The suggestive power of pictures

Caroline Series

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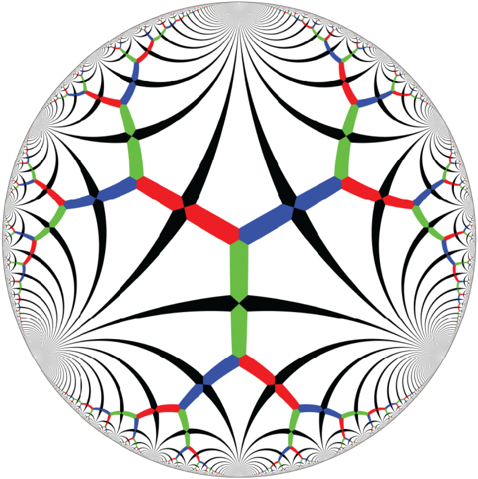
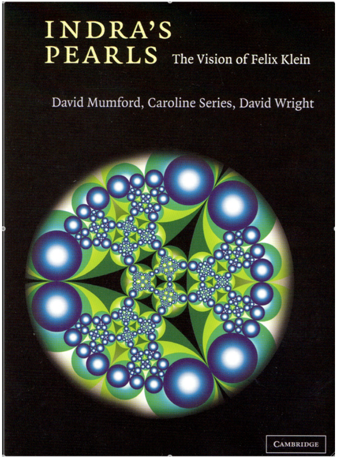
Graphics mainly by David Wright (Oklahoma) and Yasushi Yamashita (Nara)
Poster picture by Roice Nelson



Mumford's project

Inspired by some of the first pictures of the Mandelbrot set, around 1980 Mumford conceived the idea of a systematic computer investigation of limit sets of Kleinian groups. This means studying Möbius maps $z \mapsto \frac{az+b}{cz+d}$ acting on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Such maps can be considered as elements of $SL(2, \mathbb{C})$.

Given a group G of such maps, its limit set $\Lambda(G)$ is the set of accumulation points of G-orbits. G acts properly discontinuously on the regular set $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$ and $\Omega(G)/G$ is a finite union of Riemann surfaces.



Indra's Pearls. Mumford, Series and Wright. CUP Reprinted 2015

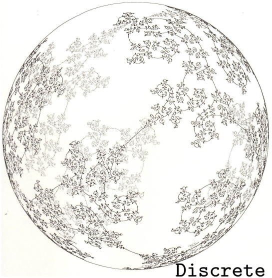
He carried out the project with David Wright. The results are reported in Indra's Pearls.

Limit sets and action on $\mathbb{H}^3$

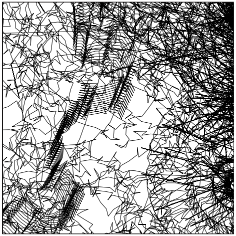
A group G of Möbius maps also acts by isometries on hyperbolic 3-space $\mathbb{H}^3$. If G is discrete then $\mathbb{H}^3/G$ is a hyperbolic manifold or orbifold. G is geometrically finite if there is a finite sided fundamental region for the $\mathbb{H}^3$ action.

If $\Lambda(G) \neq \hat{\mathbb{C}}$, then G is discrete, but the converse is in general false. If G is geometrically finite, then the components of $\Omega(G)/G$ correspond bijectively to the ends of the 3-manifold $\mathbb{H}^3/G$.

Geometrically finite groups are much easier to handle. Their theory was pretty much complete at the time of the first explorations (1980s), while geometrically infinite groups were very little understood.



Discrete



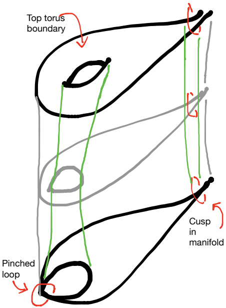
Non-discrete

The topological manifold

Mumford's strategy was to construct an explicit family of groups G depending on a single complex parameter $c \in \mathbb{C}$, in order to make a systematic computer study of their limit sets.

The family he chose is called the Maskit embedding of Teichmüller space. It represents a family of groups for which the corresponding hyperbolic manifold is a thickened punctured torus, with a second pinch point on the lower boundary, making the lower boundary a rigid triply punctured sphere.

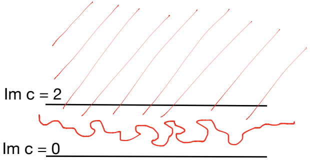
The family can be parametrised as $G_c =< A : z \mapsto c + 1/z, B : z \mapsto z + 2;\ c \in \mathbb{C} >$. The above description of $M = \mathbb{H}^3/G_c$ holds whenever G_c is free and discrete. The puncture corresponds to the fact that the commutator of A and B is parabolic. The second pinch point on the bottom boundary corresponds to B being parabolic.



The hyperbolic manifold for the Maskit embedding

The classic theory: the rough shape of $\mathcal{D}$

Let $\mathcal{D}$ denote the set of all free discrete groups G_c. The parameter c identifies it with a closed subset of $\mathbb{C}$. If $G \in \operatorname{Int}\mathcal{D}$, then it is known to be geometrically finite. We are interested in locating the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ in $\mathbb{C}$.



Since G_c is geom. finite, $\Omega(G_c)/G_c$ has two components. One is a once punctured torus $S_{1,1}$ and the other a sphere $S_{0,3}$ with 3 punctures.
We can determine the rough shape of $\mathcal{D}$ from the Teichmüller theory of these two surfaces. As a Riemann surface, $S_{0,3}$ is rigid since $3g - 3 + n = 0$, so $\mathrm{Teich}(S_{0,3}) = \{\mathrm{pt}\}$. Also $3g - 3 + n = 1$ so $\mathrm{Teich}(S_{1,1})$ is conformally the upper half plane – the same as the space of all conformal structures on a flat torus, classically identified with $\mathbb{H}^2$.

Mumford's problem is set up so that $\mathcal{D}$ looks as much like $\mathbb{H}^2$ as possible. It is invariant under $c \mapsto c + 2$. The entire region above $\Im c = 2$ is in $\mathcal{D}$, and $\partial\mathcal{D}$ is between the horizontal lines $\Im c = 0$ and $\Im c = 2$.

Locating $\mathcal{D}$ precisely: Cusp groups

Before Mumford's project, no one had any idea how to locate $\partial\mathcal{D}$ and only a few special points on $\partial\mathcal{D}$ were known. Thus the issue was to find points on $\partial\mathcal{D}$.

Bers and Maskit described one type of group, called cusp groups, on $\partial\mathcal{D}$. Pick a loop on the torus ∂M, corresponding to some $g \in G$. If the group G_c is discrete and geometrically finite and g is parabolic then $c \in \partial\mathcal{D}$.

Now the free homotopy classes of loops on the torus are enumerated by the rationals p/q. The corresponding group element $g_{p/q}$ is a particular product of the generators A, B:

$$g_{1/15} = A^{15}B, g_{2/5} = A^2BA^3B.$$

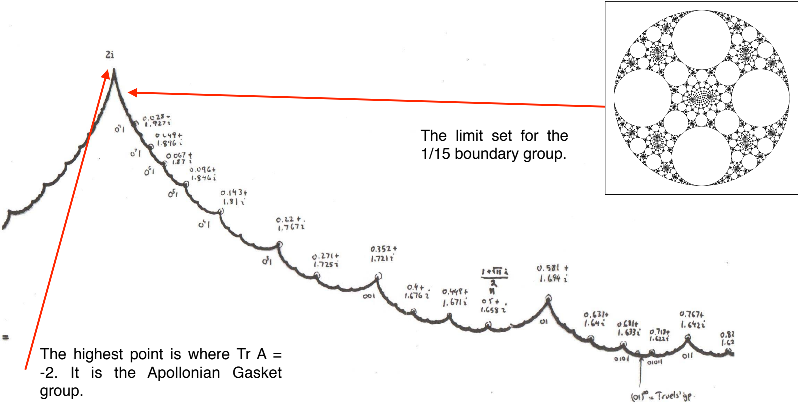
Since the matrix entries are polynomials in c, this gives a family of polynomial equations in c which can be solved to search for boundary points. For example

$$\operatorname{Tr} g_{2/5}(c) = \operatorname{Tr} A^2BA^3B(c) = \pm 2.$$

Mumford and David Wright did this systematically. Warning: There are lots of extraneous solutions which do not represent discrete groups. These are not in $\mathcal{D}$ so no good! Getting something which looked like a boundary required a lot of experimentation and skill.

The Mumford-Wright boundary plot

Each plotted point corresponds to a cusp group at which the particular word $W_{p/q}$ is parabolic. They also discovered that the limit sets of these groups are all made of beautiful patterns of tangent circles.



The limit set for the 1/15 boundary group.

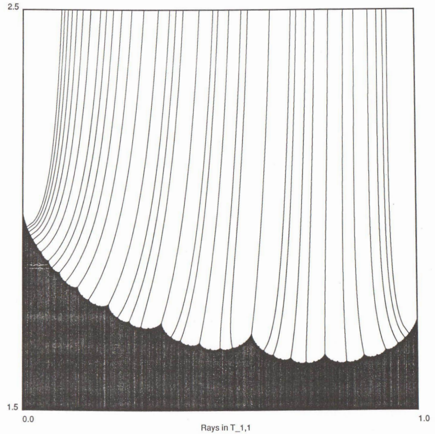
The highest point is where Tr A = -2. It is the Apollonian Gasket group.

The collection of plotted points appear to form a curve which one could well believe is a dense subset of $\partial\mathcal{D}$. But is it?

Pleating rays

Around 1990, I became interested in the Mumford-Wright pictures. Working with Linda Keen, we asked David Wright to plot the locus $\mathrm{Tr}\, W_{p/q} \in \mathbb{R}$ starting from $\mathrm{Tr}\, W_{p/q} = \pm 2$. This was the result.

The rays are pairwise disjoint. There is one for each $p/q \in \mathbb{Q}$ which starts on $\partial \mathcal{D}$ and follows the path along which $\mathrm{Tr}\, W_{p/q} \in (2, \infty)$. It is asymptotic to $\Re c = 2p/q$ at infinity.



Picture by David Wright, 1990

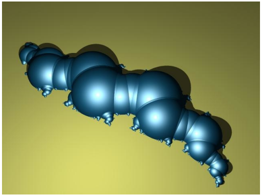
Theorem (McMullen 1991) Cusp groups are dense on the boundary.

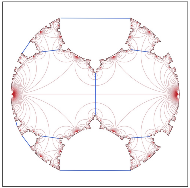
Theorem (Keen, Maskit, S. 1993) There is a unique cusp group for each p/q.

Theorem (Minsky 1998) For each irrational α there is exactly one punctured torus group with ending lamination α. Hence the boundary is a Jordan curve.

The meaning of the rays

Eventually Keen and I found the explanation of the rays and were able to confirm their apparent properties. Consider the hyperbolic convex hull $\mathcal{C}$ of $\Lambda(G)$. Suppose $\partial\mathcal{C}$ has a bending (pleating) line which is the axis of a loxodromic $T \in SL(2,\mathbb{C})$. There can be no twisting about its axis so $\mathrm{Tr}\, T \in \mathbb{R}$. The group G has lots of Fuchsian subgroups and $\Lambda(G)$ is made of overlapping circles whose intersection points are endpoints of $\mathrm{Ax}\, T$ and its conjugates.

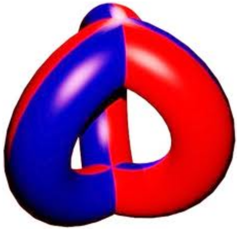




The p/q-ray $\mathcal{P}_{p/q}$ is the locus where $\partial\mathcal{C}$ is bent exactly along $\mathrm{Ax}\, W_{p/q}$ and its G-images. Along $\mathcal{P}_{p/q}$ the combinatorial arrangement of circles is fixed but their size and intersection angles vary. We proved that the rays behave exactly as appear in the pictures. (No singularities, non-empty, pairwise disjoint, asymptotic direction correct, ending on $\partial\mathcal{D}$, dense in $\mathcal{D}$, etc.)

The diagonal slice of Schottky space

If $G \subset SL(2, \mathbb{C})$ is discrete and free then the quotient $\mathbb{H}^3/G$ is a genus 2 handlebody. The space of all such groups is sometimes called Schottky space. This space has 3 complex parameters:



Proposition Any group $G = \langle A, B \rangle \subset SL(2, \mathbb{C})$ is determined up to conjugation by $(\mathrm{Tr}\, A, \mathrm{Tr}\, B, \mathrm{Tr}\, AB)$.

Tan Ser Peow, Yasushi Yamashita, and I recently studied the diagonal slice of Schottky space in which $\mathrm{Tr}\, A = \mathrm{Tr}\, B = \mathrm{Tr}\, AB$.

If G is free and discrete, the quotient $M = \mathbb{H}^3/G$ has 3-fold symmetry.

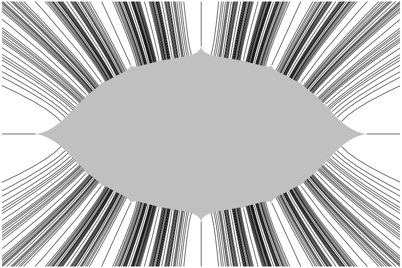
Our technique was to reduce the problem by quotienting the handlebody by its 3-fold symmetry. The resulting orbifold is a ball B with 2 order 3 cone axes, so ∂B is a sphere with 4 order 3 cone points.
The parameter space Δ is the same as that of the symmetrical handlebody. Because of the symmetry, essential simple loops on the boundary of either the handlebody or the quotient ball can be enumerated by rationals $p/q \bmod 2$.

The pleating rays

This picture made by Yasushi Yamashita represents the space Δ. The parameter is $x = \operatorname{Tr} A = \operatorname{Tr} B = \operatorname{Tr} AB$. The set of free discrete groups Δ is the complement of the central black region.

Essential simple loops on ∂B can be enumerated by rationals $p/q \bmod 2$. For each such p/q, there is a unique pleating ray which ends in a cusp group on $\partial\Delta$. The other end is asymptotic to the direction $\pi(p-q)/q$. The ray represents all orbifolds for which $\operatorname{Ax} g_{p/q}$ is the bending line of $\partial\mathcal{C}$. All the groups along a ray are discrete and free.

Theorem (S., Tan, Yamashita 2017)
The rational rays behave as appears. They are dense in Δ and are interpolated by rays corresponding to 'irrational' bending laminations.



Density of the higher dimensional analogue of rays is open: one of Thurston's few remaining unproven conjecturess.

The minimal volume hyperbolic 3-orbifold

By coincidence, the same parameter space came up in Gaven Martin's search for the minimum volume hyperbolic 3-orbifold. This means finding a discrete $G \subset PSL(2, \mathbb{C}) = \operatorname{Isom} \mathbb{H}^3$ for which $vol(\mathbb{H}^3/G)$ is minimum.

It featured in a picture, also by Yamashita, on the cover of the December 2016 AMS Notices.

As you can see the picture has some additional features – some of the rays extend into the grey central region, and there are lots of red and blue dots.

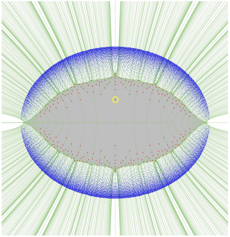
What are the dots?

Many of the groups involved in Martin's search turn out to be arithmetic. Very roughly, this means the group is discrete, the quotient has finite volume, and the matrix entries are integers in an algebraic number field.

Here is a very useful criterion for arithmeticity:

THEOREM GMMR* Let $G =< A, B | A^3 = B^3 = \mathrm{id} > \subset \mathrm{SL}(2, \mathbb{C})$. Then G is a discrete subgroup of an arithmetic group only if the trace parameter x is the root of a monic polynomial with integer coefficients with all its roots $(-1, 2)$ except possibly for one pair of complex conjugate roots.

*GMMR: Gehring, Machlachlan, Martin, Reid, 1997



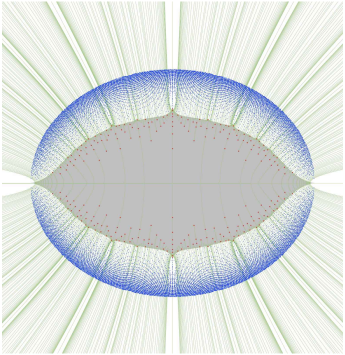
Flammang and Rhin (2005) found and listed all $15909(!)$ possible such polynomials with a complex root. Blue dots represent groups of infinite volume, red finite volume.

The minimum volume group G_0 is shown in yellow at the parameter value

$$x = \frac{1 + i\sqrt{-3 + 2\sqrt{5}}}{2}.$$

What about the extended rays?

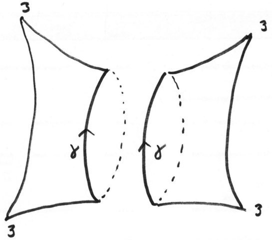
In Yasushi's picture, the green pleating rays $\mathcal{P}_{p/q}$ which densely fill Δ extend into the central grey region.



By considering values near ∞, one shows $\operatorname{Tr} g_{p/q}$ is negative on $\mathcal{P}_{p/q}$. Moving inwards along the ray, the translation length of $g_{p/q}$ decreases from ∞ to 0. At this point $g_{p/q}$ is parabolic and $\operatorname{Tr} g_{p/q} = -2$.

The rays are continued to the point where $\operatorname{Tr} g_{p/q} = 2$. Once past the point where $g_{p/q}$ is parabolic, $\operatorname{Tr} g_{p/q} \in (-2, 2)$ hence is elliptic. A group which contains an elliptic element is either not free ($g^n = \mathrm{id}$) or not discrete (g acts as irrational rotation of the circle). Most of the extended rays are so short you can't see them in the picture.

Geometry along pleating rays



Suppose we are on a pleating ray of a curve γ so that γ is a bending line of $\partial\mathcal{C}$. At points in $\operatorname{Int}\mathcal{D}$, γ is purely hyperbolic and cuts $\partial\mathcal{C}$ into two halves R, L say each of which is a totally geodesic disk with two order 3 cone points: a 'half pillow'. R and L lift to two families of hyperbolic planes in $\mathbb{H}^3$ which intersect along the lifts of $\operatorname{Ax}\gamma$.

Each plane is a hemisphere which meets $\partial\mathbb{H}^3 = \mathbb{C} \cup \infty$ in a circle. The exterior of this hemisphere is outside $\mathcal{C}$ and projects to an infinite volume part of $\mathbb{H}^3/G$.

Let $\tilde{R}, \tilde{L}$ be lifts which meet along $\operatorname{Ax} g_\gamma$ and call the corresponding circles C_R, C_L. These circles are stabilised by two Fuchsian subgroups G_R, G_L of G. C_R/G_R is a 'half pillow' so G_R has 2 order 3 generators whose product, the boundary curve, is g_γ. The order 3 generators are conjugates of the order 3 elliptics we started with. Thus G_R and G_L are Fuchsian triangle groups with infinite area.

Geometry on extended rays

At the point on the ray where $\operatorname{Tr} g_\gamma = -2$, g_γ is parabolic. Circles C_R, C_L are tangent at the single fixed point of g_γ. The planes sitting above C_R and C_L still project to infinite volume parts of $\mathbb{H}^3/G$.

G_R, G_L are Fuchsian groups with 2 order 3 generators whose product is parabolic. They are subgroups of the modular group. The corresponding group G has *cusps*.

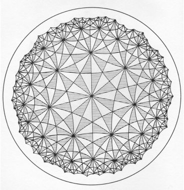
Continuing on the extended ray, C_R and C_L move apart and g_γ becomes elliptic. It still stabilises C_R and C_L, so its axis is a hyperbolic line which cuts each of the two planes above C_R and C_L orthogonally. G now only has a chance of being discrete if g_γ is elliptic of finite order n. The Fuchsian subgroups G_R, G_L become triangle groups with 2 order 3 generators whose product is elliptic. Such a group only exists if there is a hyperbolic triangle of angles $\pi/3, \pi/2, \pi/n$ with $\pi/3 + \pi/2 + \pi/n < \pi$, that is, $n \geq 7$.

There is still an infinite part of $\mathbb{H}^3/G$ 'beneath' the hemispheres above C_R and C_L, so $\mathbb{H}^3/G$ still has infinite volume.

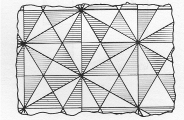
Beyond Fuchsian

What happens when we get to $n = 6$? The triangle becomes Euclidean and so do the groups G_R, G_L. This can only happen if C_R and C_L shrink to points and the endpoints of various axes come together.

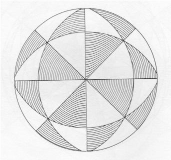
Proceeding further, the triangle becomes spherical so G_R and G_L are spherical groups and C_R and C_L become points inside $\mathbb{H}^3$ – the fixed points for G_R and G_L. For a discrete group, g must have finite order $n = 5, 4$ or 3.



Fuchsian (hyperbolic)



Euclidean



Spherical (Octahedral)

Proposition The only points on extended rays which can correspond to a finite volume orbifold $M = \mathbb{H}^3/G$ are $\mathrm{Tr}\, g = -2\cos 2\pi/n$ with $n = 6, 5, 4, 3$, that is, $\mathrm{Tr}\, g = -1, (-\sqrt{5}+1)/2, 0, +1$. If $n = 6$ there are two Euclidean subgroups giving two cusps in M; the other cases contain spherical subgroups which are icosahedral ($n = 5$); octahedral ($n = 4$); tetrahedral ($n = 3$).

Examples

- On the ray $\mathcal{P}_{1/2}$, which is the vertical line $x = 1/2 + ih$. $\operatorname{Tr} g_{1/2} = x^2 - x$.
  - $\operatorname{Tr} g_{1/2} = -1$ giving a Euclidean $(3,3,3)$ triangle subgroup at $x = (1 + i\sqrt{3})/2$. Traces are in $\mathbb{Q}(i\sqrt{3})$.
  - $\operatorname{Tr} g_{1/2} = (-\sqrt{5}+1)/2$ giving the minimum volume group G_0 at $x = (1 + i\sqrt{-3+2\sqrt{5}})/2$, volume $0.03905\ldots$. Here $(g_{1/2})^5 = 1$.
- On the ray $\mathcal{P}_{3/4}$. $\operatorname{Tr} g_{3/4} = x^4 - 3x^3 + x^2 + 3x$.
  - $\operatorname{Tr} g_{3/4} = (-\sqrt{5}+1)/2$ giving a spherical $(2,3,5)$ subgroup. (This is the group G_1 of next minimal volume $0.0408\ldots$.)
  - $\operatorname{Tr} g_{3/4} = +1$ giving a spherical $(2,3,3)$ subgroup.

The picture suggests many questions. For example:

- Are all of the groups of Proposition A arithmetic?
- Can the groups of Proposition A be used to speed up Martin's search?
- Do all the Flammang-Rhin groups lie on extended rays, and if so, why?
- Does every discrete finite volume group lie on an extended ray?

The Bowditch BQ-condition

The Bowditch or BQ-condition is a condition on the primitive elements of the free group F_2 on 2 generators, namely:

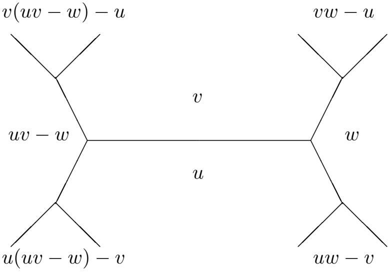
- All primitive elements are loxodromic
- $|\operatorname{Tr} U| \leq 2$ for at most finitely many primitive U.

Denote the set of groups satisfying the BQ-conditions by $\mathcal{B}$. Clearly $\mathcal{B} \supset \operatorname{Int} \mathcal{D}$ where $\mathcal{D}$ denotes the set of discrete free groups. Bowditch conjectured that if the commutator $[A, B]$ is parabolic, then $\mathcal{B} = \operatorname{Int} \mathcal{D}$.

Our motivation for studying the 'diagonal slice' Δ was to investigate the BQ-conditions computationally.

Starting from an initial triple, the traces of primitive elements can be rapidly computed using the trace relation:

$$\operatorname{Tr} U \operatorname{Tr} V = \operatorname{Tr} UV + \operatorname{Tr} UV^{-1}.$$



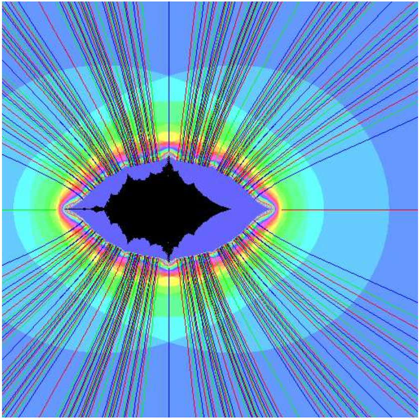
Tree of traces dual to the Farey tessellation.

The algorithm

Bowditch's results lead to a fast check of whether, for any initial triple of traces, the BQ-conditions hold.

- If z, w label the ends of an edge, put an arrow from z to w if $|z| > |w|$.

Theorem [Bowditch '98; Tan, Wong, Zhang ' 08] Define a subtree $\mathbb{T}$ as those edges on which there is a certain (given) upper bound on the adjacent traces. Then $\mathbb{T}$ is connected and any region is joined by a path of descending arrows which eventually land in $\mathbb{T}$. The parameter is in $\mathcal{B}$ iff $\mathbb{T}$ is finite.



Comparison of Δ to $\mathcal{B}$.

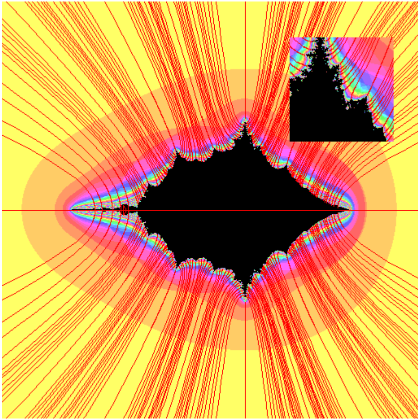
If $[A, B]$ is parabolic, then plots indicate that Bowditch's conjecture is correct. However this picture shows it fails in Δ. $\mathcal{B}$ is the entire region outside the black part. Plainly it is not equal to Δ, the part covered by the pleating rays.

The Y-rays

This remarkable picture illustrates the result of plotting certain branches of 'real trace' rays corresponding to the primitive elements in the diagonal slice. The primitive elements are *not* the possible bending lines of $\partial\mathcal{C}$ and the rays are *not* the true pleating rays.

In the course of trying to establish their geometrical meaning, I proved:

Theorem [S. 2017, claimed independently by Jaejong Lee and BinBin Xu]): Minsky's condition of primitive stability and Bowditch's BQ-condition are equivalent.



Y-rays filling the Bowditch set.

The motivation and proof involves a new condition called the bounded intersection property.

The bounded intersection property BIP

Lemmas:

• Every pair (u, v) of generators of F_2 is conjugate to a unique pair (u', v') both of which are cyclically shortest and palindromic wrt exactly one of the basic generator pairs $(a, b), (a, ba), (ab, b)$.

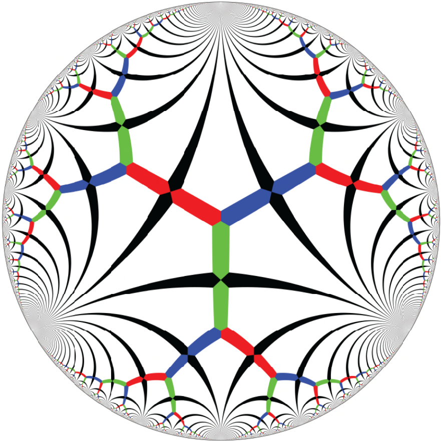
• The axis of a word which is palindromic in generators (u, v) cuts the common perpendicular of $\mathrm{Ax}\, u, \mathrm{Ax}\, v$ perpendicularly.

The bounded intersection property BIP is the condition that all axes of primitive elements palindromic wrt one of the 3 basic generator pairs meet the corresponding common perpendicular in a bounded interval.

Theorem: The Bowditch condition implies BIP.

Theorem: Bowditch and BIP imply primitive stability.

The suggestive power ....



Q. What does this picture suggest?

A. Proof by induction on the dual tree.

Q. Why isn't it quite the right picture?

A. Cyclic order of colours round a vertex should alternate with the level.