Computational complexity and 3-manifolds and zombies

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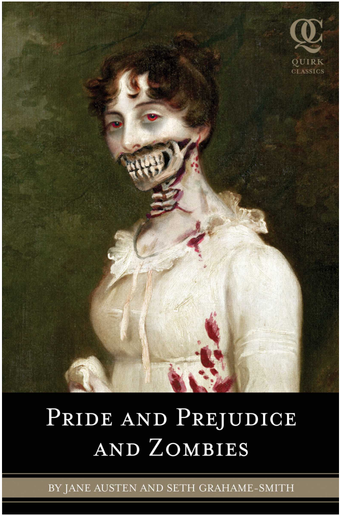
Joint with Greg Kuperberg. Based on arXiv:1707.03811 and work in preparation.



Outline

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Enumerative/coloring invariants

G: a finite group, fixed once and for all.

X: a space with some computable description, e.g. a simplicial complex, triangulated 3-manifold or knot diagram.

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What is the complexity of the problem of computing $\#H(X, G)$?

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Counting kernels

Related invariant:

$$Q(X,G) := \{\ \Gamma \lhd \pi_1(X)\ \mid\ \pi_1(X)/\Gamma \cong G\ \}.$$

The relation to $\#H(X,G)$:

$$\#H(X,G) = \sum_{J \leq G} \#\operatorname{Aut}(J) \cdot \#Q(X,J)$$

Knots

$c \in G$: a group element, fixed once and for all.

K: knot diagram

$\gamma \in \pi_1(K)$: meridian

Invariants:

$$\#H(K, \gamma, G, c) = \#\{f : \pi_1(S^3 \setminus K) \to G \mid f(\gamma) = c\}$$
$$\#Q(K, \gamma, G, c) = \#\{\Gamma \lhd \pi_1(S^3 \setminus K) \mid \exists \alpha : \pi_1/\Gamma \cong G \text{ w/ } \alpha(\gamma) = c\}$$

Main theorems

Fix G a finite, nonabelian simple group, and a nontrivial group element $c \in G$.

Theorem (Homology 3-spheres, Kuperberg-S)

Let M be a triangulation of an integer homology 3-sphere, thought of as computational input. Then the problem of computing $\#Q(M, G)$ is #P-complete via parsimonious reduction. Moreover, the reduction guarantees that $\#Q(M, J) = 0$ for all nontrivial, proper subgroups $J < G$.

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Note for M in the image of our reduction,

$$\#H(M, G) = 1 + \#\,\mathrm{Aut}(G) \cdot \#Q(M, G).$$

Corollary

Each of the following decision problems is NP-complete via Karp reduction:

- $\#Q(M, G) > 0$?
- $\#H(M, G) > 1$?
- $\#Q(K, \gamma, G, c) > 0$?
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Corollary

For each fixed $n \geq 5$, it is NP-complete to decide whether a homology 3-sphere M has a connected n-sheeted cover, even with the promise that it has no connected k-sheeted cover with $1 < k < n$.

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Remarks

- We say our reduction to $\#H(-, G)$ is "almost parsimonious," i.e. parsimonious up to the unavoidable trivial homomorphism and $\mathrm{Aut}(J)$ multiplicities.
- Krovi-Russell proved $\#H(L, A_5, c)$ is #P-complete, where L is a link and c is a conjugacy class with at least 4 fixed points. Not a (weakly) parsimonious reduction.
- Prior to our theorem, the hardness of counting/finding homomorphisms to a finite, nonabelian simple G wasn't known for finitely presented groups (much less 3-manifold groups).
- Our techniques extend to allow maps to any finite list of nonabelian simple groups.
- Expect "decoupling" results for $\#H(-, -, G, c)$ and $\#H(-, -, G, c')$ when c and c' are not outer automorphic.

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1. Results
2. Circuit reductions
3. Alphabet
4. Universality
5. Outlook

CSAT and #CSAT

Decision problem CSAT

Input: a Boolean circuit Z (see board)

Output:

$$\begin{cases} \text{YES} & \exists x : Z(x) = 1 \\ \text{NO} & \text{otherwise} \end{cases}$$

Counting analogue #CSAT

Input: a Boolean circuit Z

Output:

$$\#\{x \mid Z(x) = 1\}$$

#CSAT is #P-complete via parsimonious reduction.

Basic idea

Given Z, construct (in polynomial time) a triangulated homology 3-sphere M_Z so that

$$\#Q(M_Z, G) = \#\mathrm{CSAT}(Z)$$

and

$$\#Q(M_Z, J) = 0$$

for all nontrivial, proper $J < G$.

How? Gadget construction via combinatorial TQFT. (see board)

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#ZSAT

Issue 2: Our MCG circuits are equivariant w.r.t. $\mathrm{Aut}(G)$ action on $\hat{R}_{2g}$.

So we contrive a model to account for this, and to look like what our later theorems about the TQFT provide.

A: alphabet (large finite set)
K: finite group, acts on A
z: the only fixed point (all other orbits free). The "zombie digit."
$I \subset A$: initialization constraints
$F \subset A$: finalization constraints

$\#ZSAT_{A,K,I,F}$

Input: a planar, K-equivariant reversible circuit Z, over the alphabet A, with gates in $\mathrm{Rub}_K(A^2)$.
Output:

$$\#\{x \in (I \cup \{z\})^n \mid Z(x) \in (F \cup \{z\})^n\}$$

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The strategy

Lemma

As long as A, I and F aren't too big or too small, $\#ZSAT_{A,K,I,F}$ is $\#P$-complete via almost parsimonious reduction.

Take $K = \mathrm{Aut}(G)$ and pick $I, F \subset A \subset \hat{R}_g$ so that the reduction

$$\#\mathrm{ZSAT}_{A,K,I,F} \to \#H(-, G)$$

is "obvious."

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Schur invariant of a homomorphism

Orient Σ_g.

Then every $f \in R_g$ yields a $\mathrm{MCG}_*(\Sigma_g)$-invariant:

$$\mathrm{sch}(f) = f_*[\Sigma_g] \in H_2(G).$$

Key property [Livingston]: Suppose f_1 and f_2 are surjective. Then $\mathrm{sch}(f_1) = \mathrm{sch}(f_2)$ if and only if f_1 and f_2 are stably equivalent. (G can be any finite group here.)

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The alphabet

Let

$$R_g^0 = \{f : \pi_1(\Sigma_g) \twoheadrightarrow G \mid \mathrm{sch}(f) = 0\}.$$

R_g^0 is a free Aut(G)-set.

Our zombie symbol is the trivial homomorphism

$$z : \pi_1(\Sigma_g) \to G,$$

which is indeed fixed by every element in Aut(G).

Our alphabet is

$$\boxed{A = \{z\} \cup R_g^0}$$

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Remark on the knot case: Schur-type invariants for branched covers

Let C be the conjugacy class of c. Following Brand and Ellenberg-Venkatesh-Westerland, there exists a classifying space for (concordance classes of) C-branched G-covers of smooth manifolds:

$$BG_C = BG \bigsqcup_{ev: L^C BG \times S^1 \to BG} L^C BG \times D^2.$$

If S is an oriented surface and $f : \pi_1(S \setminus \{n \text{ points}\}) \to G$ a homomorphism with $f(\text{puncture}) \in C \cup C^{-1}$, then there is a corresponding C-branched Schur invariant

$$\text{sch}_C(f) = f_*[S] \in H_2(BG_C).$$

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Two goals

After that aside, let's remind ourselves:

$$A = \{z\} \cup R_g^0$$
$$= \{\text{trivial homomorphism}\} \cup \{\text{surjections with sch} = 0\}$$

1. Show the image of the Torelli subgroup under $\mathrm{MCG}_*(\Sigma_{2g}) \to \mathrm{Sym}_{\mathrm{Aut}(G)}(A^2)$ contains $\mathrm{Rub}_{\mathrm{Aut}(G)} A^2$.
2. Do more work to ensure $\pi_1(M_Z)$ has no spurious homomorphisms to G with "digits" in $\hat{R}_g \setminus A$, where $\hat{R}_g$ is the set of all homomorphisms $\pi_1(\Sigma_g) \to G$. SPOILER ALERT: we make them zombies too.

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An important theorem

Theorem (Dunfield-Thurston)

Let g be large enough (depends on G). Then the image of $\mathrm{MCG}_*(\Sigma_g)$ *inside*

$$\mathrm{Sym}(R_g^0/\operatorname{Aut}(G))$$

contains $\mathrm{Alt}(R_g^0/\operatorname{Aut}(G))$.

Great! But: need to understand the action on R_g^0, not its $\operatorname{Aut}(G)$-quotient. Also want some control over the action on the spurious homomorphisms in $\hat{R}_g \setminus A$.

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Refining the Dunfield-Thurston theorem I

The two actions of $\mathrm{MCG}_*(\Sigma_g)$ and $\mathrm{Aut}(G)$ on

$$\hat{R}_g = \{\pi_1(\Sigma_g) \to G\}$$

commute. Equivalently, the image of $\mathrm{MCG}_*(\Sigma_g)$ is contained in

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Similarly, the action on A is contained in

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Refining the Dunfield-Thurston theorem II

Let

$$R_g = \{\pi_1(\Sigma_g) \twoheadrightarrow G\}.$$

We don't need to worry about elements of

$$R_g \setminus R_g^0,$$

because they will never factor through a handlebody.

Refining the Dunfield-Thurston theorem III

So, we consider the action of $\mathrm{MCG}_*(\Sigma_g)$ on

$$R_g^0 \;\sqcup\; \hat{R}_g \setminus R_g \;\sqcup\; H_1(\Sigma_g).$$

Theorem

Let g be large enough. Then the image of $\mathrm{MCG}_*(\Sigma_g)$ inside

$$\mathrm{Sym}_{\mathrm{Aut}(G)}(R_g^0) \times \mathrm{Sym}_{\mathrm{Aut}(G)}(\hat{R}_g \setminus R_g) \times \mathrm{Sp}(2g, \mathbb{Z})$$

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Gadgets

We now apply our theorem to construct Torelli mapping classes that serve as gadgets for the binary gates in

$$\mathrm{Rub}_{\mathrm{Aut}(G)}(A^2).$$

Importantly: Our theorem allows us to treat all spurious homomorphisms as zombies, not just the zombie digit.

Final reduction

- Take $K = \mathrm{Aut}(G)$, A as above, let $I \subset A$ be the H^0 constraint, $F \subset A$ the H^1 constraint.
- Check these A, I and F aren't too big or too small to guarantee #P-hardness.
- Given Z, replace every gate with the appropriate $\mathrm{MCG}_*(\Sigma_{2g})$ gadget. Call the wired up mapping class $\phi_Z \in \mathrm{MCG}_*(\Sigma_{ng})$.
- Let $M_Z = H_0 \sqcup_{\phi_Z} H_1$. Triangulate.

Everything we've done guarantees:

- M_Z is a homology sphere.
- M_Z is constructed in linear time.
- Treating spurious digits as zombies ensures they can't both initialize and finalize, hence

$$\begin{aligned}\#H(M_Z, G) &= 1 + \#\mathrm{Aut}(G) \cdot \#Q(M_Z, G) \\ &= \#\mathrm{ZSAT}(Z).\end{aligned}$$

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$$\begin{aligned}\#H(M_Z, G) &= 1 + \#\mathrm{Aut}(G) \cdot \#Q(M_Z, G) \\ &= \#\mathrm{ZSAT}(Z).\end{aligned}$$

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