = \lambda - \alpha_i^2$ and

$$\frac{\int_{\partial B_r} |\nabla u_i|^2 \,\mathrm{d}\mathcal{H}^{d-1}}{\int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}} \,\mathrm{d}x} \overset{(2.20)}{\geq} 2\frac{\alpha_i}{r}.$$

thus leading to

$$\frac{J'(r)}{J(r)} \geq \frac{2}{r}\,(\alpha_1 + \alpha_2) - \frac{4}{r} \overset{(2.18)}{\geq} 0.$$

The last conclusion of the theorem follows from (2.19). □

A consequence of the ACF-monotonicity formula is the following identification of the least possible frequency at free boundary points.

Corollary 2.4.3. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then,*

$$I_u(x_0, 0^+) \in \{3/2\} \cup [2, +\infty) \quad \forall\, x_0 \in \Gamma(u).$$

Proof. From Corollary 2.3.2 it is not restrictive to assume that u is a homogeneous solution in $\mathbb{R}^{n+1}$ with exponent $\lambda = I_u(x_0, 0^+)$. From the $C^{1,\alpha}$ regularity of u and the fact that $\nabla u(x_0) = 0$ for a free boundary point x_0, we deduce that $\lambda > 1$. Moreover, we can consider the horizontal derivatives $\partial_e u \in C^\alpha(B^+)$ for every $e \in \mathbb{S}^n$, with $e \cdot e_{n+1} = 0$.

It is easy to verify that $(\partial_e u)^\pm$ are subharmonic functions with disjoint supports and $(\partial_e u)^\pm(0) = 0$ (see Exercise 2.6.1). Therefore, we can apply Theorem 2.4.2 to infer that for all $r \in (0, 1]$

$$J(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla(\partial_e u)^+(x)|^2}{|x|^{n-1}} \,\mathrm{d}x \int_{B_r} \frac{|\nabla(\partial_e u)^-(x)|^2}{|x|^{n-1}} \,\mathrm{d}x < C < +\infty.$$

Considering that $\nabla(\partial_e u)^\pm$ is $(\lambda - 2)$-homogeneous, we deduce that

$$J(r) = r^{4\,\lambda - 8} J(1).$$

Therefore, either $\lambda \geq 2$ or we must have $J(1) = 0$. Note that this is possible if and only if at least one between $(\partial_e u)^+$ and $(\partial_e u)^-$ is constantly zero. In particular, for every $e \in \mathbb{S}^n$ with $e \cdot e_{n+1} = 0$, we have that u is monotone in the direction e, which is equivalent to say that u is a function of a two variables (see Exercise 2.6.2):

$$u(x) = v(x \cdot \bar{e}, x_{n+1}) \quad \text{for some } \bar{e} \in \mathbb{S}^n, \quad \bar{e} \cdot e_{n+1} = 0,$$

and $v : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the Signorini problem, which can be easily classified (see Exercise 2.6.3). By direct inspection the only frequency $\lambda \in (1, 2)$ is given by the value $\frac{3}{2}$. $\square$

2.5. Optimal regularity: $C^{1,1/2}$.

THEOREM 2.5.1. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, $u \in C^{1,1/2}(B_1^+ \cup B_1')$ with*

$$\|u\|_{C^{1,1/2}(B^+_{1/2} \cup B'_{1/2})} \leq C(K) \|u\|_{L^2(B^+)}.$$

PROOF. For every $x \in B^+_{1/2} \cup B'_{1/2}$ we denote by $d(x)$ the distance from the free boundary:

$$d(x) := \mathrm{dist}\big(x, \Gamma(u)\big).$$

Note that either $B_{d(x)}(x) \cap B_1' \subset \Lambda(u)$ or $B_{d(x)}(x) \cap B_1' \subset B_1' \setminus \Lambda(u)$: in particular, in the first case the odd reflection of u, in the second one the even reflection of u, are harmonic functions. We denote such harmonic functions (in both cases) with U.

In order to prove the theorem, it is enough to show that

$$|\nabla u(x_1) - \nabla(x_2)| \leq C \, \|u\|_{L^2(B)} \, |x_1 - x_2|^{1/2}$$

for all $x_1, x_2 \in B^+_{1/2} \cup B'_{1/2}$ with $|x_1 - x_2| < 1/8$. We consider three cases.

Case 1: $d(x_1) \geq 1/4$. In this case, $x_2 \in B_{d(x_1)/2}(x_1)$ and by the interior estimates for the harmonic function U we have that

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\leq C \, \frac{\|U\|_{L^2(B_{d(x_1)}(x_1))}}{d(x_1)^{2+\frac{n+1}{2}}} \, |x_1 - x_2| \\ &\leq C \, \|u\|_{L^2(B_1^+)} \, |x_1 - x_2|^{1/2}. \end{aligned}$$

Case 2: $d(x_2) \leq d(x_1) < 1/4$ and $|x_1 - x_2| \geq d(x_1)/2$. From Corollary 2.3.3 and Corollary 2.4.3 we have that

$$\|U\|_{L^2(B_{d(x_1)}(x_1))} \leq C \, \|u\|_{L^2(B^+)} \, d(x_1)^{n/2+4}.$$

In particular, considering that U is harmonic, we have that

$$\begin{aligned} \|U\|_{L^\infty(B_{d(x_1)/2}(x_1))} &\leq C \, \frac{\|U\|_{L^2(B_{d(x_1)}(x_1))}}{d(x_1)^{\frac{n+1}{2}}} \leq C \frac{\|u\|_{L^2(B_1^+)} \, d(x_1)^{\frac{n+4}{2}}}{d(x_1)^{\frac{n+1}{2}}} \\ &= \|u\|_{L^2(B_1^+)} \, d(x_1)^{\frac{3}{2}} \end{aligned}$$

Since the same can be done for x_2, we get

$$\begin{aligned}|\nabla u(x_1) - \nabla u(x_2)| &\le |\nabla u(x_1)| + |\nabla u(x_2)| \\ &\le C\,\|u\|_{L^2(B^+)}\,d(x_1)^{1/2} \le C\,\|u\|_{L^2(B_1^+)}\,|x_1-x_2|^{1/2}.\end{aligned}$$

Case 3: $d(x_2) \le d(x_1) < 1/4$ and $|x_1 - x_2| < d(x_1)/2$. Arguing as before via interior estimates for harmonic functions and Corollary 2.3.3, we have

$$\|D^2 U\|_{L^\infty(B_{d(x_1)/2}(x_1))} \le C\,\frac{\|U\|_{L^2(B_{d(x_1)}(x_1))}}{d(x_1)^{\frac{n+5}{2}}} \le \|u\|_{L^2(B^+)}\,d(x_1)^{-1/2}.$$

Therefore

$$\begin{aligned}|\nabla u(x_1) - \nabla(x_2)| &\le \|D^2 u\|_{L^\infty(B_{d(x_1)/2}(x_1))}|x_1-x_2| \\ &\le C\,\|u\|_{L^2(B)}\,d(x_1)^{-1/2}|x_1-x_2| \\ &\le C\,\|u\|_{L^2(B)}\,|x_1-x_2|^{1/2}.\end{aligned}$$

□

2.6. Exercises.

Exercise 2.6.1. Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Show that, for every $e \in \mathbb{S}^n$ with $e \cdot e_{n+1} = 0$, the functions $(\partial_e u)^\pm$ are subharmonic.

Hint. *Consider the approximations $v_\delta^+ := \phi_\delta(\partial_e u)$ with ϕ_δ a regularization of t^+ and $\phi_\delta(0) = 0$. Similarly, consider $v_\delta^- := \phi_\delta(-\partial_e u)$. Note that $v_\delta^\pm$ are zero in a neighborhood of $\Lambda(u)$.*

Exercise 2.6.2. Let $v \in C^1(B_1)$, $B_1 \subset \mathbb{R}^d$, be monotone in each direction $e \in \mathbb{S}^{d-1}$. Show that v is a functions of a single variable: *i.e.* there exists $\phi : \mathbb{R} \to \mathbb{R}$ and $\bar{e} \in \mathbb{S}^{d-1}$ such that $v(x) = \phi(\bar{e} \cdot x)$.

Exercise 2.6.3. Show that the only homogeneous solutions to the Signorini problem in $\mathbb{R}^2$ (*i.e.* $n = 1$) are given by the following formulas:

$$\begin{aligned}u_{2m}(x_1,x_2) &= C\,\mathrm{Re}(x_1 + i|x_2|)^{2m}, \quad m \in \mathbb{N}\setminus\{0\},\ C \ge 0 \\ u_{2m-1/2}(x_1,x_2) &= C\,\mathrm{Re}(x_1 + i|x_2|)^{2m-1/2}, \quad m \in \mathbb{N}\setminus\{0\},\ C \ge 0 \\ u_{2m+1}(x_1,x_2) &= C\,\mathrm{Im}(x_1 + i|x_2|)^{2m+1}, \quad m \in \mathbb{N},\ C \ge 0,\end{aligned}$$

where the determination of the square root for $u_{2m-1/2}$ is chosen in such a way that the $u_{2m-1/2}(x_1, 0) \ge 0$.

Hint. *Use polar co-ordinates.*

3. The free boundary: the regular points

We start now the study of the regularity points of the free boundary. To this aim, it can be useful to recall the definitions of contact set $\Lambda(u)$ and of free boundary $\Gamma(u)$ of a solution u to the Signorini problem:

$$\Lambda(u) := \big\{(x',0) \in B_1' : u(x',0) = 0\big\} \quad \text{and} \quad \Gamma(u) := \partial_{B_1'}\Lambda(u),$$

where $\partial_{B_1'}$ denotes the boundary in the (relative) topology of B_1'.

3.1. The regular part of the free boundary. In this chapter we consider the so called *regular part* $\Gamma_{3/2}(u)$ of the free boundary, defined as the set with least frequency:

$$\Gamma_{3/2}(u) := \big\{x \in \Gamma(u) : I_u(x,0^+) = 3/2\big\}.$$

The reason why this subsets of the free boundary is called *regular* has mostly to do with the results we are going to discuss next.

Theorem 3.1.1. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, $\Gamma_{3/2}(u)$ is a relatively open subset of $\Gamma(u)$ and is an analytic $(n-1)$-dimensional submanifold of $\mathbb{R}^{n+1}$.*

Comments 3.1.2. (i) $\Gamma_{3/2}(u) \subset \Gamma(u)$ open is easily seen as follows: from Almgren's monotonicity formula in Proposition 2.3.1

$$I_u(x,0^+) = \lim_{r\to 0^+} I_u(x,r) = \inf_{r>0} I_u(x,r)$$

is an upper semicontinuous function (greatest lower bound of continuous functions $x \mapsto I_u(x,r)$), and therefore taking into account Corollary 2.4.3 we infer that

$$\Gamma_{3/2}(u) := \big\{x \in \Gamma(u) : I_u(x,0^+) < 2\big\} \subset \Gamma(u) \quad \text{is relatively open.}$$

(ii) The main breakthrough is due to Athanasopoulos, Caffarelli and Salsa [**3**] (see Caffarelli, Salsa and Silvestre [**6**] for the case of the fractional Laplacian), where the authors prove the $C^{1,\alpha}$ regularity of $\Gamma_{3/2}(u)$.

(iii) The higher regularity has been recently obtained in [**7, 16**] via bootstrap methods and hodograph transformation.

Here we present a proof of the $C^{1,\alpha}$ regularity of $\Gamma_{3/2}(u)$ as a consequence of the *epiperimetric inequality* established by Focardi and Spadaro [**8**].

3.2. The epiperimetric inequality. We introduce a family of *boundary adjusted energies*: namely, for every $u \in H^1(B_1)$ even symmetric with respect to x_{n+1}, for every $x_0 \in \Gamma_{3/2}(u)$ and for every $r \in (0, 1-|x_0|)$, we set

$$W_{x_0}(r,u) := \frac{1}{r^{n+2}}\int_{B_r(x_0)} |\nabla u|^2 \,\mathrm{d}x - \frac{3}{2\,r^{n+3}}\int_{\partial B_r(x_0)} u^2 \,\mathrm{d}\mathcal{H}^n.$$

We omit to write the point x_0 if it is the origin.

Remark 3.2.1. The introduction of the boundary adjusted energies goes back to the work by Weiss for the classical obstacle problem [**21**] and has been generalized to the Signorini problem by Garofalo and Petrosyan [**12**].

The main result is now the following

Theorem 3.2.2 (Epiperimetric inequality Focardi-S. [**8**]). *There exists a dimensional constant $\kappa \in (0,1)$ such that if $c \in H^1(B_1)$ is a $3/2$-homogeneous function with $c \geq 0$ on B', then*

$$\inf_{v \in \mathcal{A}_c} W(1,v) \leq (1-\kappa)\, W(1,c). \tag{3.1}$$

Recall the definition

$$\mathcal{A}_c := \left\{ v \in c + H^1_0(B_1) : v|_{B_1'} \geq 0,\ v(x', x_{n+1}) = v(x', -x_{n+1}) \right\}.$$

Remark 3.2.3. A similar inequality has also been proved by Garofalo, Petrosyan and Smit Vega [**?**].

3.2.4. *Decay of the boundary adjusted energy.* The main consequence of the epiperimetric inequality in Theorem 3.2.2 is the decay of the boundary adjusted energy.

Proposition 3.2.5. *There exists a dimensional constant $\gamma > 0$ with this property. For every $x_0 \in \Gamma_{3/2}(u)$ there exists a constant $C > 0$ such that*

$$0 \leq W_{x_0}(r,u) \leq C\, r^{\gamma} \quad \forall\, 0 < r < 1 - |x_0|. \tag{3.2}$$

Proof. Without loss of generality, we consider $x_0 = 0$. We start computing:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} W(r,u) &= -\frac{n+2}{r^{n+3}} D(r) + \frac{D'(r)}{r^{n+3}} - \frac{3}{2\, r^{n+3}} H'(r) + \frac{3(n+3)}{2\, r^{n+4}} H(r) \\
&= -\frac{n+2}{r}\, W(r,u) - \frac{3(n+2)}{2\, r^{n+4}} H(r) \\
&\quad + \underbrace{\frac{D'(r)}{r^{n+3}} + \frac{9}{2r^{n+4}} H(r) - 3\, \frac{D(r)}{r^{n+3}}}_{=:I}.
\end{aligned} \tag{3.3}$$

where we used the formula (2.14)

$$H'(r) = \frac{n}{r}\, H(r) + 2\, D(r).$$

In order to treat the terms in I, we introduce the rescaled functions

$$u_r(x) := \frac{u(rx)}{r^{3/2}} \tag{3.4}$$

and deduce by simple computations that

$$\begin{aligned}
I &= \frac{1}{r} \int_{\partial B_1} \left(|\nabla u_r|^2 - 3 u_r \nabla u_r \cdot \nu + \frac{9}{2} u_r^2 \right) d\mathcal{H}^n \\
&= \frac{1}{r} \int_{\partial B_1} \left[\left(\nabla u_r \cdot \nu - \frac{3}{2} u_r \right)^2 + |\nabla_\tau u_r|^2 + \frac{9}{4} u_r^2 \right] d\mathcal{H}^n,
\end{aligned} \tag{3.5}$$

where we denoted by $\nabla_\tau u_r$ the (covariant) derivative of u_r in the directions tangent to ∂B_1. Let c_r be the $3/2$-homogeneous extension of $u_r|_{\partial B_1}$, i.e.

$$c_r(x) := |x|^{3/2} u_r\Big(x/|x|\Big).$$

It is simple to verify that

$$\int_{\partial B_1}\left(|\nabla_\tau u_r|^2 + \frac{9}{4}u_r^2\right) = (n+2)\int_{B_1}|\nabla c_r|^2\,dx.$$

We then conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}W(r,u) = \frac{n+2}{r}\Big(W_{3/2}(1,c_r) - W_{3/2}(1,u_r)\Big)\\ + \frac{1}{r}\int_{\partial B_1}\left(\nabla u_r\cdot\nu - \frac{3}{2}u_r\right)^2 d\mathcal{H}^n.\end{aligned} \tag{3.6}$$

By the epiperimetric inequality in Theorem 3.2.2

$$\frac{\mathrm{d}}{\mathrm{d}r}W(r,u) \geq 2\,\frac{n+2}{r}\,\frac{\kappa}{1-\kappa}W(1,u_r) = 2\,\frac{n+2}{r}\,\frac{\kappa}{1-\kappa}W(r,u) \quad \forall\, 0<r<r_0.$$

Integrating this inequality we get

$$W(r,u) \leq W(1,u)\,r^\gamma \quad \forall\, 0<r<r_0,$$

with $\gamma := 2(n+1)\kappa/(1-\kappa)$. $\square$

3.3. Regularity of the free boundary. In this section we show how to derive the regularity of the free boundary around points of least frequency as a simple consequence of the epiperimetric inequality. We divide the argument in different steps.

3.3.1. *Rescaled profiles.* Assume that $0 \in \Gamma_{3/2}(u)$ and set

$$u_r(x) := \frac{u(rx)}{r^{3/2}}. \tag{3.7}$$

A first consequence of Corollary 2.3.3 of Chapter 2 is that the rescaled profiles u_r have equi-bounded Dirichlet energies:

$$\begin{aligned}\int_{B_1}|\nabla u_r|^2\,dx &= \frac{\int_{B_r}|\nabla u|^2\,dx}{r^{n+2}} = \frac{r\int_{B_r}|\nabla u|^2\,dx}{\int_{\partial B_r}u^2\,d\mathcal{H}^n}\,\frac{\int_{\partial B_r}u^2\,d\mathcal{H}^n}{r^{n+3}}\\ &\overset{\text{Ch.2 (2.16)}}{\leq} I_u(r)\,H_u(1) \leq I_u(1)\,H_1(1).\end{aligned} \tag{3.8}$$

Therefore, for every infinitesimal sequence of radii $r_k \downarrow 0$ there exists a subsequence $r_{k'} \downarrow 0$ such that $u_{r_{k'}} \to u_0$ in $L^2(B_1)$. Note however that this does not allow to deduce that there exists a limiting function u_0 which is not identically 0. As an application of the epiperimetric inequality and the related decay of the energy in Proposition 3.2.5 we can deduce that this is actually the case for every such limiting profiles u_0.

PROPOSITION 3.3.2 (Nondegeneracy). *Let $u \in H^1(B_1)$ be a solution to the Signorini problem and assume that $0 \in \Gamma_{3/2}(u)$. Then there exists a constant $H_0 > 0$ such that*

$$H(r) \geq H_0\, r^{n+3} \quad \forall\, 0 < r < 1. \tag{3.9}$$

PROOF. The starting point is the computation of $H'(r)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\log\frac{H(r)}{r^{n+3}}\right) = 2\,\frac{D(r)}{H(r)} - \frac{3}{r} = \frac{2\,r^{n+2}}{H(r)}W(r,u). \tag{3.10}$$

Let $\gamma > 0$ be the constant in Proposition 3.2.5: by Corollary 2.3.3, there exists a constant $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\log\frac{H(r)}{r^{n+3}}\right) \leq C\, r^{\gamma/2-1} \quad \forall\, 0 < r < 1. \tag{3.11}$$

Integrating this differential inequality we get that the function

$$\frac{H(r)}{r^{n+3}e^{\frac{2C}{\gamma}r^{\gamma/2}}}$$

is nonincreasing. In particular, there exists the limit

$$H_0 := \lim_{r\to 0}\frac{H(r)}{r^{n+3}e^{Cr^{\gamma/2}}} = \lim_{r\to 0}\frac{H(r)}{r^{n+3}} > 0.$$

Since the function $\frac{H(r)}{r^{n+3}}$ is monotone increasing by (3.10), we conclude the proof. □

Note now that by (3.9) we then deduce that

$$\int_{\partial B_1} u_r^2\, d\mathcal{H}^n \geq H_0.$$

Therefore, since from (3.8) we also deduce the convergence of the traces of u_r on ∂B_1, we finally get that

$$\int_{\partial B_1} u_0^2\, d\mathcal{H}^n \geq H_0 > 0$$

for every limiting profile u_0, thus showing that $u_0 \not\equiv 0$.

3.3.3. *Uniqueness of blowups.* A key ingredient of the analysis of the free boundary we are going to perform is to show that

(i) the blowup u_0 is actually *unique*, meaning that the *whole* sequence $u_r \to u_0$ in $L^2(B_1)$ as $r \to 0$,
(ii) there is a rate of convergence of u_r to u_0.

This is again an easy consequence of the epiperimetric inequality.

PROPOSITION 3.3.4. *Let u be a solution to the Signorini problem and $K \subset\subset B_1'$. Then there exist a constant $C > 0$ such that for every $x_0 \in \Gamma_{3/2}(u) \cap K$ there exists a unique blowup u^{x_0} and*

$$\int_{\partial B_1} |u_r^{x_0} - u^{x_0}|\, d\mathcal{H}^n \leq C\, r^{\gamma/2} \quad \textit{for all } \; 0 < r < \mathrm{dist}(K, \partial B_1), \tag{3.12}$$

where $\gamma > 0$ is the constant in Proposition 3.2.5.

Proof. Consider the case $x_0 = 0 \in \Gamma_{3/2}(u)$. Let $0 < s < r < 1$ be fixed radii; we can then compute as follows:

$$\begin{aligned}
\int_{\partial B_1} |u_r - u_s|\,\mathrm{d}\mathcal{H}^n &\le \int_{\partial B_1}\int_s^r t^{-1}\Big|\nabla u_t\cdot\nu - \frac{3}{2}u_t\Big|\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n\\
&\le \sqrt{n\,\omega_n}\int_s^r t^{-1/2}\left(t^{-1}\int_{\partial B_1}\Big|\nabla u_t\cdot\nu - \frac{3}{2}u_t\Big|^2\,\mathrm{d}\mathcal{H}^n\right)^{1/2}\mathrm{d}t\\
&\overset{(3.6)}{\le} \sqrt{n\,\omega_n}\int_s^r t^{-1/2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}W_{3/2}(t,u)\Big)^{1/2}dt\\
&\le \sqrt{n\,\omega_n}\,\log\big(r/s\big)^{1/2}\big(W(r,u)-W(s,u)\big)^{1/2}. \qquad (3.13)
\end{aligned}$$

By (3.2) and a simple dyadic argument (applying (3.13) to $s = r/2 = 2^{-k}$ for $k \in \mathbb{N}$ sufficiently large) we easily deduce that for every $0 < s < r < 1$

$$\int_{\partial B_1} |u_r - u_s|\,d\mathcal{H}^n \le C\,r^{\gamma/2}$$

for a constant $C > 0$ which in turn depends only on the constants in Proposition 3.2.5. Sending s to 0 and eventually changing the value of the constant C, we then conclude. The same holds for every other $x_0 \in \Gamma_{3/2}(u) \cap K$. $\square$

3.3.5. *$C^{1,\alpha}$ regularity of the free boundary $\Gamma_{3/2}$.* In view of the uniqueness result in Proposition 3.3.4 we are in the position to prove the $C^{1,\alpha}$ regularity $\Gamma_{3/2}$.

Proposition 3.3.6. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then there exists a dimensional constant $\alpha > 0$ such $\Gamma_{3/2}(u)$ is a $C^{1,\alpha}$ regular submanifold of dimension $n-1$.*

Proof. Without loss of generality it is enough to prove that if $0 \in \Gamma_{3/2}(u)$ then $\Gamma_{3/2}(u)$ is a $C^{1,\alpha}$ submanifold in a neighborhood of 0. To this aim we start noticing that by the openness of $\Gamma_{3/2}(u)$ there exists $s > 0$ such that $B_s \cap \Gamma(u) = B_s \cap \Gamma_{3/2}(u)$. From the complete characterization of the homogeneous $3/2$ solutions, we have that for every $x_0 \in B_s \cap \Gamma_{3/2}(u)$ the unique blowup $u_0^{x_0}$

$$u_0^{x_0} = \lambda_{x_0} h_{e(x_0)},$$

for some $\lambda_{x_0} > 0$ and $|e(x_0)| = 1$ with $e(x_0)\cdot e_{n+1} = 0$, where

$$h_{e(x_0)}(x) = u_{3/2}(x\cdot e(x_0), x_{n+1})$$

– cf. Chapter 2 Exercise 2.6.3.

We first prove the Hölder continuity of $x_0 \mapsto \lambda_{x_0}$. To this aim we start observing that, thanks to Proposition 3.2.5 and Proposition 3.3.2 we can further estimate (3.10) in the following way

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\log\frac{H(x_0,r)}{r^{n+3}}\right) = \frac{2\,r^{n+2}}{H(x_0,r)}W_{x_0}(r,u) \le C\,r^{\gamma-1} \quad \forall r \in (0,1). \qquad (3.14)$$

Notice that by the strong convergence in $L^2(B_1)$ of the rescaled functions it follows that

$$\lambda_{x_0}^2 = c_0 \lim_{r\to 0} \frac{H^{x_0}(r)}{r^{n+3}}$$

for some *dimensional* constant $c_0 > 0$. Integrating (3.14) we can then deduce that

$$c_0 \frac{H^{x_0}(r)}{r^{n+3}} - \lambda_{x_0}^2 \le C\, r^{\gamma} \quad \forall\, r \in (0,1). \tag{3.15}$$

Notice moreover that for $x_0, y_0 \in B_s \cap \Gamma_{3/2}(u)$ and $r = |x_0 - y_0|^{1-\theta}$ with $\theta = \gamma/(1+\gamma)$ it holds that

$$\begin{aligned}
&\int_{\partial B_1} |u_r^{x_0} - u_r^{y_0}|\, \mathrm{d}\mathcal{H}^n \\
&\le r^{-3/2} \int_{\partial B_1} \int_0^1 \big|\nabla u\big(s(x_0 + rx) + (1-s)(y_0 + rx)\big)\big|\, |y_0 - x_0|\, ds\, d\mathcal{H}^n(x) \\
&\le C r^{-1}\, |y_0 - x_0| \le C\, |y_0 - x_0|^{\theta}.
\end{aligned} \tag{3.16}$$

Therefore we can conclude that for $r = |x_0 - y_0|^{1-\theta}$ with $\theta = \gamma/(1+\gamma)$ it holds that

$$\begin{aligned}
\big|\lambda_{x_0}^2 - \lambda_{y_0}^2\big| &\le \Big|\lambda_{x_0}^2 - c_0 \frac{H(x_0,r)}{r^{n+3}}\Big| + c_0 \Big|\frac{H(x_0,r)}{r^{n+3}} - \frac{H(y_0,r)}{r^{n+3}}\Big| + \Big|c_0 \frac{H(y_0,r)}{r^{n+3}} - \lambda_{y_0}^2\Big| \\
&\le C\, r^{\gamma} + C \int_{\partial B_1} \big|(u_r^{x_0})^2 - (u_r^{y_0})^2\big|\, d\mathcal{H}^n \\
&\le C\, r^{\gamma} + C \int_{\partial B_1} \big|u_r^{x_0} - u_r^{y_0}\big|\, d\mathcal{H}^n \le C\, r^{\theta}
\end{aligned} \tag{3.17}$$

where we used the uniform L^∞ (actually $C^{1,1/2}$) bound on $u_r^{x_0}$ for every $x_0 \in \Gamma_{3/2}(u) \cap B_s$.

By Proposition 3.3.4 and a similar computation we can show that

$$\begin{aligned}
\int_{\partial B_1} |u_0^{x_0} - u_0^{y_0}|\, d\mathcal{H}^n &\le \int_{\partial B_1} |u_0^{x_0} - u_r^{x_0}|\, d\mathcal{H}^n + \int_{\partial B_1} |u_r^{x_0} - u_r^{y_0}|\, d\mathcal{H}^n \\
&\quad + \int_{\partial B_1} |u_r^{y_0} - u_0^{y_0}|\, d\mathcal{H}^n \\
&\overset{(3.12)\,\&\,(3.16)}{\le} C\, r^{\gamma/2} + C\, |x_0 - y_0|^{\theta} \le C\, |x_0 - y_0|^{\gamma\theta/2}
\end{aligned} \tag{3.18}$$

Note finally that there exists a geometric constant $\bar{C} > 0$ such that

$$|e(x_0) - e(y_0)| \le \bar{C} \int_{\partial B_1} \big|h_{e(x_0)} - h_{e(y_0)}\big|\, d\mathcal{H}^n.$$

Therefore from (3.17) and (3.18) we easily deduce that

$$|e(x_0) - e(y_0)| \le C\, |x_0 - y_0|^{\gamma\theta/2}. \tag{3.19}$$

Next we show that the vectors $e(x_0)$ do actually encode a geometric property of the free boundary. To this aim we introduce the following notation for cones centered at points $x_0 \in \Gamma_{3/2}(u)$: for any $\varepsilon > 0$ we set

$$C^{\pm}(x_0, \varepsilon) := \{x \in \mathbb{R}^n \times \{0\} \,:\, \pm\langle x - x_0, e(x_0)\rangle \geq \varepsilon|x - x_0|\}\,.$$

The main claim are then the following two: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x_0 \in \Gamma_{3/2}(u) \cap B_{s/2}$,

$$u > 0 \quad \text{in} \;\; C^{+}(x_0, \varepsilon) \cap B_{\delta}(x_0). \tag{3.20}$$

$$u = 0 \quad \text{in} \;\; C^{-}(x_0, \varepsilon) \cap B_{\delta}(x_0). \tag{3.21}$$

Assume by contradiction that there exist $x_j \in \Gamma_{3/2}(u) \cap B_{s/2}$ with $x_j \to x_0 \in \Gamma_{3/2}(u) \cap \bar{B}_{s/2}$, and $y_j \in C^+(x_j, \varepsilon)$ with $y_j - x_j \to 0$ such that $u(y_j) = 0$. By the $C^{1,1/2}$ regularity of the solution, (3.12) and (3.18), the rescalings $u^{x_j}_{r_j}$ with $r_j := |y_j - x_j|$, converge uniformly to $u^{x_0}_0$. Up to subsequences, by the Hölder continuity of the normals proved in (3.19) we can assume that $r_j^{-1}(y_j - x_j) \to z \in C^+(x_0, \varepsilon) \cap \mathbb{S}^{n-1}$ and by uniform convergence $u^{x_0}_0(z) = 0$. This contradicts the fact that $x_0 \in \Gamma_{3/2}(u)$ and $u^{x_0}_0 > 0$ on $C^+(x_0, \varepsilon) \setminus \{0\}$.

For what concerns (3.21), we argue as above: assume by contradiction that there exist $x_j \to x_0 \in \Gamma_{3/2}(u) \cap \bar{B}_{s/2}$ as above and $y_j \in C^-(x_j, \varepsilon)$ with $y_j - x_j \to 0$ such that $u(y_j) > 0$, which implies $\partial_{n+1} u(y_j) = 0$. By the $C^{1,1/2}$ regularity of the solution, (3.12) and (3.18), the rescalings $u^{x_j}_{r_j}$ with $r_j := |y_j - x_j|$, converge uniformly to $u^{x_0}_0$. Up to subsequences, by the Hölder continuity of the normals proved in (3.19) we can assume that $r_j^{-1}(y_j - x_j) \to z \in C^-(x_0, \varepsilon) \cap \mathbb{S}^{n-1}$ and by uniform convergence $\partial_{n+1} u^{x_0}_0(z) = 0$. This contradicts the fact that $x_0 \in \Gamma_{3/2}(u)$ and $\partial_{n+1} u^{x_0}_0 > 0$ on $C^-(x_0, \varepsilon) \setminus \{0\}$.

We can now conclude that $\Gamma_{3/2}(u) \cap B_\rho$ is the graph of a function g, for a suitably chosen small $\rho > 0$. Without loss of generality assume that $e(0) = e_n$ and set

$$g(x'') := \max\big\{t \in \mathbb{R} \,:\, (x'', t, 0) \in \Lambda(u)\big\}$$

for all points $x' \in \mathbb{R}^{n-1}$ with $|x'| \leq \delta\sqrt{1 - \varepsilon^2}$. Note that by (3.20) this maximum exists and belongs to $[-\varepsilon\delta, \varepsilon\delta]$. Moreover $u(x', t, 0) = 0$ for every $-\varepsilon\,\delta < t < g(x')$ and $u(x', t, 0) > 0$ for every $g(x') < t < \varepsilon\,\delta$. Eventually, by applying (3.20) with respect to arbitrary ε, we deduce that g is differentiable and in view of (3.19) we can conclude that g is $C^{1,\alpha}$ regular for a suitable $\alpha > 0$. □

3.4. Proof of the epiperimetric inequality. In this section we give a sketch of the proof of the epiperimetric inequality Theorem 3.2.2. To simplify the notation in the proof below we shall denote $W(1, \cdot)$ by $\mathscr{G}$.

3.4.1. *Proof by contradiction.* We argue by contradiction: we start off assuming the existence of numbers $\kappa_j \downarrow 0$ and of functions $c_j \in H^1(B_1)$ that are even symmetric with respect to x_{n+1}, $3/2$-homogeneous, positive on B_1' and such that

$$(1-\kappa_j)\mathscr{G}(c_j) \leq \inf_{v\in\mathscr{A}_{c_j}} \mathscr{G}(v), \tag{3.22}$$

where we recall

$$\mathscr{A}_{c_j} := \left\{u \in H^1(B_1) : \; u \geq 0 \;\text{ on } B_1',\, u = c_j \;\text{ on } (\partial B_1)^+\right\}.$$

Note that (3.22) is invariant if we replace c_j with λc_j and $\lambda > 0$: in particular, we can assume that

$$\mathrm{dist}_{H^1}\big(c_j, \mathscr{H}_{3/2}\big) = 1 \qquad \text{for all } j \in \mathbb{N}, \tag{3.23}$$

where $\mathscr{H}_{3/2}$ denotes the closed convex cone of $3/2$-homogeneous solutions

$$\mathscr{H}_{3/2} := \big\{\lambda\, u_{3/2}(x\cdot e, x_{n+1}) : \lambda > 0,\; |e| = 1,\; e\cdot e_{n+1} = 0\big\},$$

– cf. Exercise 2.6.3 Chapter 2. Moreover, by a change of coordinates depending on j, we can also assume that

$$\|c_j - \lambda_j\, h\|_{H^1} = 1,$$

where $h := h_{e_n}$. We divide the rest of the proof in some intermediate steps.

3.4.2. *Introduction of a family of auxiliary functionals.* We rewrite inequality (3.22) conveniently and interpret it as an almost minimality condition for a sequence of new functionals.

We start noticing that, for every $\psi \in \mathscr{H}_{3/2}$ and for every $\varphi \in H^1(B_1)$, a simple integration by parts yields

$$\begin{aligned}\int_{B_1^+} \nabla\psi\cdot\nabla\varphi\, dx &= \int_{B_1^+} \mathrm{div}(\varphi\nabla\psi)dx \\ &= \int_{(\partial B_1)^+} \varphi\frac{\partial\psi}{\partial\nu}d\mathcal{H}^n - \int_{B_1'} \varphi\,\frac{\partial\psi}{\partial x_{n+1}}(x',0^+)\mathrm{d}x' \\ &= \frac{3}{2}\int_{(\partial B_1)^+} \varphi\,\psi\, d\mathcal{H}^n - \int_{B_1'} \varphi\,\frac{\partial\psi}{\partial x_{n+1}}(x',0^+)\mathrm{d}x',\end{aligned}$$

where $\nu = \frac{x}{|x|}$ and we used that ψ is $3/2$-homogeneous and $\Delta\psi = 0$ in B_1^+. Therefore, by the even symmetry of ψ we conclude

$$\int_{B_1} \nabla\psi\cdot\nabla\varphi\, dx = \frac{3}{2}\int_{\partial B_1} \varphi\,\psi\, d\mathcal{H}^n - 2\int_{B_1'} \varphi\,\frac{\partial\psi}{\partial x_{n+1}}(x',0^+)d\mathcal{H}^n. \tag{3.24}$$

In particular, (3.24) yields that the first variation of $\mathscr{G}_{3/2}$ at $\psi \in \mathscr{H}_{3/2}$ in the direction $\varphi \in H^1(B_1)$, formally defined as

$$\delta\mathscr{G}_{3/2}(\psi)[\varphi] := 2\int_{B_1} \nabla\psi\cdot\nabla\varphi\, dx - 3\int_{\partial B_1} \psi\,\varphi\, d\mathcal{H}^n,$$

satisfies

$$\delta\mathscr{G}_{3/2}(\psi)[\varphi] = -4\int_{B_1'} \varphi\,\frac{\partial\psi}{\partial x_{n+1}}(x',0^+)d\mathcal{H}^n. \tag{3.25}$$

Furthermore, by taking into account the Signorini boundary conditions for ψ and (3.24) applied to $\varphi = \psi$, we get

$$\mathscr{G}_{3/2}(\psi) = 0 \qquad \text{for all } \psi \in \mathscr{H}_{3/2}. \tag{3.26}$$

For any fixed j, let $v \in \mathscr{A}_{c_j}$ and use (3.25) (applied twice to $\psi_j = \lambda_j h$ with test functions $\varphi = c_j - \psi_j$ and $\varphi = v - \psi_j$) and (3.26), in order to rewrite (3.22) in the following form

$$\begin{aligned}(1-\kappa_j)\Big(\mathscr{G}(c_j)-\mathscr{G}(\psi_j)-\delta\mathscr{G}(\psi_j)[c_j-\psi_j]-4\int_{B_1'}(c_j-\psi_j)\,\frac{\partial\psi_j}{\partial x_{n+1}}(x',0^+)\,\mathrm{d}x'\Big)\\ \leq \mathscr{G}(v) - \mathscr{G}(\psi_j) - \delta\mathscr{G}(\psi_j)[v-\psi_j] - 4\int_{B_1'}(v-\psi_j)\,\frac{\partial\psi_j}{\partial x_{n+1}}(x',0^+)\,\mathrm{d}x'.\end{aligned}$$

Simple algebraic manipulations then lead to

$$\begin{aligned}(1-\kappa_j)\Big(\mathscr{G}(c_j-\psi_j) - 4\int_{B_1'}(c_j-\psi_j)\frac{\partial\psi_j}{\partial x_{n+1}}(x',0^+)\,\mathrm{d}x'\Big)\\ \leq \mathscr{G}(v-\psi_j) - 4\int_{B_1'}(v-\psi_j)\frac{\partial\psi_j}{\partial x_{n+1}}(x',0^+)\,\mathrm{d}x',\end{aligned} \tag{3.27}$$

for all $v \in \mathscr{A}_{c_j}$.

Next we introduce the following notation. We set

$$z_j := c_j - \lambda_j\, h, \tag{3.28}$$

$$\mathscr{B}_j := \left\{ z \in z_j + H^1_0(B_1) : (z+\lambda_j h)|_{B_1'} \geq 0 \right\}. \tag{3.29}$$

Then we define the functionals $\mathscr{G}_j : L^2(B_1) \to (-\infty, +\infty]$ given by

$$\mathscr{G}_j(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2}\int_{\partial B_1} z_j^2\, d\mathcal{H}^n - 4\,\lambda_j \int_{B_1'} z\,\frac{\partial h}{\partial x_{n+1}}(x',0^+)\,\mathrm{d}x' & \text{if } z \in \mathscr{B}_j,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.30}$$

Note that the second term in the formula does not depend on z but only on the boundary conditions $z_j|_{\partial B_1}$.

Therefore, (3.27) reduces to

$$(1-\kappa_j)\mathscr{G}_j(z_j) \leq \mathscr{G}_j(z) \qquad \text{for all } z \in \mathscr{B}_j. \tag{3.31}$$

Moreover, note that by (3.23) and (3.28)

$$\|z_j\|_{H^1(B_1)} = 1. \tag{3.32}$$

This implies that we can extract a subsequence (not relabeled) such that

(a) $(z_j)_{j\in\mathbb{N}}$ converges weakly in $H^1(B_1)$ to some z_∞;
(b) the corresponding traces $(z_j|_{\partial B_1^+})_{j\in\mathbb{N}}$ converge strongly in $L^2(\partial B_1^+)$;
(c) $(\lambda_j)_{j\in\mathbb{N}}$ has a limit $\lambda \in [0, \infty]$.

Now we establish the equi-coercivity and some further properties of the family of the auxiliary functionals $(\mathscr{G}_j)_{j\in\mathbb{N}}$. Notice that for all $w \in \mathscr{B}_j$, being $w|_{\partial B_1} = z_j|_{\partial B_1}$, it holds that

$$\begin{aligned} -\int_{B_1'} w \frac{\partial h}{\partial x_{n+1}}(x', 0^+)\, \mathrm{d}x' &= \int_{B_1'} -(w + \lambda_j\, h) \frac{\partial h}{\partial x_{n+1}}(x', 0^+)\, \mathrm{d}x' \\ &\quad + \lambda_j \int_{B_1'} h \frac{\partial h}{\partial x_{n+1}}(x', 0^+)\, \mathrm{d}x' \geq 0, \end{aligned} \tag{3.33}$$

where we used $(w + \lambda_j\, h)|_{B_1'} \geq 0$. Therefore, we deduce from the very definition (3.30) that for all $w \in \mathscr{B}_j$

$$\int_{B_1} |\nabla w|^2 dx - \frac{3}{2} \int_{\partial B_1} z_j^2 \leq \mathscr{G}_j(w), \tag{3.34}$$

thus establishing the equi-coercivity of the sequence $(\mathscr{G}_j)_{j\in\mathbb{N}}$.

By taking into account (3.32), if $\lambda \in [0, +\infty)$ then

$$\liminf_j \mathscr{G}_j(z_j) \geq 1 - \int_{B_1} z_\infty^2 - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n - 4\lambda \int_{B_1'} z_\infty \frac{\partial h}{\partial x_{n+1}}(x', 0^+)\, \mathrm{d}x'. \tag{3.35}$$

Instead, if $\lambda = +\infty$ then (3.32) and (3.34) yield

$$\liminf_j \mathscr{G}_j(z_j) \geq 1 - \int_{B_1} z_\infty^2 - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n.$$

Hence in all instances, it is not restrictive (up to passing to a further subsequence which we do not relabel) to assume that $(\mathscr{G}_j(z_j))_{j\in\mathbb{N}} \subset \mathbb{R}$ has a limit in $(-\infty, +\infty]$. Finally, note that

$$\lim_j \mathscr{G}_j(z_j) = +\infty \quad \Longleftrightarrow \quad \lim_j \lambda_j \int_{B_1'} z_j \frac{\partial h}{\partial x_{n+1}}(x', 0^+)\, \mathrm{d}x' = -\infty. \tag{3.36}$$

3.4.3. *Asymptotic analysis of $(\mathscr{G}_j)_{j\in\mathbb{N}}$: Γ-convergence.* Next step of the proof is to upgrade the convergence of z_j to z_∞ to strongly $H^1(B_1)$ and to characterize the limiting functions z_∞.

Here we prove a Γ-convergence result for the family of energies $\mathscr{G}_j$. To this aim, we recall some basic definitions of this important notion introduced by De Giorgi.

Definition 3.4.4. Let (X, d) be a metric space and functionals $F_j : X \to \mathbb{R}$ for $j \in \mathbb{N} \cup \{\infty\}$. We say that a sequence of functionals F_j Γ-converge to F_∞ (and we write $F_\infty = \Gamma\text{-}\lim F_j$) if

(a) for all $(w_j)_{j\in\mathbb{N}}$ and $w \in X$ such that $w_j \to w$

$$\liminf_{j\to+\infty} F_j(w_j) \geq F_\infty(w); \tag{3.37}$$

(b) for all $w \in X$ there exists $(w_j)_{j\in\mathbb{N}} \subset X$ such that $w_j \to w$ and

$$\limsup_{j\to+\infty} F_j(w_j) \leq F_\infty(w). \tag{3.38}$$

$(w_j)_{j\in\mathbb{N}}$ is called a *recovery sequence*

This is a simple consequence of the definition.

LEMMA 3.4.5. *Let (X,d) be a metric space and $F_j, F_\infty : X \to \mathbb{R}$ functionals such that $F_\infty = \Gamma\text{-}\lim F_j$. If $(x_j)_{j\in\mathbb{N}} \subset X$ is a sequence such that*

$$\lim_{j\to+\infty} F_j(x_j) = \lim_{j\to+\infty} \inf_X F_j,$$

then every accumulation point x_∞ of $(x_j)_{j\in\mathbb{N}}$ is a minimum point of F_∞ and

$$\min_X F_\infty = F_\infty(x_\infty) = \lim_{j\to+\infty} F_j(x_j).$$

PROOF. Assume that $x_{j_k} \to x_\infty$. Then, for every $w \in X$, by using (b) ((w_j) and $(\bar x_j)$ are recovery sequence for w and x_∞) and (a) we have that

$$\begin{aligned} F_\infty(w) &\overset{(b)}{\geq} \limsup_{j\to+\infty} F_j(w_j) \geq \lim_{j\to+\infty} \inf_X F_j = \lim_{j\to+\infty} F_j(x_j) \\ &= \lim_{k\to+\infty} F_{j_k}(x_{j_k}) \overset{(a)}{\geq} F_\infty(x_\infty) \overset{(b)}{\geq} \limsup_{j\to+\infty} F_j(\bar x_j) \geq \lim_{j\to+\infty} \inf_X F_j. \end{aligned}$$

□

We prove the following proposition.

PROPOSITION 3.4.6. *We have the following Γ-convergence result.*

(1) *If $\lambda \in [0,+\infty)$, then $(z_\infty + \lambda\, h)|_{B_1'} \geq 0$ and $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(1)}_\infty$, where*

$$\mathscr{G}^{(1)}_\infty(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n & \\ \displaystyle\qquad\qquad -4\lambda \int_{B_1'} z\, \frac{\partial h}{\partial x_{n+1}}(\hat x, 0^+)\, \mathrm{d}x' & \text{if } z \in \mathscr{B}^{(1)}_\infty, \\ +\infty & \text{if } z \in L^2(B_1)\setminus \mathscr{B}^{(1)}_\infty, \end{cases}$$

where

$$\mathscr{B}^{(1)}_\infty := \Big\{ z \in z_\infty + H^1_0(B_1) : (z + \lambda\, h)|_{B_1'} \geq 0 \Big\}.$$

(2) *If $\lambda = +\infty$ and $\lim_j \mathscr{G}_j(z_j) < +\infty$, then $z_\infty|_{B_1^{\prime,-}} = 0$ with $B_1^{\prime,-} = B_1' \cap \{x_n \leq 0\}$ and $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(2)}_\infty$, where*

$$\mathscr{G}^{(2)}_\infty(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n & \text{if } z \in \mathscr{B}^{(2)}_\infty, \\ +\infty & \text{if } z \in L^2(B_1)\setminus \mathscr{B}^{(1)}_\infty, \end{cases}$$

where

$$\mathscr{B}^{(2)}_\infty := \Big\{ z \in z_\infty + H^1_0(B_1) : z|_{B_1^{\prime,-}} = 0 \Big\}.$$

(3) *if $\vartheta = +\infty$ and $\lim_j \mathscr{G}_j(z_j) = +\infty$, then $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(3)}_\infty$, where $\mathscr{G}^{(3)}_\infty \equiv +\infty$ on the whole $L^2(B_1)$.*

If $\lim_j \mathscr{G}_j(z_j) < +\infty$, we show that actually $(z_j)_{j\in\mathbb{N}}$ converges strongly to z_∞ in $H^1(B_1)$. The equi-coercivity of $(\mathscr{G}_j)_{j\in\mathbb{N}}$ established in (3.34), the Poincarè inequality and the condition $\|z_j\|^2_{H^1} = 1$ in (3.32) imply the existence of an absolute minimizer ζ_j of $\mathscr{G}_j$ on L^2 with fixed $i \in \{1, 2\}$. Next note that by (3.31) z_j is an almost minimizer of $\mathscr{G}_j$, in the following sense:

$$0 \le \mathscr{G}_j(z_j) - \mathscr{G}_j(\zeta_j) \le \kappa_j \, \mathscr{G}_j(z_j) \le \kappa_j \cdot \sup_j \mathscr{G}_j(z_j).$$

Hence, recalling that we have assumed the existence of the limit $\mathscr{G}(z_j)$, we can apply Lemma 3.4.5 to infer that z_∞ is the unique (due to the strict convexity of $\mathscr{G}^{(i)}_\infty$) minimizers of $\mathscr{G}^{(i)}_\infty$ for $i = 1, 2$. In particular, using the strong convergence of the traces in $L^2(\partial B_1^+)$ we infer that

$$\int_{B_1} |\nabla z_j|^2 \, \mathrm{d}x \to \int_{B_1} |\nabla \zeta_\infty|^2 \, \mathrm{d}x,$$

in turn implying the strong convergence of $(z_j)_{j\in\mathbb{N}}$ to z_∞ in $H^1(B_1)$.

3.4.7. *Characterization of z_∞ in case (1).* We recall what we have achieved so far about z_∞, namely

(i) $\|z_\infty\|_{H^1} = 1$,
(ii) z_∞ is $3/2$-homogeneous and even with respect to $x_{n+1} = 0$,
(iii) z_∞ is the unique minimizer of $\mathscr{G}^{(1)}_\infty$ with respect to its own boundary conditions,
(iv) $z_\infty \in \mathscr{B}^{(1)}_\infty$, i.e. $z_\infty + \lambda\, h \ge 0$ on B_1'.

As an easy consequence of the properties above, we show now that

$$w_\infty := z_\infty + \lambda h$$

is a solution of the Signorini problem. To show this claim, for every $z \in \mathscr{B}^{(1)}_\infty$ we set $w := z + \vartheta\, h$ and by means of (3.25) we write

$$\begin{aligned}
\mathscr{G}^{(1)}_\infty(z) &= \int_{B_1} |\nabla w|^2 dx - \vartheta^2 \int_{B_1} |\nabla h|^2 dx - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n \\
&\quad - 2\lambda \int_{B_1} \nabla z \cdot \nabla h \, dx - 4\lambda \int_{B_1'} z \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, \mathrm{d}x' \\
&\overset{(3.25)}{=} \int_{B_1} |\nabla w|^2 dx - \lambda^2 \int_{B_1} |\nabla h|^2 dx - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^n \\
&\quad - 3\vartheta \int_{\partial B_1} z_\infty \, h \, d\mathcal{H}^n.
\end{aligned}$$

Therefore, since z_∞ is the unique minimizer of $\mathscr{G}^{(1)}_\infty$ and $w_\infty \ge 0$ on B_1', it follows from the previous computation that w_∞ is a solution of the Signorini problem. Using now the $3/2$-homogeneity of w_∞ and the classification of global solutions of the thin obstacle problem with such homogeneity, we deduce that w_∞, and hence z_∞, belongs to $\mathscr{H}_{3/2}$. This is a contradiction because $z_j \to z_\infty \in \mathscr{H}_{3/2}$ but $\mathrm{dist}_{H^1}(z_j, \mathscr{H}_{3/2}) = 1$.

3.4.8. *Discussion of case (3).* The heuristic idea to rule out case (3) is to correct the scaling of the energies in order to get a non-trivial Γ-limit for the rescaled functionals.

More in details, we start recalling that by (3.36) if $\lim_j \mathscr{G}_j(z_j) = +\infty$, then

$$\gamma_j := -4\lambda_j \int_{B_1'} z_j \frac{\partial h}{\partial x_{n+1}}(\hat{x}, 0^+)\, \mathrm{d}x' \uparrow +\infty. \tag{3.39}$$

Further, the convergence $z_j \to z_\infty$ in $L^2(B_1')$ and (3.33) yield

$$\begin{aligned}\lim_j \frac{\gamma_j}{\lambda_j} &= -4 \lim_j \int_{B_1'} z_j \frac{\partial h}{\partial x_n}(\hat{x}, 0^+)\, \mathrm{d}x' \\ &= -4 \int_{B_1'} z_\infty \frac{\partial h}{\partial x_n}(\hat{x}, 0^+)\, \mathrm{d}x' \in [0, +\infty),\end{aligned}$$

so that

$$\lambda_j\, \gamma_j^{-1/2} \to\uparrow +\infty. \tag{3.40}$$

It is then immediate to deduce that the right rescaling of the functionals $\mathscr{G}_j$ is obtained by dividing by a factor γ_j^{-1}: namely, for every $z \in \mathscr{B}_j$ we consider $\gamma_j^{-1}\mathscr{G}_j(z)$ and notice that

$$\gamma_j^{-1}\mathscr{G}_j(z) = \widetilde{\mathscr{G}_j}\Big(\gamma_j^{-1/2} z\Big), \tag{3.41}$$

where the functional $\widetilde{\mathscr{G}_j}$ is given by

$$\widetilde{\mathscr{G}_j}(w) := \begin{cases} \displaystyle\int_{B_1} |\nabla w|^2 dx - \frac{3}{2}\int_{\partial B_1} w^2\, d\mathcal{H}^n - 4\,\frac{\lambda_j}{\gamma_j^{1/2}} \int_{B_1'} w\, \frac{\partial h}{\partial x_n}(\hat{x}, 0^+)\, d\mathcal{H}^n & \\ & \text{if } w \in \widetilde{\mathscr{B}_j}, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.42}$$

where

$$\widetilde{\mathscr{B}}_j := \Big\{ w \in \gamma_j^{-1/2} z_j + H^1_0(B_1) :\ \big(w + \lambda_j\, \gamma_j^{-1/2}\, h\big)|_{B_1'} \geq 0 \Big\}. \tag{3.43}$$

Setting $\widetilde{z_j} := \gamma_j^{-1/2} z_j$, by (3.32) and $\gamma_j \uparrow +\infty$ we get $\widetilde{z_j} \to 0$ in $H^1(B_1)$. In addition, (3.41) and the very definition of γ_j in (3.39) imply that

$$\widetilde{\mathscr{G}_j}(\widetilde{z_j}) = 1 + O(\gamma_j^{-1}). \tag{3.44}$$

Furthermore, (3.31) rewrites as

$$(1-\kappa_j)\widetilde{\mathscr{G}_j}(\widetilde{z_j}) \leq \widetilde{\mathscr{G}_j}(\widetilde{z}) \qquad \text{for all } \widetilde{z} \in \widetilde{\mathscr{B}_j}.$$

In particular, by taking into account (3.40), $\tilde{z}_j \to 0$ in $H^1(B_1)$ and (3.44), namely $\lim_j \widetilde{\mathscr{G}_j}(\widetilde{z_j}) < +\infty$, we can argue exactly as in case (2) to deduce that

$$\Gamma(L^2(B_1))\text{-}\lim_j \widetilde{\mathscr{G}_j} = \widetilde{\mathscr{G}_\infty}$$

with

$$\widetilde{\mathscr{G}_\infty}(\widetilde{z}) := \begin{cases} \int_{B_1} |\nabla \widetilde{z}|^2 dx & \text{if } \widetilde{z} \in \widetilde{\mathscr{B}_\infty}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathscr{B}_\infty} := \{\widetilde{z} \in H^1_0(B_1) : \widetilde{z}|_{B_1^{\prime,-}} = 0\}$.

By the convergence $\widetilde{z_j} \to 0$ in $H^1(B_1)$, the null function turns out to be the unique minimizer of $\widetilde{\mathscr{G}_\infty}$ and $\lim_j \widetilde{\mathscr{G}_j}(\widetilde{z_j}) = \widetilde{\mathscr{G}_\infty}(0) = 0$, thus leading to a contradiction to (3.44).

3.4.9. *Characterization of z_∞ in case (2).* To this aim, as already pointed out, we need to investigate more closely the properties of the limit z_∞. From now on we assume that we are in the setting of case (2): i.e. $\lambda = +\infty$ and $\lim_j \mathscr{G}_j(z_j) < +\infty$.

We exploit the fact that ψ_j is a point of minimal distance of c_j from $\mathscr{H}_{3/2}$ to deduce that z_∞ is orthogonal to the tangent space $T_h\mathscr{H}_{3/2}$. We start noticing that $\lambda = +\infty$ implies that $\lambda_j > 0$ for all j large enough. Moreover, by the minimal distance condition (3.23) we infer that, for all $\nu \in \mathbb{S}^{n-1}$ and $\lambda \geq 0$,

$$\|z_j\|_{H^1} \leq \|\psi_j - \lambda h_\nu + z_j\|_{H^1},$$

or, equivalently,

$$-\|\psi_j - \lambda h_\nu\|^2_{H^1(B_1)} \leq 2\langle z_j, \psi_j - \lambda h_\nu\rangle. \tag{3.45}$$

Therefore, assuming $(\lambda_j, e_n) \neq (\lambda, \nu)$ and renormalizing (3.45), we get

$$-\|\psi_j - \lambda h_\nu\|_{H^1} \leq 2\langle z_j, \frac{\psi_j - \lambda h_\nu}{\|\psi_j - \lambda h_\nu\|_{H^1}}\rangle,$$

and by taking the limit $(\lambda, \nu) \to (\lambda_j, e_{n-1})$ we conclude

$$\langle z_j, \zeta\rangle = 0 \qquad \text{for all } \zeta \in T_{\psi_j}\mathscr{H}_{3/2} = T_h\mathscr{H}_{3/2}, \tag{3.46}$$

where

$$T_h\mathscr{H}_{3/2} = \{\alpha\, h + v_{e_n,\xi} :\ \xi \cdot e_{n+1} = \xi \cdot e_n = 0,\ \alpha \in \mathbb{R}\}, \tag{3.47}$$

where we have set

$$v_{e,\xi}(x) := (\hat{x} \cdot \xi)\sqrt{\sqrt{(\hat{x} \cdot e)^2 + x_{n+1}^2} + \hat{x} \cdot e} \tag{3.48}$$

Note moreover that

$$v_{e,\xi}(x) = \sqrt{2}\,(\hat{x} \cdot \xi)\,\mathrm{Re}\big[(\hat{x} \cdot e + i\, x_{n+1})^{1/2}\big],$$

where the determination of the complex square root is chosen in such a way that $v_{e,\xi} \geq 0$ in $\{x_{n+1} = 0\}$.

Now letting $j \uparrow \infty$ in the equality above we get that

$$\langle z_\infty, \zeta\rangle = 0 \qquad \text{for all } \zeta \in T_h\mathscr{H}_{3/2}. \tag{3.49}$$

A consequence of Let $z_\infty : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the following:

(a) z_∞ solves the boundary value problem

$$\begin{cases} \Delta z_\infty = 0 & \text{in } B_1 \setminus \{x_n \leq 0\}, \\ z_\infty = 0 & \text{on } B_1'^{,-}; \end{cases} \tag{3.50}$$

(b) $z_\infty(x', x_{n+1}) = z_\infty(x', -x_{n+1})$ for every $(x', x_{n+1}) \in B_1$;
(c) z_∞ is $3/2$-homogeneous,

is that

$$z_\infty(x) = a_0\, h(x) + \left(\sum_{i=1}^{n-1} a_i\, x_i\right) \sqrt{\sqrt{x_n^2 + x_{n+1}^2} + x_n}, \tag{3.51}$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{R}$, i.e. $z_\infty \in T_h \mathscr{H}_{3/2}$ (cp. (3.47)). The proof is left as an exercise.

We finally reach a contradiction: since z_∞ has the form in (3.53), we can choose h as test function in (**??**) to deduce $a_0 = 0$. Then take $\zeta = v_{e_{n-1},\xi}$ (cp. (3.48)) to deduce $a_1 = \ldots = a_{n-2} = 0$ by the arbitrariness of $\xi \in \mathbb{S}^{n-1}$ with $\xi \cdot e_n = \xi \cdot e_{n-1} = 0$.

Therefore, z_∞ is the null function, contradicting the strong convergence $1 = \|z_j\| \to \|z_\infty\|$.

3.4.10. *Proof of the* Γ*-convergence.* We refer to the paper [8].

3.5. Exercises.

Exercise 3.5.1. Let $z_\infty : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the following:

(a) z_∞ solves the boundary value problem

$$\begin{cases} \Delta z_\infty = 0 & \text{in } B_1 \setminus \{x_n \leq 0\}, \\ z_\infty = 0 & \text{on } B_1'^{,-}; \end{cases} \tag{3.52}$$

(b) $z_\infty(x', x_{n+1}) = z_\infty(x', -x_{n+1})$ for every $(x', x_{n+1}) \in B_1$;
(c) z_∞ is $3/2$-homogeneous.

Then,

$$z_\infty(x) = a_0\, h(x) + \left(\sum_{i=1}^{n-1} a_i\, x_i\right) \sqrt{\sqrt{x_n^2 + x_{n+1}^2} + x_n}, \tag{3.53}$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{R}$.

Hint. *Follow the three steps:*

(I) *to show the Hölder regularity of* z_∞ *and of all its transversal derivatives in the sense of distributions*

$$v_\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-1}} z_\infty}{\partial x_{n-1}^{\alpha_{n-1}}} \quad \text{with } \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-2};$$

(II) *the use of a bidimensional conformal transformation in the variable* (x_n, x_{n+1}) *to reduce the problem to the upper half ball* B_1^+*;*

(III) *the classification of all* $3/2$*-homogeneous solutions.*

4. The free boundary: the singular points

In this chapter we investigate the structure of another class of free boundary points, the *singular points.*

DEFINITION 4.0.1. Let $u \in H^1(B_1)$ be a solution to the thin obstacle problem. A point of the free boundary $x_0 \in \Gamma(u)$ is called *singular* it the coincidence set $\Lambda(u)$ has Lebesgue density zero at x_0:

$$\lim_{r\to 0^+} \frac{\mathcal{H}^n(\Lambda(u)\cap B'_r(x_0))}{r^n} = 0.$$

4.1. Frequency characterization. The singular points are also characterize by the value of their frequency.

LEMMA 4.1.1. *A point of the free boundary x_0 is singular if and only if its frequency equals $2m$ for some $m \in \mathbb{N}\setminus\{0\}$.*

PROOF. We start showing that if $x_0 \in \Gamma(u)$ is a singular point, then its frequency is an even natural number. Without loss of generality, assume that $x_0 = 0$ and consider the blowup rescalings

$$u_r(y) := \frac{r^{n/2} u(r\, y)}{\int_{\partial B_r(x_0)} u^2 \mathrm{d}\mathcal{H}^n}.$$

Let u_0 be any blowup of u at 0. We claim that u_0 is a harmonic function. Indeed, consider the Signorini boundary condition for the rescaled solution u_r:

$$\Delta u_r := 2\,\partial_{n+1} u_r\, \mathcal{H}^n \llcorner \Lambda(u_r).$$

We note next that by definition of singular point we have that

$$\lim_{r\to 0^+} \mathcal{H}^n\big(\Lambda(u_r)\cap B'_1\big) = \lim_{r\to 0^+} \frac{\mathcal{H}^n(\Lambda(u)\cap B'_r(x_0))}{r^n} = 0,$$

which implies that $\mathcal{H}^n \llcorner \Lambda(u_r)$ converges to zero as a measure in B'_1. Moreover, since u_r converges $C^1(B_1 + \cup B'_1)$ to u_0, we also infer that

$$\Delta u_r = 2\,\partial_{n+1} u_r\, \mathcal{H}^n \llcorner \Lambda(u_r) \rightharpoonup^* 2\,\partial_{n+1} u_0\, \mathcal{H}^n \llcorner \Lambda(u_0) = \Delta u_0,$$

as measure, thus proving that u_0 is harmonic in B_1 and therefore, by its homogeneity, u_0 is harmonic in the entire space. A homogeneous (hence with polynomial growth) harmonic function is by Liouville theorem a polynom, and therefore its homogeneity equals an integer $k \in \mathbb{N}$. Finally, considering that u_0 is positive on $\mathbb{R}^n \times \{0\}$ and even symmetric with respect to x_{n+1}, one easily infers that the degree of u_0 is indeed even (cf. Exercise 4.4.1).

For the reverse implication, let 0 be a point of the free boundary with frequency $2m$ for some $m \in \mathbb{N}\setminus\{0\}$, and let u_0 be any blowup of u at 0. We claim that u_0 is indeed a harmonic polynom. To this aim, we consider the harmonic polynoms

$$p_l(x) := \Re\big[(x_l + i x_{n+1})^{2m}\big] \quad l = 1, \dots, n,$$

and we consider a radial cut-off functions $\psi(x) := \varphi(|x|)$ with $\psi \in C_c^\infty(B_1)$. Recalling that $\Delta u_0 = 2\partial_{n+1}u_0\mathcal{H}^n \llcorner \Lambda(u_0)$ we can test as follows:

$$\begin{aligned}2\int_{\Lambda(u_0)} \partial_{n+1}u_0\,\psi\,p_l\,\mathrm{d}x' &= -\int_{B_1} \nabla u_0 \cdot \nabla\big(\psi\,p_l\big)\,\mathrm{d}x\\ &= -\int_{B_1} p_l\,\nabla u_0\cdot\nabla\psi\,\mathrm{d}x - \int_{B_1}\psi\,\nabla u_0\cdot\nabla p_l\,\mathrm{d}x\\ &= \int_{B_1}\big(-p_l\,\nabla u_0\cdot\nabla\psi + u_0\nabla\psi\cdot\nabla p_l\big)\,\mathrm{d}x, \qquad (4.1)\end{aligned}$$

where we used in (4.1) that $\Delta p_l = 0$. We consider next that $\nabla\psi(x) = \varphi'(|x|)\,\frac{x}{|x|}$ and infer that

$$\begin{aligned}2\int_{\Lambda(u_0)} \partial_{n+1}u_0\,\psi\,p_l\,\mathrm{d}x' &= \int_{B_1}\varphi'(|x|)\left(-p_l\,\nabla u_0\cdot\frac{x}{|x|} + u_0\,\nabla p_l\cdot\frac{x}{|x|}\right)\mathrm{d}x\\ &= \int_{B_1}\frac{\varphi'(|x|)}{|x|}\varphi'(|x|)\,\left(-2m\,p_l\,u_0 + 2m\,u_0\,p_l\right)\,\mathrm{d}x = 0,\end{aligned} \qquad (4.2)$$

where we also used the homogeneity of u_0 and p_l to infer that

$$\nabla p_l\cdot x = 2m\,p_l \quad \text{and} \quad \nabla u_0\cdot x = 2mu_0.$$

Summing now the equations (4.2) for $l = 1,\dots,n$ and setting $P := \sum_{l=1}^n p_l$, we infer that

$$\int_{\Lambda(u_0)} \partial_{n+1}u_0\,\psi\,P\,\mathrm{d}x' = 0. \qquad (4.3)$$

Since $P(x) > 0$ for every $x \neq 0$ and $\partial_{n+1}u_0 \leq 0$, it follow from (4.3) that $\partial_{n+1}u_0 \equiv 0$, *i.e.* $\Delta u_0 = 0$.

In particular, u_0 is a harmonic polynom of degree $2m$: this implies that

$$\mathcal{H}^n\big(\Lambda(u_0)\cap B_1\big) = 0,$$

(cp. Exercise 4.4.2). From the uniform convergence of u_r to u_0 we infer that

$$\limsup_{r\to 0^+}\Lambda(u_r) := \cap_{r>0}\cup_{0<s<r}\Lambda(u_r) \subset \Lambda(u_0),$$

and therefore

$$\lim_{r\to 0^+}\frac{\mathcal{H}^n(\Lambda(u)\cap B_r'(x_0))}{r^n} = \lim_{r\to 0^+}\mathcal{H}^n\big(\Lambda(u_r)\cap B_1'\big) = \mathcal{H}^n\big(\Lambda(u_0)\cap B_1'\big) = 0,$$

i.e. 0 is a singular point. □

4.2. Uniqueness of blowups. In this section we show that the blowup at any singular point is unique. This is a consequence of the following monotonicity formula of Monneau-type (see [**12**]).

PROPOSITION 4.2.1. *Let $u \in H^1(B_1)$ be a solution to the thin obstacle problem and let $x_0 \in \Gamma(u)$ be a singular point with frequency $2m$ for some*

$m \in \mathbb{N} \setminus \{0\}$. *Then, for every homogeneous harmonic polynom of degree* $2m$ *with*

$$p(x', 0) \geq 0 \quad \textit{and} \quad p(x', x_{n+1}) = p(x', -x_{n+1}),$$

the function

$$r \mapsto M_u(x_0, r, p) := \frac{1}{r^{n+4m}} \int_{\partial B_r(x_0)} (u-p)^2 \, d\mathcal{H}^n$$

is monotone nondecreasing for $r \in \big(0, \mathrm{dist}(x_0, \partial B_1)\big)$

Proof. Without loss of generality we assume that $x_0 = 0$ and compute the derivative of $M(r) := M_u(x_0, r, p)$: set for simplicity $w := u - p$,

$$\begin{aligned}
M'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{4m}} \int_{\partial B_1} w^2(ry)\, \mathrm{d}\mathcal{H}^n(y)\right) \\
&= -\frac{4m}{r^{4m+1}} \int_{\partial B_1} w^2(ry)\, \mathrm{d}\mathcal{H}^n(y) + \frac{2}{r^{4m}} \int_{\partial B_1} w(ry)\, \partial_\nu w(ry)\, \mathrm{d}\mathcal{H}^n(y) \\
&= -\frac{4m}{r^{n+4m+1}} \int_{\partial B_r} w^2\, \mathrm{d}\mathcal{H}^n + \frac{2}{r^{n+4m}} \int_{\partial B_r} w\, \partial_\nu w\, \mathrm{d}\mathcal{H}^n \\
&= -\frac{4m}{r^{n+4m+1}} \int_{\partial B_r} w^2\, \mathrm{d}\mathcal{H}^n + \frac{2}{r^{n+4m}} \int_{B_r} |\nabla w|^2\, \mathrm{d}x \\
&\quad + \frac{2}{r^{n+4m}} \int_{B_r} w\, \Delta w\, \mathrm{d}x. \qquad (4.4)
\end{aligned}$$

We notice next that

$$w\, \Delta w = (u-p)\, \Delta(u-p) = -p\, \Delta u = -p\, \partial_{n+1} u\, \mathcal{H}^n \llcorner \Lambda(u) \geq 0.$$

Moreover,

$$\begin{aligned}
\int_{B_r} |\nabla w|^2\, \mathrm{d}x &= \int_{B_r} |\nabla u|^2\, \mathrm{d}x + \int_{B_r} |\nabla p|^2\, \mathrm{d}x + 2\int_{B_r} \nabla u \cdot \nabla p\, \mathrm{d}x \\
&= \int_{B_r} |\nabla u|^2\, \mathrm{d}x + \int_{B_r} |\nabla p|^2\, \mathrm{d}x - 2\int_{B_r} u\, \Delta p\, \mathrm{d}x \\
&\quad + 2\int_{\partial B_r} u\, \partial_n p\, \mathrm{d}\mathcal{H}^n \\
&= \int_{B_r} |\nabla u|^2\, \mathrm{d}x + \int_{B_r} |\nabla p|^2\, \mathrm{d}x + \frac{4m}{r} \int_{\partial B_r} u\, p\, \mathrm{d}\mathcal{H}^n,
\end{aligned}$$

where we use the homogeneity of p:

$$\nabla p \cdot \frac{x}{|x|} = \frac{2m}{|x|}\, p.$$

Therefore, we derive from (4.4)

$$\begin{aligned} M'(r) &= \frac{2}{r^{n+4m}} \int_{B_r} |\nabla u|^2 \, \mathrm{d}x - \frac{4m}{r^{n+4m+1}} \int_{\partial B_r} u^2 \, \mathrm{d}\mathcal{H}^n \\ &\quad + \frac{2}{r^{n+4m}} \int_{B_r} |\nabla p|^2 \, \mathrm{d}x - \frac{4m}{r^{n+4m+1}} \int_{\partial B_r} u^2 \, \mathrm{d}\mathcal{H}^n \\ &= \frac{2}{r^{n+4m}} \int_{B_r} |\nabla u|^2 \, \mathrm{d}x - \frac{4m}{r^{n+4m+1}} \int_{\partial B_r} u^2 \, \mathrm{d}\mathcal{H}^n \geq 0, \end{aligned}$$

where we used that

$$I_u(r) = \frac{r \int_{B_r} |\nabla u|^2 \, \mathrm{d}x}{\int_{\partial B_r} u^2} \geq I_u(0^+) = 2m \equiv I_p(r) = \frac{r \int_{B_r} |\nabla p|^2 \, \mathrm{d}x}{\int_{\partial B_r} p^2}.$$

□

A simple corollary is now the uniqueness of the blowup.

Corollary 4.2.2. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, for every singular point x_0 of the free boundary there exists a unique blowup u^{x_0}.*

Proof. Without loss of generality let $x_0 = 0$ be a singular point. Assume by contradiction that there are two sequences of radii $(r_i)_{i \in \mathbb{N}}$ and $(s_i)_{i \in \mathbb{N}}$ such that

$$\lim_i u_{r_i} = p_1 \neq p_2 = \lim_i u_{s_i}.$$

Then, since p_1 is an admissible polynom for the Monneau-type monotonicity, we can consider $M(r) := M_u(0, r, p_1)$. Note that

$$\lim_{r_i \to 0^+} M(r_i) = 0 < \lim_{r_i \to 0^+} M(r_i) = \int_{\partial B_1} (p_2 - p_1)^2 \mathrm{d}\mathcal{H}^n = \lim_{s_i \to 0^+} M(s_i),$$

against the monotonicity of M. □

Another consequence of the Monneau-type monotonicity is the following nondegeneracy.

Proposition 4.2.3. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, for every singular point x_0 of the free boundary with frequency $2m$ we have that*

$$H(x_0, r) \geq C\, r^{n+4m} \quad \forall\, r \in \big(0, \mathrm{dist}(x_0, \partial B_1)\big). \tag{4.5}$$

Proof. Without loss of generality we assume that $x_0 = 0$ and we argue by contradiction: there is a sequence of radii $r_k \downarrow 0$ such that

$$\frac{H(r_k)}{r_k^{n+4m}} \to 0.$$

Let u_0 be the blowup of u at 0. Then,

$$M_u(0,r,u_0) = \frac{1}{r^{n+4m}} \int_{\partial B_r} (u-u_0)^2 \,\mathrm{d}\mathcal{H}^n = \frac{H(r)}{r^{n+4m}} + \frac{1}{r^{n+4m}} \int_{\partial B_r} u_0^2 \,\mathrm{d}\mathcal{H}^n - \frac{2}{r^{n+4m}} \int_{\partial B_r} u\, u_0 \,\mathrm{d}\mathcal{H}^n.$$

Therefore,

$$\lim_{k\to+\infty} M_u(0,r_k,u^0) = \int_{\partial B_1} u_0^2 \,\mathrm{d}\mathcal{H}^n.$$

In particular, by the monotonicity of the Monneau-type formula, we have that $M_u(0,r,u_0) \geq \int_{\partial B_1} u_0^2 \,\mathrm{d}\mathcal{H}^n$ for all $r>0$, which reads as

$$\begin{aligned}
0 &\leq \frac{1}{r^{n+4m}} \int_{\partial B_r} \big(u^2 - 2\,u\,u_0\big) \,\mathrm{d}\mathcal{H}^n \\
&= \frac{1}{r^{4m}} \int_{\partial B_1} \big(u^2(ry) - 2\,u(ry)\,u_0(ry)\big) \,\mathrm{d}\mathcal{H}^n(y) \\
&= \frac{1}{r^{4m}} \int_{\partial B_1} \big(u^2(ry) - 2\,r^{2m}\,u(ry)\,u_0(y)\big) \,\mathrm{d}\mathcal{H}^n(y) \\
&= \int_{\partial B_1} \big(\frac{H(r)}{r^{n+4m}} u_r^2 - 2\,\frac{H^{1/2}(r)}{r^{n/2+2m}}\, u_r\, u_0\big) \,\mathrm{d}\mathcal{H}^n(y).
\end{aligned}$$

where we used that

$$u_r(y) = \frac{r^{n/2} u(ry)}{H^{1/2}(r)}.$$

Dividing by $r^{-n/2-2m} h(r)^{1/2}$ and taking the limit along the sequence $r_k \to 0^+$ we infer that

$$-2 \int_{\partial B_1} u_0^2 \,\mathrm{d}\mathcal{H}^n \geq 0,$$

which gives the desired contradiction. □

In particular, for every singular point x_0 with frequency $2m$, one can also consider these new rescalings

$$\tilde{u}_{x_0,r}(y) := \frac{u(x_0+ry)}{r^m}.$$

As a corollary of Proposition 4.2.3 we get the following.

Corollary 4.2.4. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, for every singular point x_0 of the free boundary with frequency $2m$ there exists a harmonic polynom q_{x_0} of degree $2m$ which is the unique limit as $r \to 0^+$ of the rescalings $\tilde{u}^{x_0,r}$.*

Proof. From Corollary 2.3.3 and Proposition 4.2.3 the rescalings $\tilde{u}_{x_0,r}(y)$ have equibounded L^2 norms at the boundary which are uniformly away from zero. Therefore, the conclusion follows from the compactness (implied for example by Theorem 2.5.1) and Monneau monotonicity formula. □

4.3. Stratification of singular points. In this section we prove a stratification for the singular part of the free boundary following [**12**]. The key ingredients is the following uniform estimate. For simplicity we denote by Γ_{2m} the set of singular free boundary points with frequency $2m$.

PROPOSITION 4.3.1. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, the map $\Gamma_{2m} \ni x_0 \mapsto q_{x_0} \in L^2(\partial B_1)$ is continuous and for every compact subset $K \subset B_1'$ there exists a modulus of continuity σ_K such that*

$$\big|u(x) - q_{x_0}(x - x_0)\big| \le \sigma_K(|x - x_0|)\,|x - x_0|^{2m} \tag{4.6}$$

$\forall\, x_0 \in \Gamma_{2m} \cap K$, $\forall\, x \in B_1$.

PROOF. For every $x_0 \in \Gamma_{2m} \cap K$ and for every $\varepsilon > 0$, there exists $r_{x_0,\varepsilon} > 0$ such that

$$\big\|\tilde u_{x_0,r} - q_{x_0}\big\|_{L^2(\partial B_1)} \le \varepsilon \quad \forall\, r \le r_{x_0,\varepsilon}.$$

In particular, by continuity we deduce that there exists $\delta_{x_0,\varepsilon} > 0$ such that

$$\big\|\tilde u_{y_0,r_{x_0,\varepsilon}} - q_{x_0}\big\|_{L^2(\partial B_1)} \le \varepsilon \quad \forall\, y_0 \in \Gamma_{2m} \cap B_{\delta_{x_0,\varepsilon}}.$$

Using Monneau's monotonicity formula, we infer that

$$\big\|\tilde u_{y_0,r} - q_{x_0}\big\|_{L^2(\partial B_1)} \le \varepsilon \quad \forall\, r \le r_{x_0,\varepsilon}, \forall\, y_0 \in \Gamma_{2m} \cap B_{\delta_{x_0,\varepsilon}}.$$

and hence $\|q_{y_0} - q_{x_0}\|_{L^2(\partial B_1)} \le \varepsilon$, thus proving the first claim of the proposition.

Now, covering the compact set $K \cap \Gamma_{2m}$ with finitely many balls $B_{\delta_{x_0,\varepsilon}(x_0)}$, we infer that (set $\bar r_\varepsilon := \min_{x_0} r_{x_0,\varepsilon}$)

$$\big\|\tilde u_{y_0,r} - q_{y_0}\big\|_{L^2(\partial B_1)} \le \varepsilon \quad \forall\, r \le \bar r_\varepsilon, \forall\, y_0 \in \Gamma_{2m} \cap K.$$

Finally, we notice that, since $\tilde u_{y_0,r}$ are solution to the Signorini problem, then $(\tilde u_{y_0,r})^\pm$ are subharmonic function (cp. Exercise 2.6.1). Therefore, we can use the usual L^∞ estimate for subharmonic functions to conclude that

$$\big\|\tilde u_{y_0,r} - q_{y_0}\big\|_{L^\infty(B_{1/2})} \le C\big\|\tilde u_{y_0,r} - q_{y_0}\big\|_{L^2(\partial B_1)} \le C\,\varepsilon,$$

$\forall\, r \le \bar r_\varepsilon, \forall\, y_0 \in \Gamma_{2m} \cap K$. From the arbitrariness of ε, one easily concludes (4.6). □

Next we introduce the notion of invariant space for harmonic polynomials which are nonnegative on B_1', even symmetric with respect to x_{n+1} and of degree $2m$:

$$S(p) := \big\{y \in \mathbb{R}^n \times \{0\} : p(x + y) = p(x)\ \forall\, x \in \mathbb{R}^{n+1}\big\}.$$

It is easy to verify that, for every polynom $p \neq 0$ as above $\dim(S(p)) \le n - 1$ (cp. Exercise 4.4.3)

The main result is now the following (cp. [**12**]).

THEOREM 4.3.2. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, for every $m \in \mathbb{N}$ and for every $k \in \{0, \dots, n-1\}$ the set*

$$\Gamma_{2m}^{(k)} := \left\{x_0 \in \Gamma_{2m} : \dim(S(q_{x_0})) = k\right\}$$

is locally a subset of a C^1 regular submanifold of dimension k.

PROOF. 1. We consider the sets A_l of all points $x_0 \in \Gamma_{2m} \cap B_{1-1/l}$ such that

$$\frac{r^{2m}}{l} \le \max_{|x-x_0|=r} u(r) \le l\, r^{2m} \quad \forall\, r \in (0, 1-|x_0|). \tag{4.7}$$

In particular, by Proposition 4.2.3 (and Corollary 2.3.3) we have that $\Gamma_{2m} = \cup_{l\ge 1} A_l$. Moreover, the sets A_l are closed: indeed, if $A_l \ni x_k \to x_0$, then (4.7) holds for x_0 too, by upper semicontinuity $\lambda(x_0) \ge 2m$ and actually the equality holds because by Corollary 2.3.3 we have that $\lambda(x_0) > 2m$ implies $|u(x)| \le c|x-x_0|^{\lambda(x_0)}$ against (4.7).

2. Whitney data. We fix now a given $l \ge 1$. We show that the functions $f_\alpha : A_l \to \mathbb{R}$,

$$f_\alpha(x) := D^\alpha q_x(0), \quad |\alpha| \le 2m,\ \forall\, x \in A_l,$$

are set of Whitney data, i.e.

$$\begin{cases} \left|f_\alpha(x) - \sum_{|\beta|\le 2m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}\,(x-y)^\beta\right| = o\Big(|x-y|^{2m-|\alpha|}\Big) \\ \text{for } \ x, y \in A_l, \ \text{ as } \ |x-y| \to 0. \end{cases} \tag{4.8}$$

The case $|\alpha| = 2m$ follows from the continuity of the blowups in Proposition 4.3.1: indeed, in this case we have

$$|f_\alpha(x) - f_\alpha(y)| = |D^\alpha p_x(0) - D^\alpha p_y(0)| \le C\|p_x - p_y\|_{L^\infty(B_1)} = o(|x-y|).$$

On the other hand, if $|\alpha| < 2m$, noting that $f_\gamma \equiv 0$ for $|\gamma| < 2m$, we need to show that

$$\left|\sum_{|\beta|=2m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}\,(x-y)^\beta\right| = |D^\alpha q_y(x-y)| = o\Big(|x-y|^{2m-|\alpha|}\Big).$$

Assuming this is not the case, there exist points $x_j, y_j \in A_l$ with $|x_j - y_j| =: r_j \to 0$ such that

$$|D^\alpha q_{y_j}(x_j - y_j)| \ge \delta\, |x_j - y_j|^{2m-|\alpha|},$$

for some $\delta > 0$. Up to passing to a subsequence, we can assume that $y_j \to y_0 \in A_l$, $(x_i - y_i)/r_i \to z_0$; therefore we obtain

$$|D^\alpha q_{y_0}(z_0)| \ge \delta. \tag{4.9}$$

Consider now the rescalings: $v_j(x) := r_i^{-2m} u(y_i + r_i x)$. Note that, from (4.6) we have that $\|v_j - q_{y_j}\|_{L^\infty(B_1)} = o(1)$ and therefore $v_j \to q_{y_0}$ locally uniformly. Moreover, since $x_j \in A_l$, i.e.

$$\frac{r^{2m}}{l} \le \sup_{|x-x_j|=r} u(x) \le l\, r^{2m} \quad \forall\, 0 < r < 1 - \frac{1}{l},$$

we infer that
$$\frac{r^{2m}}{l} \leq \sup_{|x-z_0|=r} q_{y_0}(x) \leq l\, r^{2m} \quad \forall\, r > 0.$$
In particular, q_{y_0} is a homogeneous polynom of degree $2m$ around z_0, thus contradicting (4.9) since $|\alpha| < 2m$.

3. Whitney extension theorem. We can now apply the extension theorem by Whitney [**22**] to infer that there exists a function $g \in C^{2m}(\mathbb{R}^{n+1})$ such that
$$D^\alpha g(x) = f_\alpha(x) \quad \forall\, x \in A_l.$$
In particular, $A_l \subset \{g = 0\}$.

4. To conclude the proof, we notice that for every point $x_0 \in \Gamma^{(k)}_{2m}$ we have that the blowup q_{x_0} is a polynom of degree $2m$ depending only on $n+1-k$ variables, say $q_{x_0}(y_1, \ldots, y_{n+1-k}, 0, \ldots, 0)$ with co-ordinates $x = (y_1, \ldots, y_{n+1-k}, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. This means that there exists a multi-index $\beta \in \mathbb{N}^{n+1}$ with $|\beta| = 2m$ such that
$$\beta = (\beta_1, \ldots, \beta_{n+1-k}, 0, \ldots, 0) \quad \text{and} \quad \beta_i \neq 0 \quad \forall\, i = 1, \ldots, n+1-k.$$
Consider the multi-indexes $\alpha_i := (\beta_1, \ldots, \beta_i - 1, \ldots, \beta_{n+1-k}, 0, \ldots, 0)$ and the functions $F_i(x) := D^{\alpha_i} g(x)$ for $i = 1, \ldots, n+1-k$. Then, the function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n-k}$ given by $F(x) := \big(F_1(x), \ldots, F_{n+1-k}(x)\big)$ has Jacobian $D_y F(x_0) = \big(\frac{\partial F_i}{\partial y_j}(x_0)\big)_{i,j=1,\ldots,n+1-k}$ invertible. Note that $F(x_0) = 0$; therefore, by the implicit function theorem there is a neighborhood U of x_0 such that $\{F = 0\} \cap U$ is a C^1-regular submanifold. Recalling that $F(x) = 0$ for every point $x \in \Gamma_{2m}$, we conclude the proof. □

4.4. Exercise.

Exercise 4.4.1. Let p be a harmonic polynom in $\mathbb{R}^{n+1}$, homogeneous of degree $k \in \mathbb{N}$, with
$$p(x', 0) \geq 0 \quad \text{and} \quad p(x', x_{n+1}) = p(x', -x_{n+1}).$$
Show that $k = 2m$ for some $m \in \mathbb{N}$.

Exercise 4.4.2. Let p be a nontrivial harmonic polynom in $\mathbb{R}^{n+1}$, homogeneous of degree $2m \in \mathbb{N}$, with
$$p(x', 0) \geq 0 \quad \text{and} \quad p(x', x_{n+1}) = p(x', -x_{n+1}).$$
Show that
$$\mathcal{H}^n\big(\Lambda(u_0) \cap B_1\big) = 0,$$

Exercise 4.4.3. Let p be a nontrivial harmonic polynom of degree $2m$, which is nonnegative on B_1' and even symmetric with respect to x_{n+1}. Show that $\dim(S(p)) \leq n - 1$.

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