A computer search for ribbon alternating knots

(joint work with Frank Swenton)

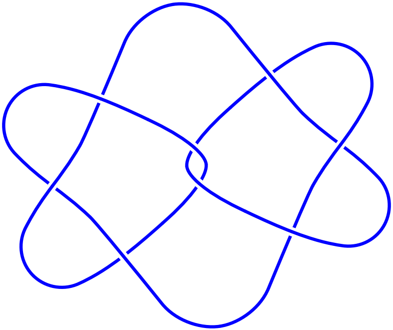
Brendan Owens

University of Glasgow

Warwick, 12th December 2017

Knots

A knot in S^3 is a smooth simple closed curve (up to smooth isotopy).



Slice knots

A knot is slice if it bounds a smoothly embedded 2-disk in the 4-ball. (Artin 1925, Fox-Milnor 1957)

These are useful in 4-dimensional topology: for example existence of a topologically slice, smoothly nonslice knot implies existence of an exotic structure on $\mathbb{R}^4$. (Hence Jake Rasmussen used Khovanov homology to show existence of exotic $\mathbb{R}^4$.)

One can exhibit an embedded surface in the 4-ball using a movie: a sequence of maxima (adding circles), saddles (band moves), and minima (removing circles).

Ribbon knots

A knot is ribbon if any of the following hold:

- it bounds a slice disk without maxima;
- a set of n band moves converts it to the $(n+1)$-component unlink;
- it bounds a ribbon-immersed disk in S^3.

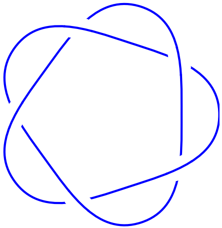
Examples: symmetric unions are ribbon.

Ribbon knots are slice; it is unknown if slice knots are ribbon.

Deciding if a knot is slice (or ribbon) is a hard problem.

Alternating knots

A knot or link in S^3 is alternating if it admits an alternating diagram.



We can immediately tell many geometric properties from an alternating diagram.

Example:

The knot above is prime, has no mutants, has genus 2, and has unknotting number 2. (Menasco 1982)(Crowell 1959, Murasugi 1958)(McCoy 2016)

Wishful thinking

Goals:

- find an algorithm to decide if a given alternating knot is slice (or ribbon);
- classify slice (or ribbon) alternating knots in some sensible way (a recursive description for example);
- determine whether slice alternating knots are ribbon.

Alternating links

Theorem (Greene 2015, Howie 2015)

A knot or link L in S^3 is alternating if and only if it admits two spanning surfaces in S^3 whose Gordon-Litherland forms are definite of opposite signs.

Equivalently: L admits two spanning surfaces F_1 and F_2 in S^3, and the double cover of S^4 branched along $F_1 \cup F_2$ is a connected sum of $\mathbb{CP}^2$'s.

Algorithmically ribbon knots

With Frank Swenton, we have implemented an algorithm to find ribbon alternating knots. We call the knots that it finds algorithmically ribbon. (Details to follow shortly.)

Theorem (O.-Swenton)

All slice two-bridge knots are algorithmically ribbon, as are all but one slice prime alternating knots of 12 or fewer crossings and all connected sums $K\# -K$ of an alternating knot with its mirror reverse. There are $29,263$ algorithmically ribbon prime alternating knots of 18 or fewer crossings.

Algorithm, in brief

Given an alternating diagram of a knot, seek a sequence of band moves and isotopies resulting in an unlink.

Rough idea: look for nice simple sequences.

Starting case: look for fusion number one ribbon alternating knots. That is, look for a sequence

$$K \xrightarrow[\quad]{\text{band move, isotopy}} \bigcirc\bigcirc.$$

Fusion number one algorithm

Given alternating K, looking for

$$K \xrightarrow{\text{band move, isotopy}} \bigcirc\bigcirc.$$

Try to stay "as alternating as possible". The unlink $\bigcirc\bigcirc$ does not admit a nonsplit alternating diagram. It does admit (many) almost-alternating (AA) diagrams.

Algorithmically ribbon disks with one band:

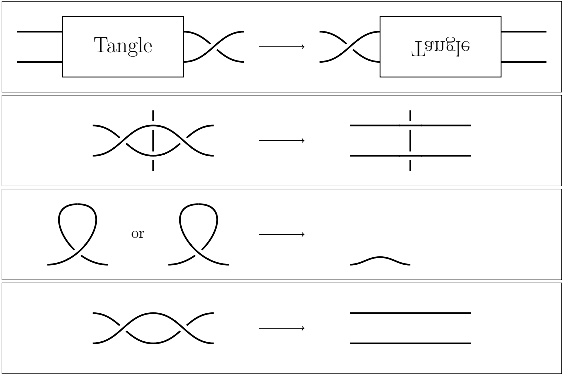
$$K \xrightarrow{\text{band move}} \text{AA} \xrightarrow{\text{isotopy}} \bigcirc\bigcirc.$$

Use Goeritz, Gordon-Litherland, and Donaldson to find band, then Tsukamoto to find the isotopy.

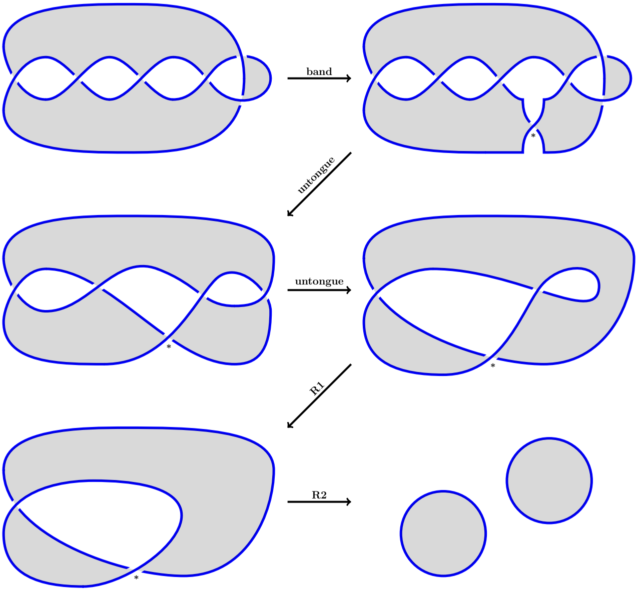
Finding isotopies

Theorem (Tsukamoto 2004)

Every almost-alternating diagram of the two-component unlink may be converted to the standard diagram $\bigcirc\ \bigcirc$ by a finite sequence of: flypes, untongue moves, Reidemeister 1's and a single Reidemeister 2.

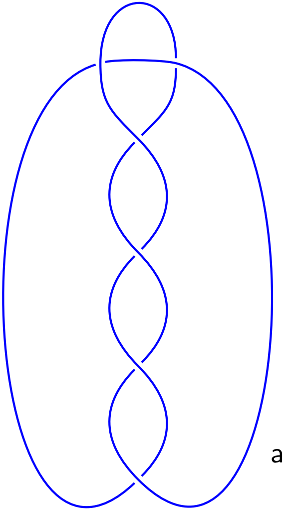


Example: stevedore is algorithmically ribbon



Next: finding bands

Goeritz (1933): associate two matrices to a chessboard-coloured knot diagram.



$$\rightsquigarrow \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 \\ -1 & 0 & 0 & 2 & -1 \end{pmatrix}$$

$$\rightsquigarrow G_b = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

and similarly $G_w = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$

Gordon-Litherland manifolds

Gordon-Litherland 1978:

$G_{b/w}$ is the intersection form of $\Sigma(D^4, F_{b/w})$ (the double cover of D^4 branched along the black/white surface).

Gordon-Litherland also generalised this to any embedded F in S^3 .

Also note, Goeritz matrices of alternating diagram are definite.

(The surfaces F_b and F_w are the definite surfaces from Greene's and Howie's characterisation of alternating links.)

Donaldson 1: obstruction to slice alternating

Theorem

Let $G_{b/w}$ be the Goeritz matrices associated to an alternating diagram of a knot K. If K is slice then there exist square integer matrices A and B with

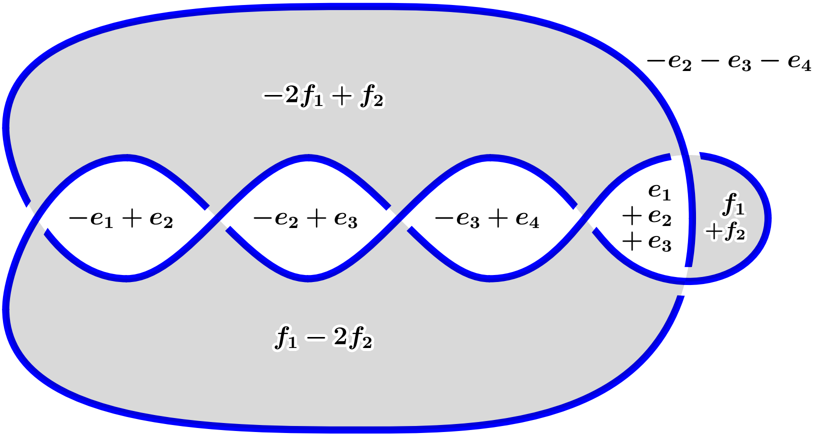
$$G_b = A^T A, \quad G_w = B^T B.$$

(Proof based on Donaldson's diagonalisation theorem; can be refined using Heegaard Floer theory, following Greene-Jabuka.)

We call an alternating diagram for which A and B exist bifactorisable. Theorem says: slice alternating $\implies$ bifactorisable.

Lisca 2007: for 2-bridge knots, slice $\iff$ bifactorisable.
(Not true for alternating.)

Example: stevedore is bifactorisable



Donaldson 2: obstruction to band moves

Applying a band move to a diagram gives rise to new ribbon-immersed surfaces.

Greene-O.-Strle (in preparation) generalise Gordon-Litherland pairing to ribbon surfaces. Adding a band gives one more generator.

If a band move is part of a ribbon disk, then get bifactorisation of the form

$$\begin{bmatrix} G_b & * \\ * & * \end{bmatrix} = \left[\begin{array}{c} A^T \\ \hline v^T \end{array}\right] \left[\begin{array}{c|c} A & v \end{array}\right], \begin{bmatrix} G_w & * \\ * & * \end{bmatrix} = \left[\begin{array}{c} B^T \\ \hline w^T \end{array}\right] \left[\begin{array}{c|c} B & w \end{array}\right].$$

Donaldson 3: obstruction to slice links

We say a diagram of a link L is MNA (minimally nonalternating) if it has k nonalternating crossings, l diagram components, and the nullity of the link is $k + l - 1$.
Examples: alternating diagrams, almost-alternating diagrams of ○ ○, $P(-2, 3, 6)$.

Theorem

Let $G_{b/w}$ be the Goeritz matrices associated to an MNA diagram of a link L. If L is slice then there exist $(m - k - l + 1) \times m$ and $(n - k - l + 1) \times n$ integer matrices A and B with

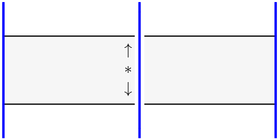
$$G_b = A^T A, \quad G_w = B^T B.$$

In other words,

$$\text{slice MNA} \implies \text{bifactorisable.}$$

Algorithmic bands

An algorithmic band is a length two, untwisted band move on an MNA diagram which passes Donaldson obstruction(s).



Algorithmic bands preserve MNA condition; for these bands Donaldson 2 and 3 are equivalent.

Fusion number one algorithmic ribbon disks

An alternating knot is fusion number one algorithmically ribbon if it can be converted to $\bigcirc\bigcirc$ by a single algorithmic band followed by a sequence of Tsukamoto moves.

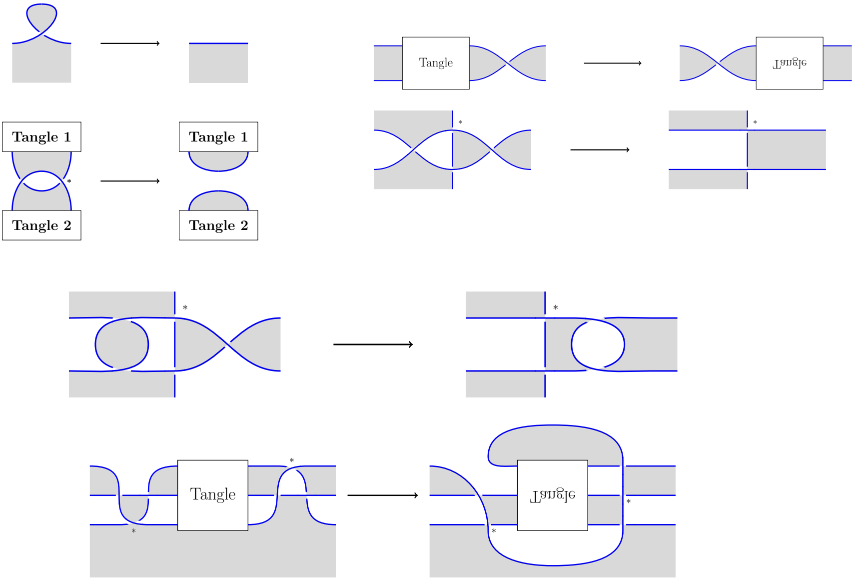
Theorem (O.-Swenton)

All slice two-bridge knots are fusion number one algorithmically ribbon. There are $22,444$ fusion number one algorithmically ribbon prime alternating knots of 18 or fewer crossings.

There are no known fusion number one alternating knots which are not algorithmically ribbon.

Tsukamoto's theorem gives a nice classification of fusion number one algorithmically ribbon knots.

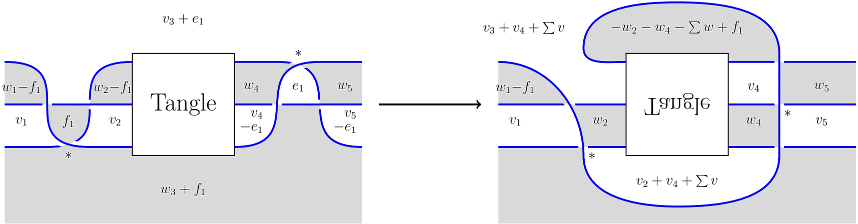
More bands need more Tsukamoto moves



Generalised Tsukamoto moves

A generalised Tsukamoto move is either R2 or flype or a connected tangle replacement $\Gamma \mapsto \Gamma'$ satisfying

1. $\Gamma \simeq \Gamma'$ (isotopic rel boundary);
2. Γ' has same number of nonalternating crossings, strictly fewer alternating crossings than Γ;
3. bifactorisability is preserved.



Some conjectures

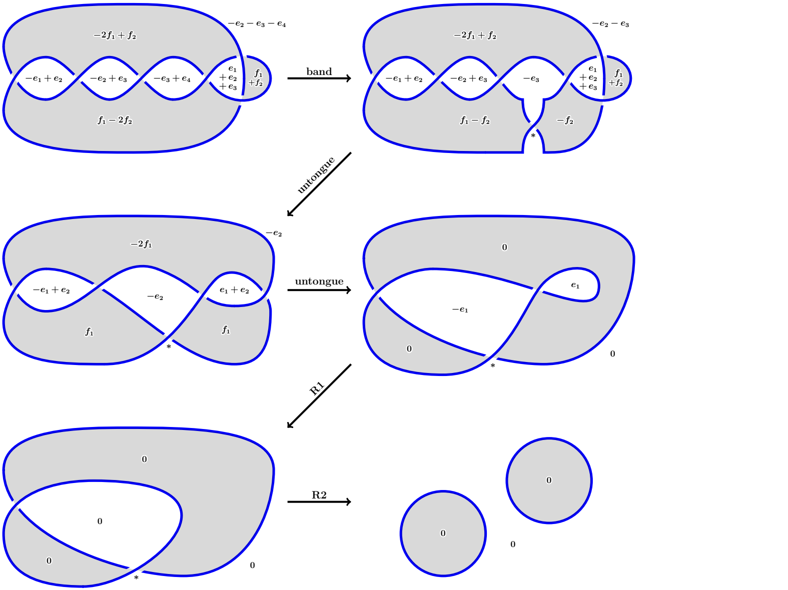
Conjecture 1

For each $n \in \mathbb{N}$ there exists a finite set $\mathcal{U}_n$ of generalised Tsukamoto moves such that any MNA diagram of the $(n+1)$-component unlink can be coverted to a crossingless diagram by a finite sequence of $\mathcal{U}_n$ moves.

Conjecture 2

For each $n \in \mathbb{N}$ there exists a finite set $\mathcal{T}_n$ of generalised Tsukamoto moves such that any MNA diagram of a link of nullity n can be coverted to a minimal crossing MNA diagram by a finite sequence of $\mathcal{T}_n$ moves.

Example: stevedore is algorithmically ribbon



Search results

Search implemented using Kirby Calculator, by Frank Swenton. Prime Alternating Knot Generator by Flint, Rankin, and de Vries used to generate input data.

$\times$'s	**PAK**	$\square$ **det**	**Bifac**	**AR**	**Bands**
to 11	563	36	28	28	1^{28}
12	1,228	62	51	48	$1^{47} \cdot 2^{1}$
13	4,878	175	138	118	1^{118}
14	19,536	567	409	305	$1^{301} \cdot 2^{4}$
15	85,263	1,921	1,245	850	$1^{805} \cdot 2^{45}$
16	379,799	6,888	3,724	2,330	$1^{2,033} \cdot 2^{297}$
17	1,769,979	24,828	11,259	6,513	$1^{5,384} \cdot 2^{1,139}$
18	8,400,285	91,486	34,197	19,071	$1^{13,731} \cdot 2^{5,333} \cdot 3^{7}$
Total	10,615,531	125,963	51,051	29,263	$1^{22,444} \cdot 2^{6,819} \cdot 3^{7}$

What are we missing?

Using a computer search for symmetric union diagrams, Axel Seeliger found 3 ribbon alternating knots which are not algorithmically ribbon. We have since found around 100 examples; all have 2-band disks, with one nonalgorithmic band followed by an isotopy back to MNA, rejoining the algorithm.

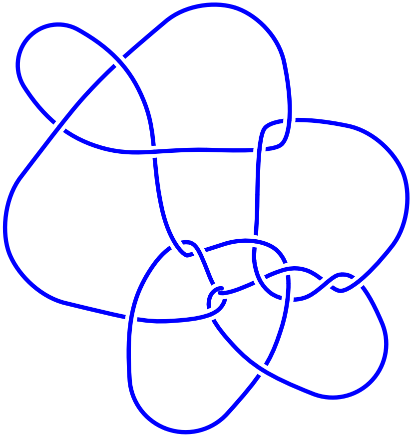
It may be reasonable to conjecture that fusion number one ribbon alternating knots are algorithmic.

Question: is there a reasonable way to enlarge the algorithm to capture all known examples?

An 18-crossing example

Is the following knot slice?

18a1959505: has determinant 69^2 and is bifactorisable.



Thanks!