Statistics of Veering Triangulations

William Worden
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joint with Dave Futer and Sam Taylor

layered

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Main characters:

1. Cusped hyperbolic 3-manifolds

$$\begin{array}{c} \mathbb{H}^3 \\ \big\downarrow \\ M = \mathbb{H}^3/\Gamma \end{array}$$

$\Gamma \in \mathbf{Isom}(\mathbb{H}^3)$ discrete, torsion free, w/ rank 2 parabolic subgroup(s)

2. Ideal triangulations

- decomposition of M into ideal tetrahedra
- if ∃ a solution to Thurston's gluing equations, then *geometric*, otherwise *non-geometric*.
- **or**: pull edges to geodesics, if some tetrahedron turns inside out, the triangulation is non-geometric.

Q1: Can the combinatorial information of a veering triangulation tell us something about the geometry of the manifold?

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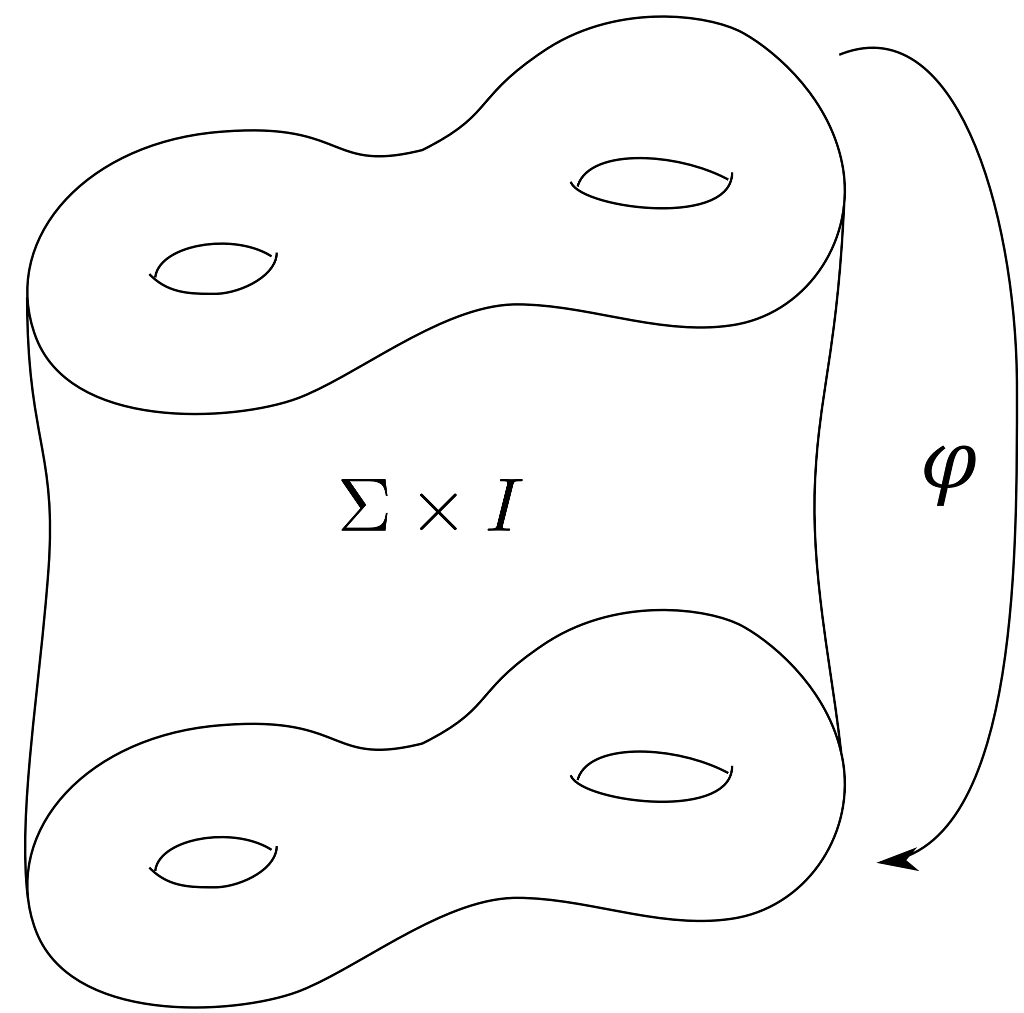
Q2: How well-behave are they? I.e., how often are veering triangulations realized geometrically?

Fibered 3-Manifolds:

- Let $\Sigma = \Sigma_{g,n}$ be a hyperbolic surface (i.e., $\chi(\Sigma) < 0$).
- Let $\varphi \in \mathrm{Mod}(\Sigma) \cong \mathrm{Homeo}^{+}(\Sigma)/\mathrm{isotopy}$.

If we cross Σ with an interval, then glue the top to the bottom by φ, we get a *mapping torus*:

$$M_\varphi = \Sigma \times I /_{(x,1)\sim(\varphi(x),0)}$$



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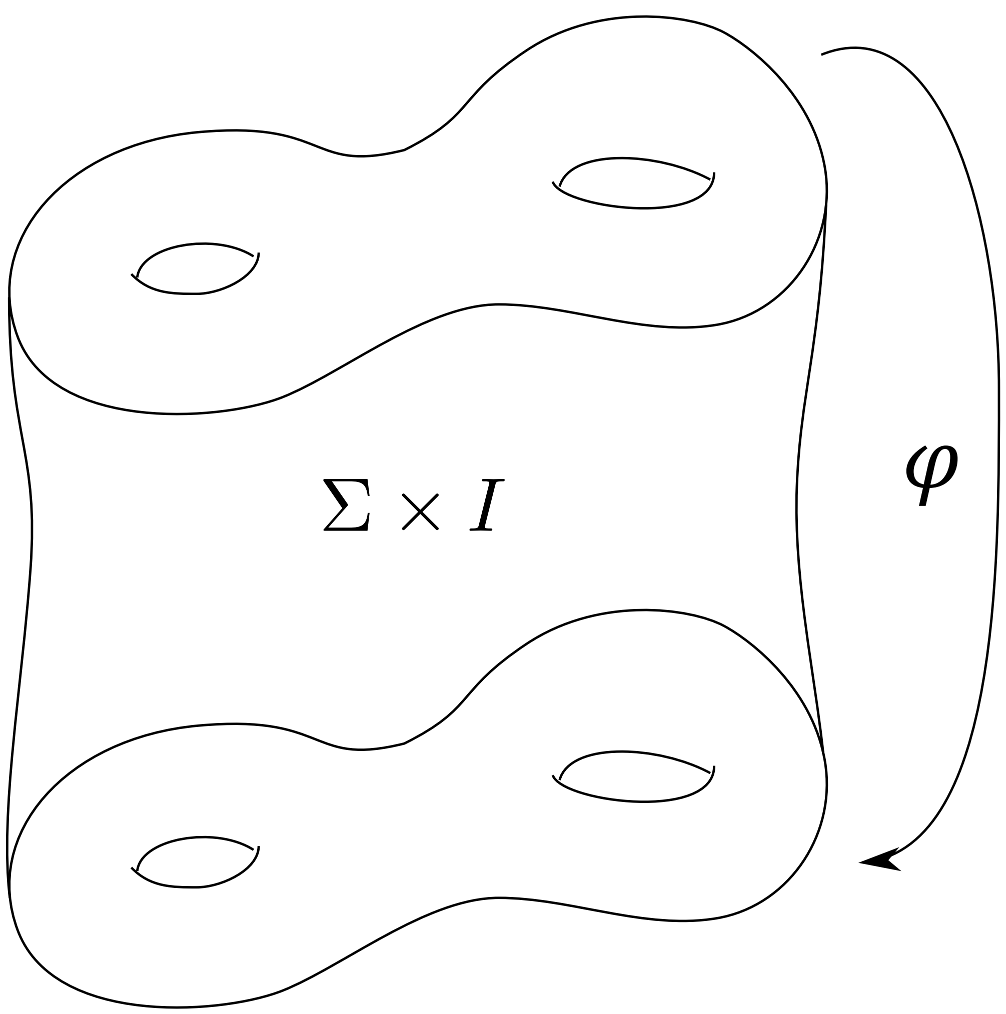

Theorem (Thurston):
M_φ is hyperbolic $\Leftrightarrow$ φ is *pseudo-Anosov*.

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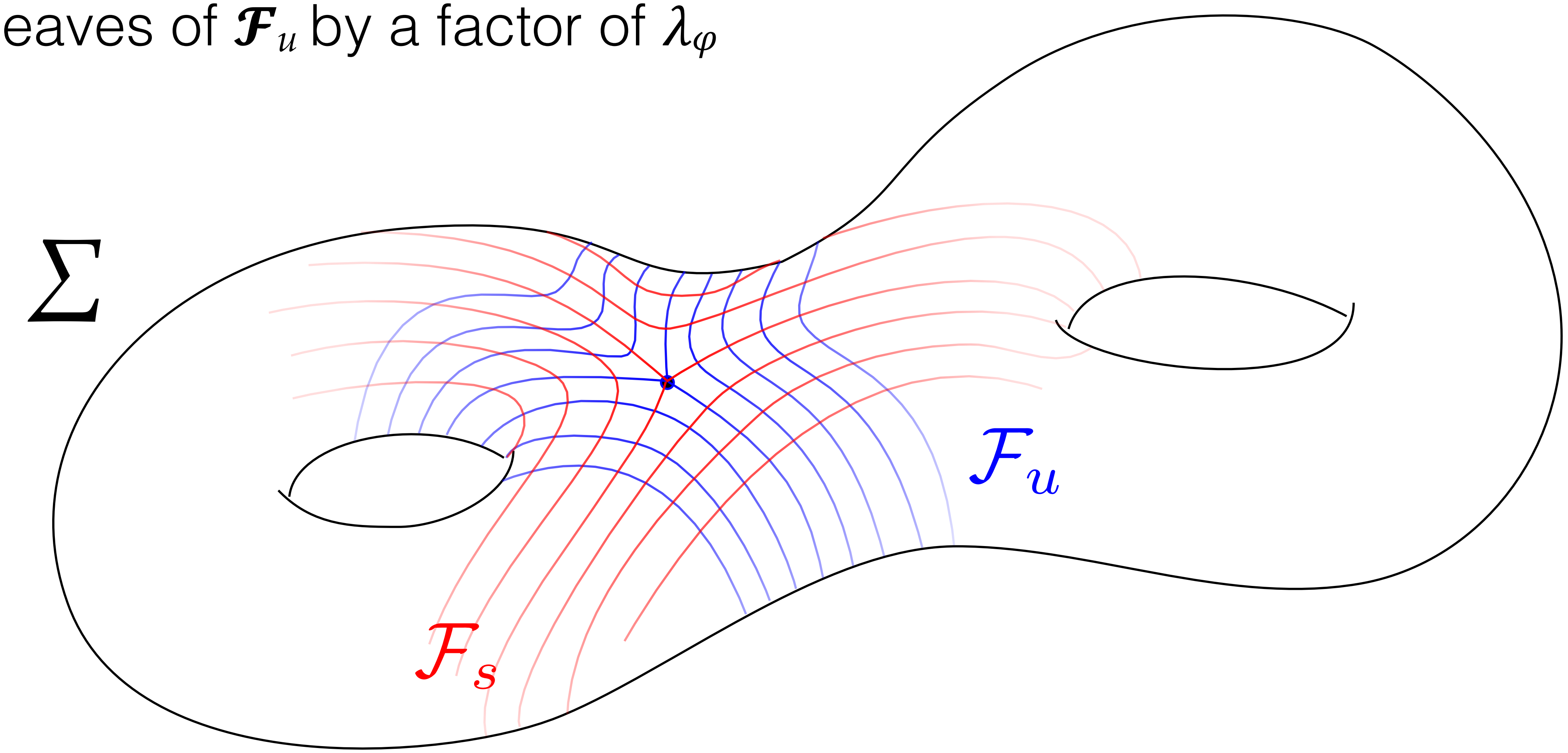
More precisely:

- there are transverse *measured foliations* $(\mathcal{F}_s, \mu_s)$ and $(\mathcal{F}_u, \mu_u)$ on Σ and dilatation $\lambda_\varphi > 1$ so that φ stretches the leaves of $\mathcal{F}_s$ and compresses the leaves of $\mathcal{F}_u$ by a factor of λ_φ



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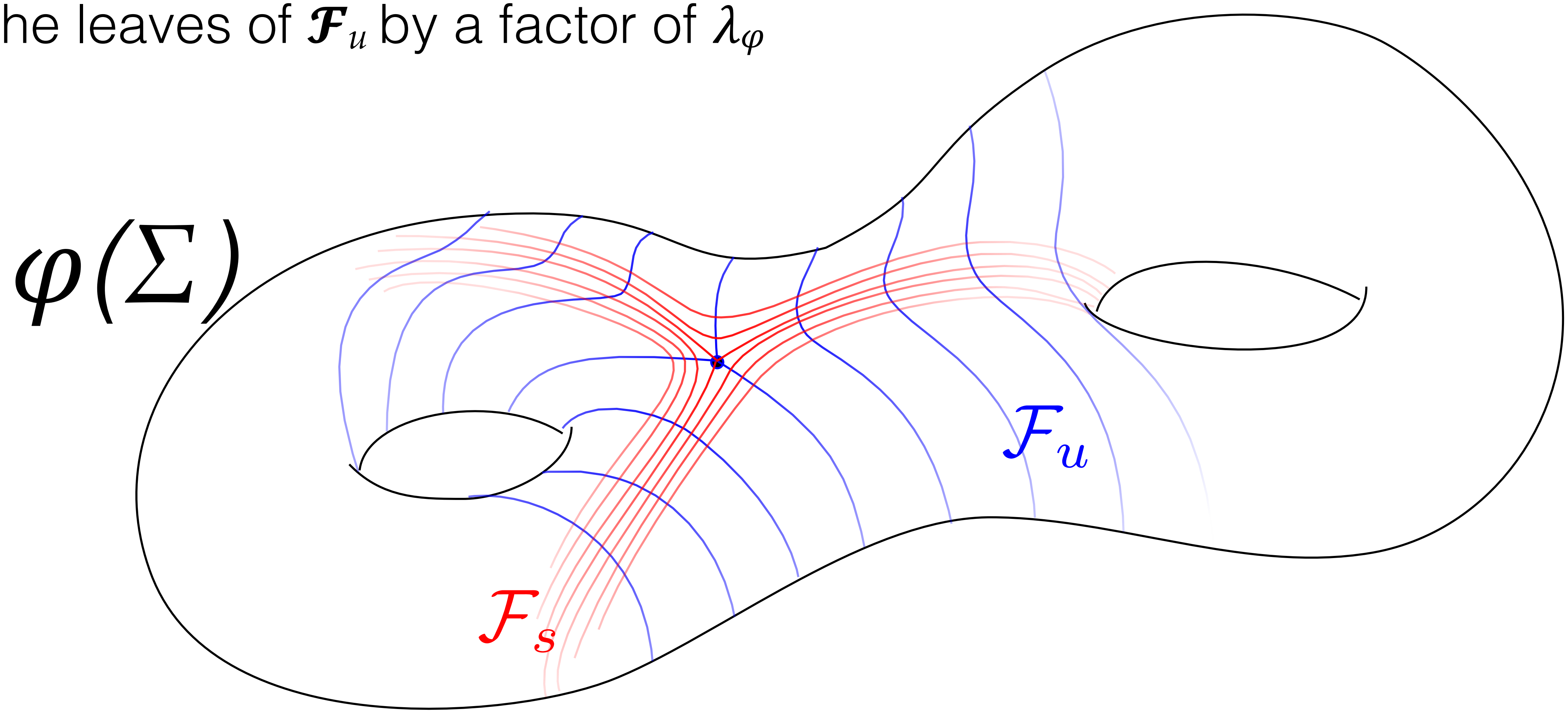

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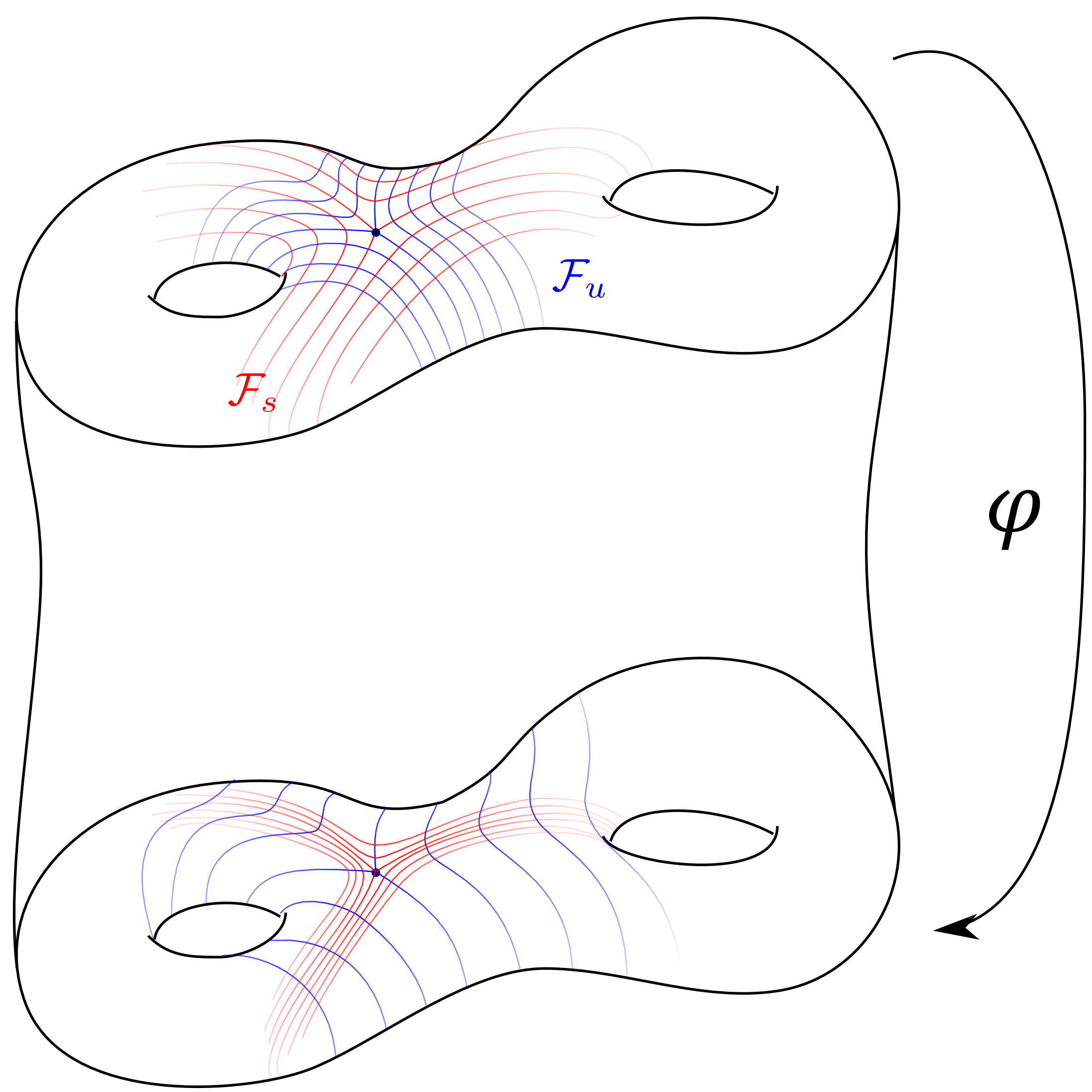

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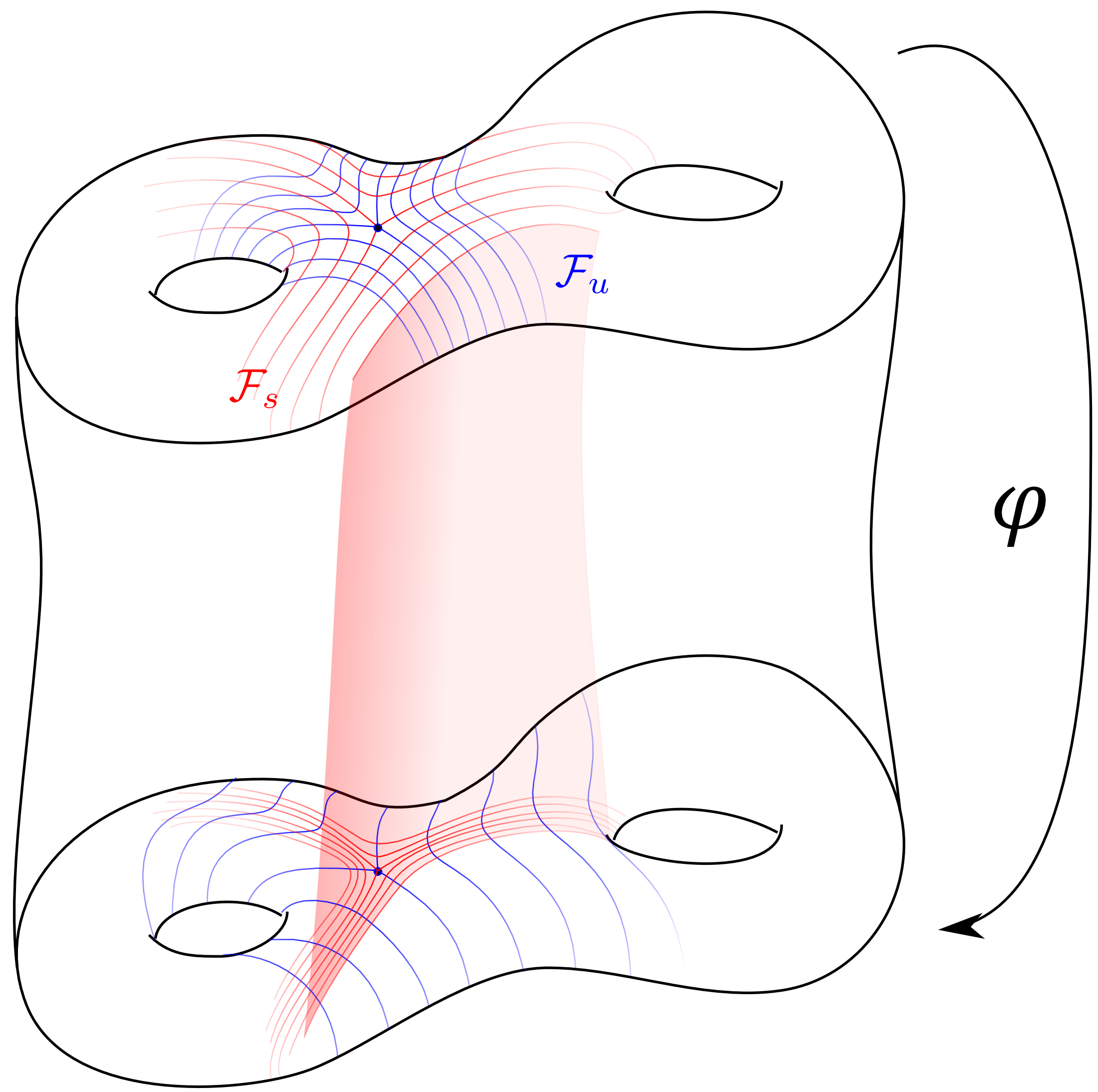
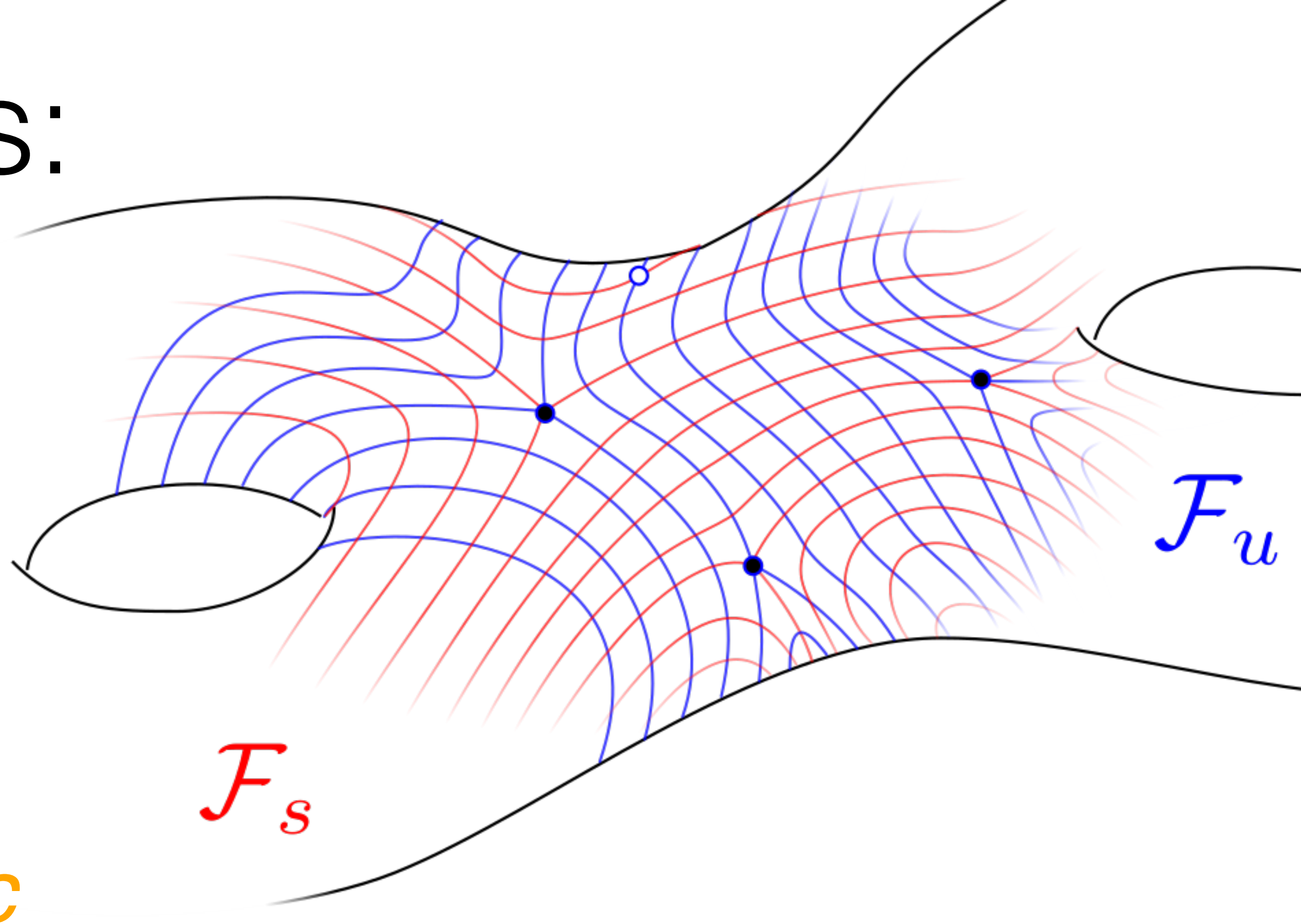


quadratic differentials:



- Associated to the stable and unstable foliations, there is a unique *quadratic differential* q [Hubbard-Masur ’79]

- q induces a *singular flat metric* on Σ, with respect to which the leaves of $\mathcal{F}_s$ and $\mathcal{F}_u$ are geodesics.

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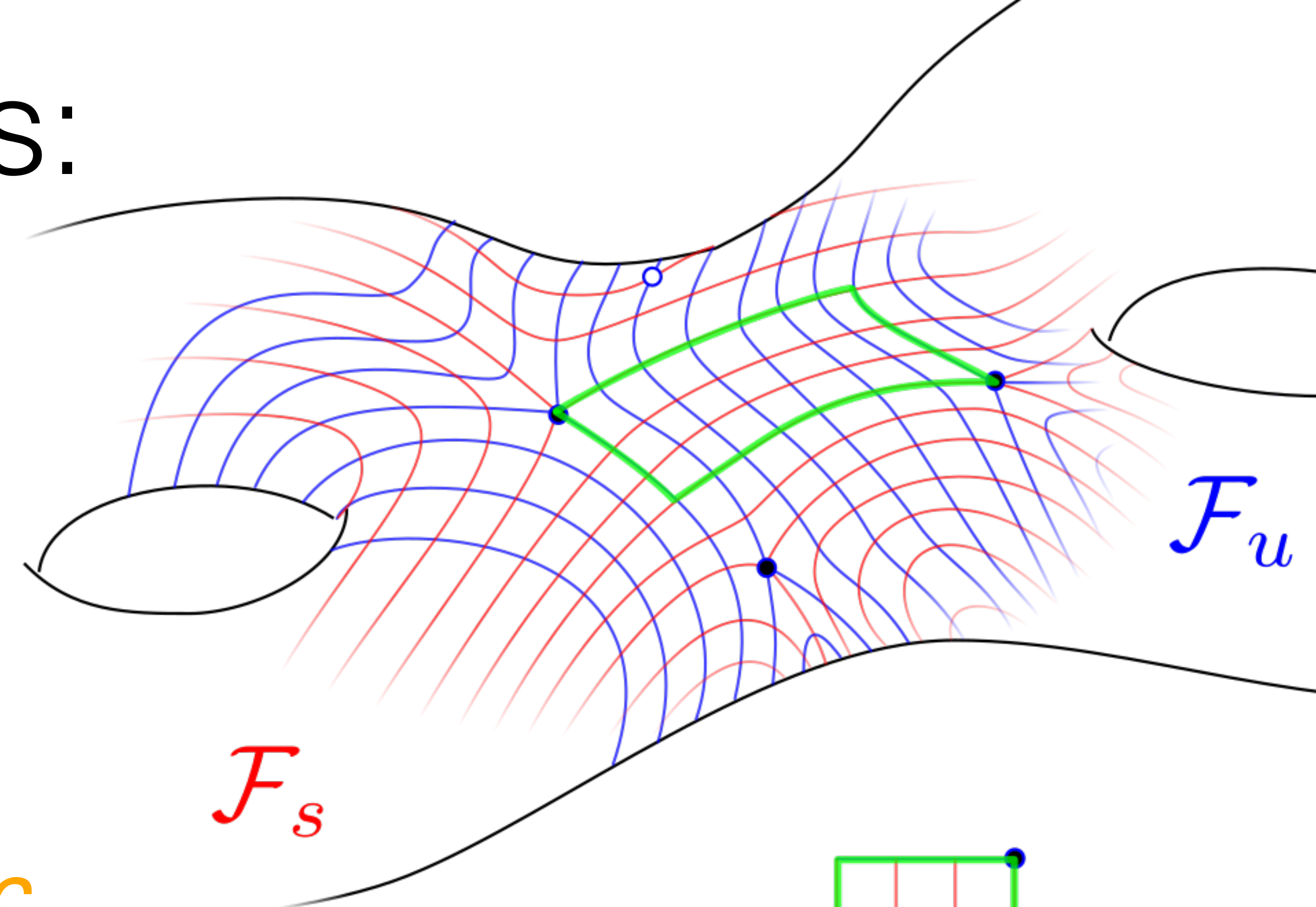

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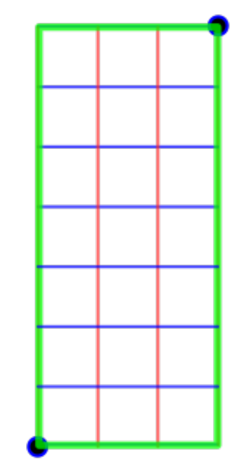

- in this context we refer to $\boldsymbol{\mathcal{F}}_u$ and $\boldsymbol{\mathcal{F}}_s$ as the *horizontal* and *vertical* foliations, resp.

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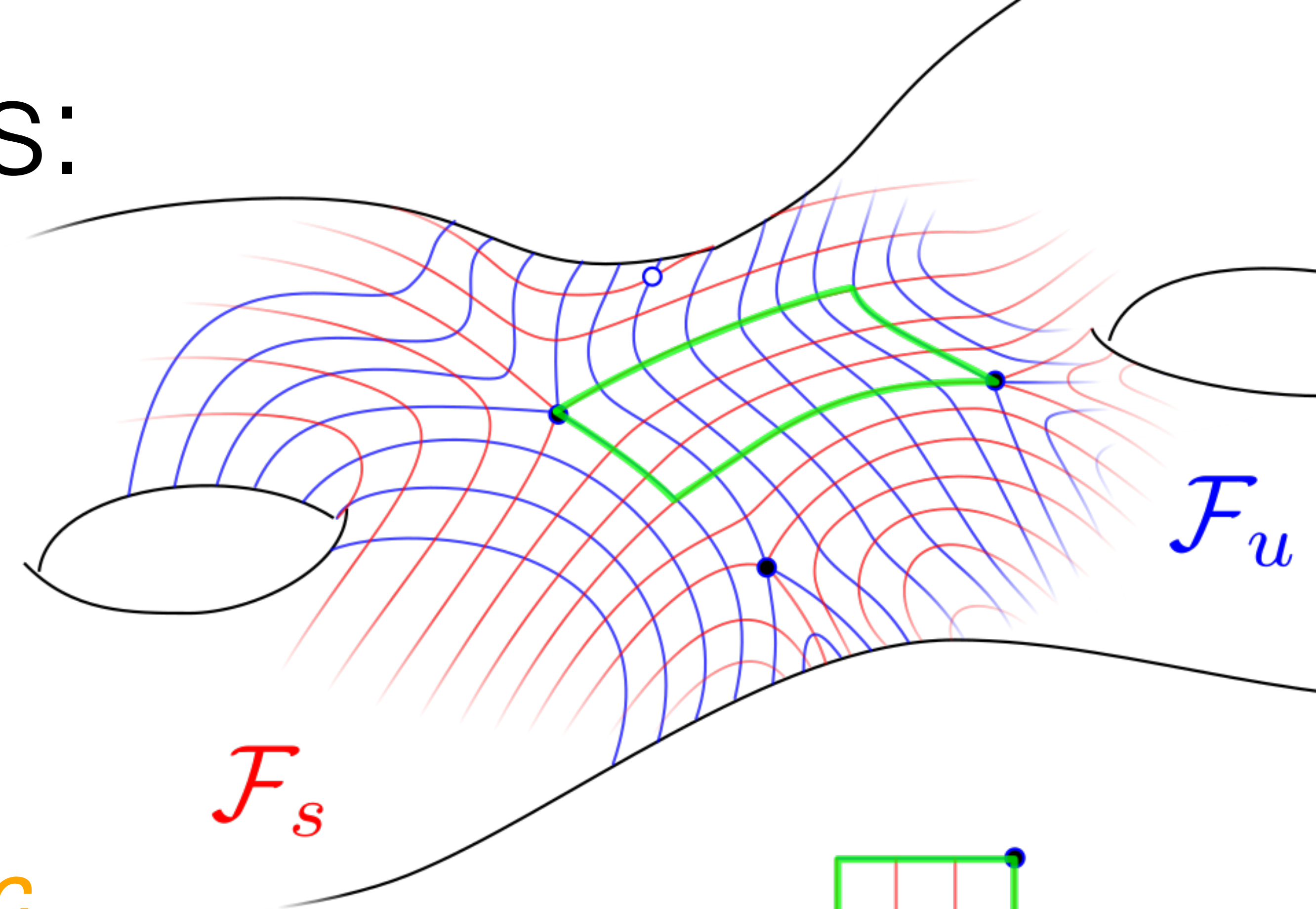

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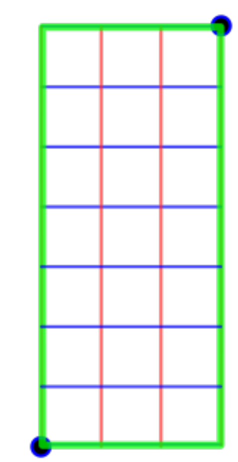

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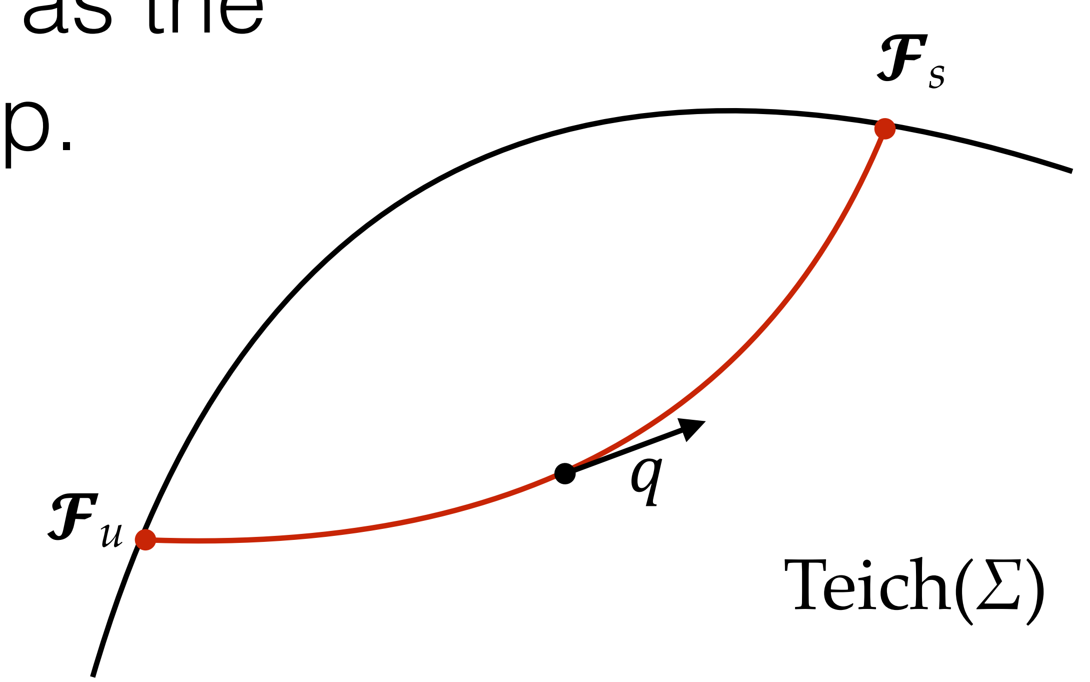
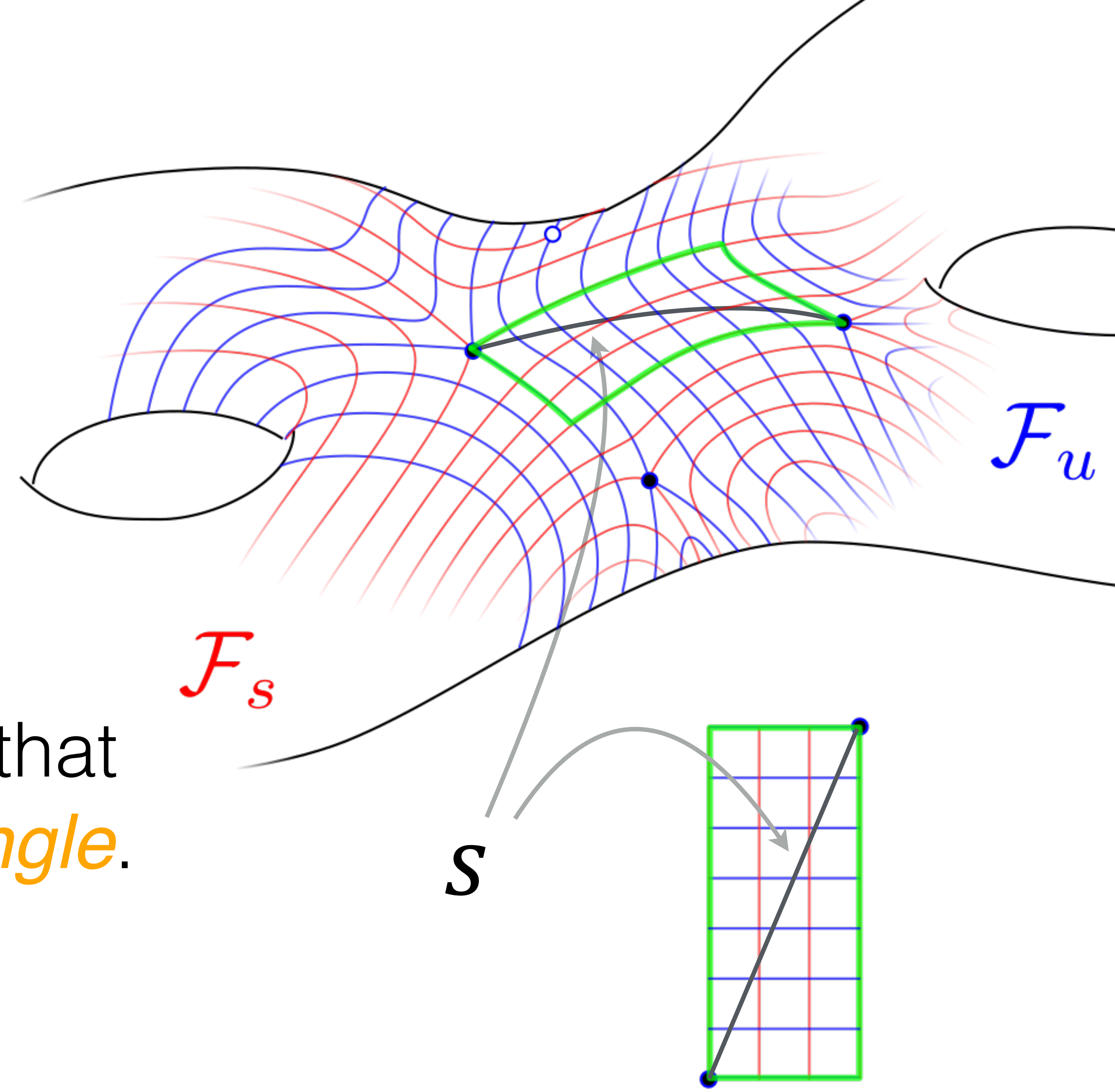


- In the context of Teichmuller space, q is a (co)tangent vector, which is tangent to the Teichmuller geodesic $(\boldsymbol{\mathcal{F}}_u, \boldsymbol{\mathcal{F}}_s)$.

maximal rectangles:

- A *saddle connection* is a geodesic arc (in the flat metric) connecting two singularities (or punctures)

- Let s be a saddle connection that spans a *singularity-free rectangle*.



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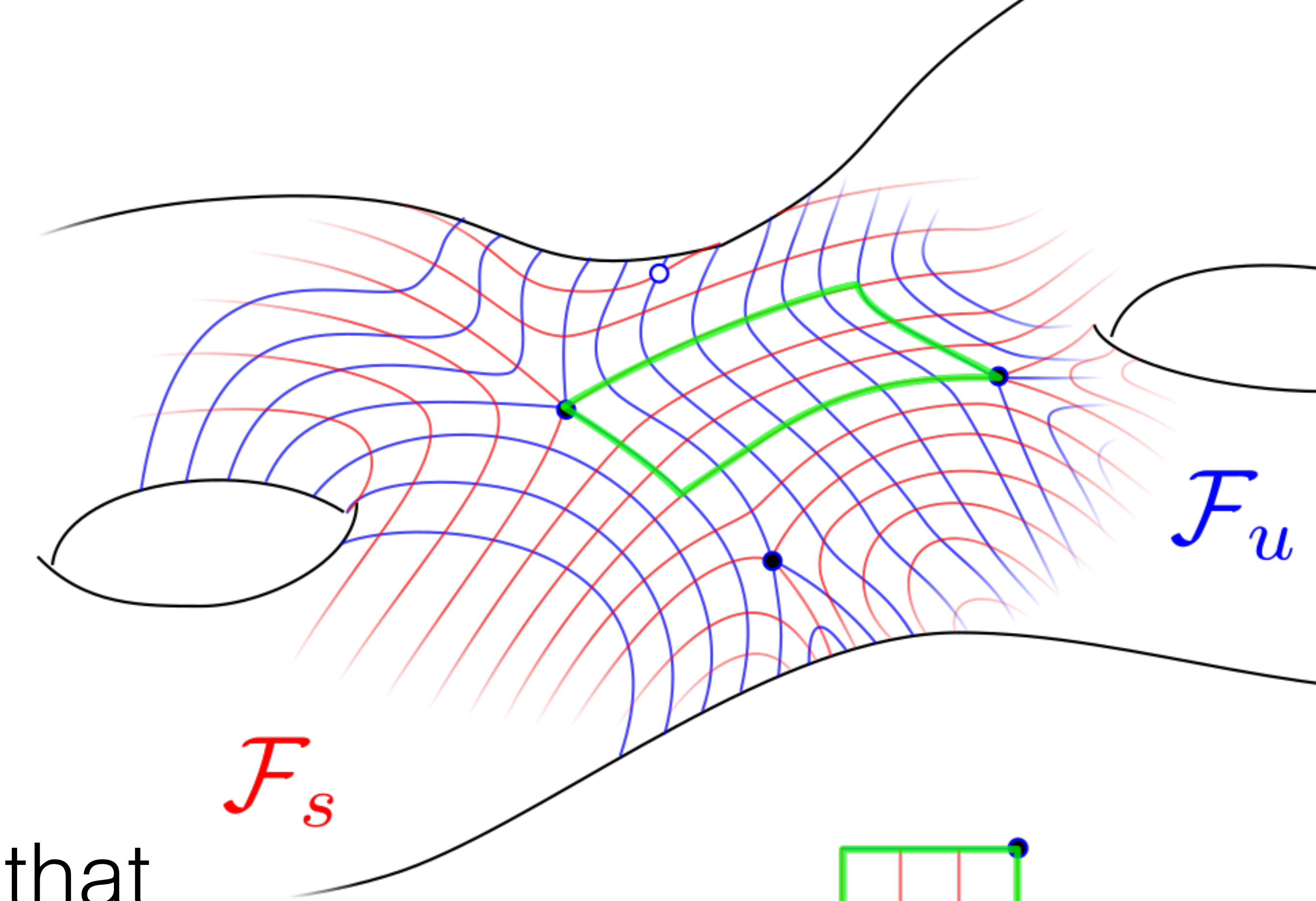

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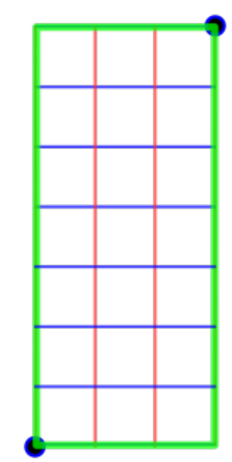

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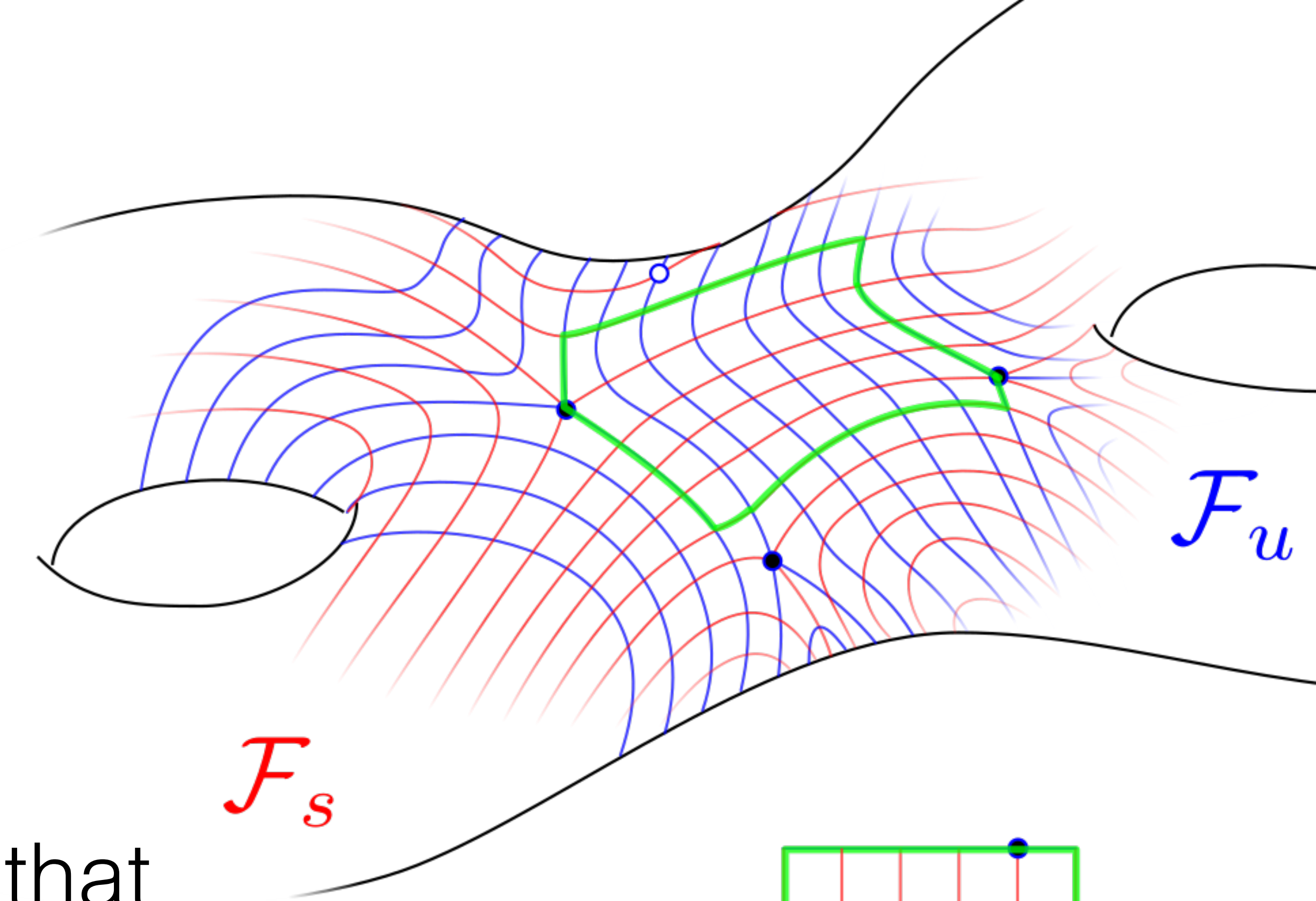

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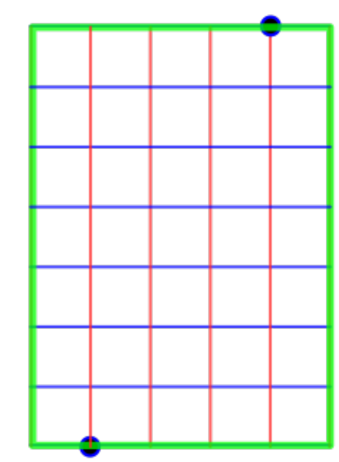

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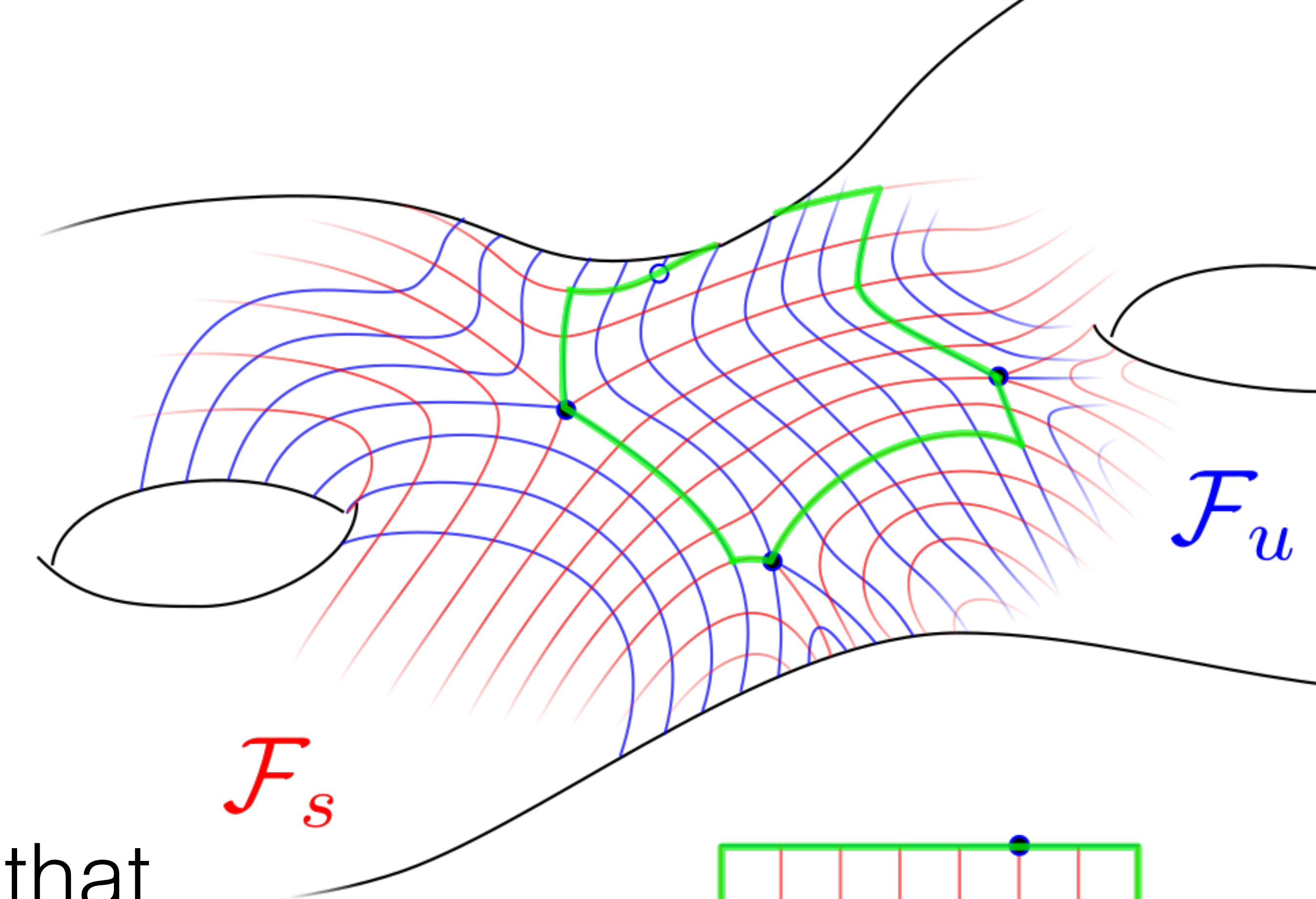

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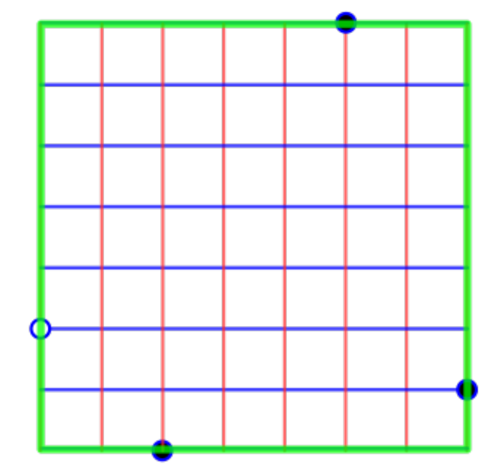

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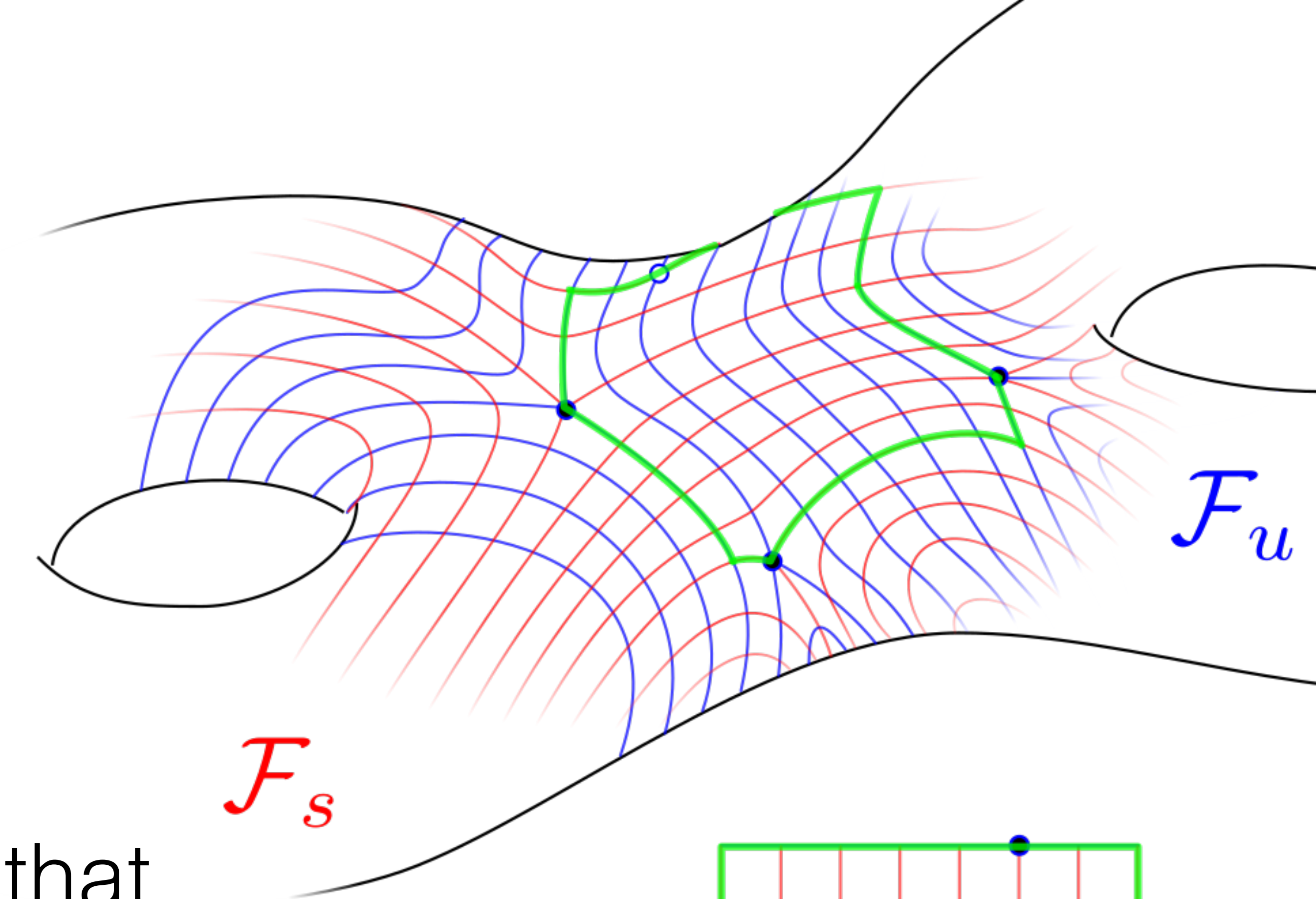

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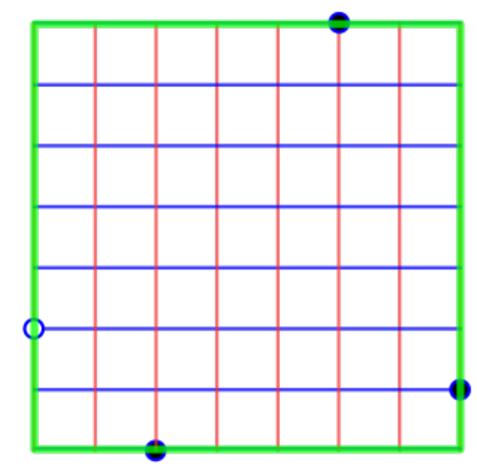

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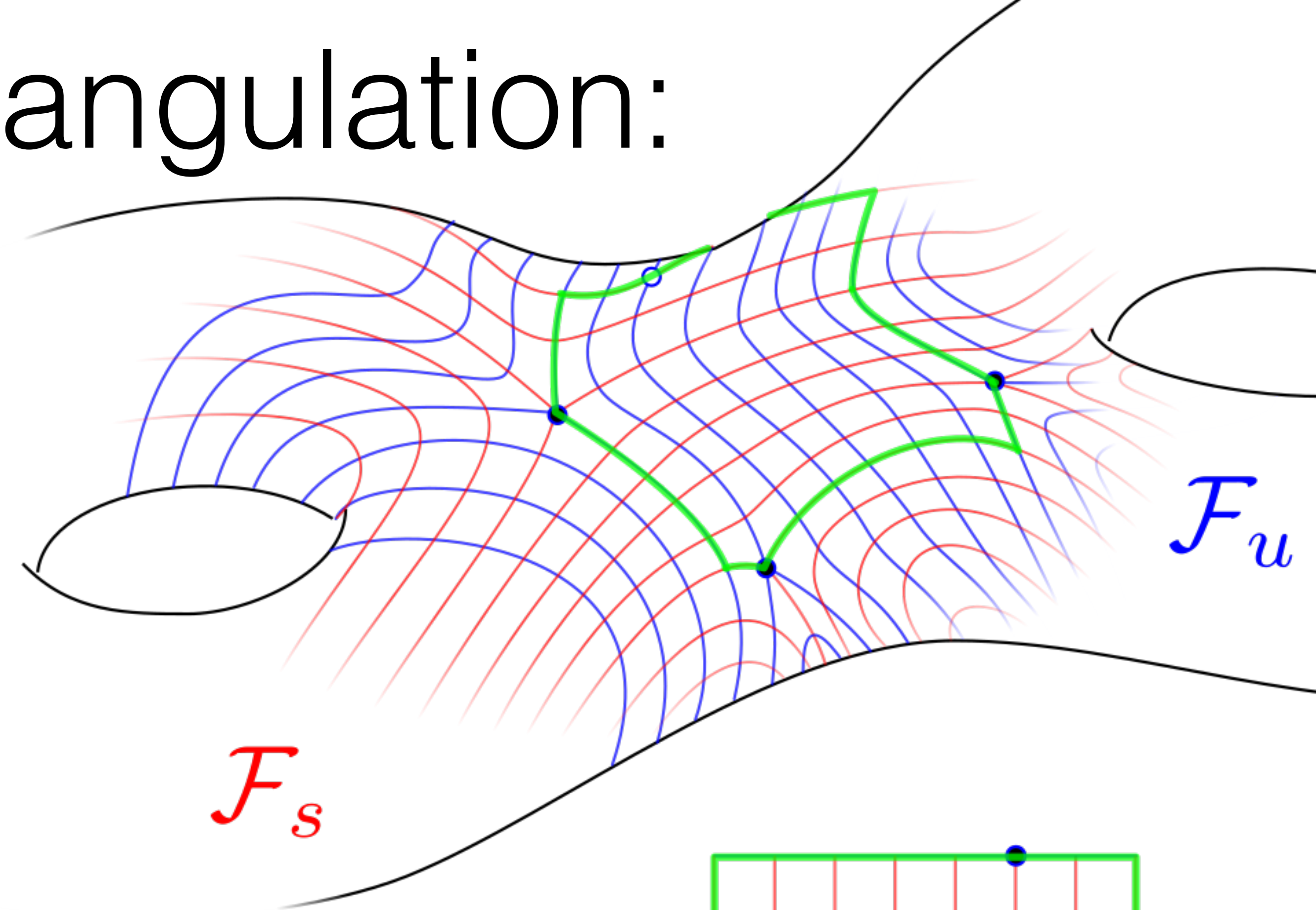
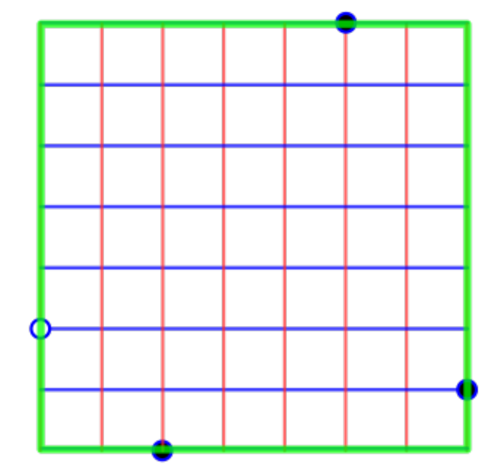
- Note: since φ is pseudo-Anosov, no leaf of $\mathcal{F}_s$ or $\mathcal{F}_u$ can be a saddle connection ($\Longrightarrow$ no singularities at corners, and only one per side)

the layered veering triangulation:

- first, puncture Σ at every singularity of the invariant foliations, to obtain Σ°





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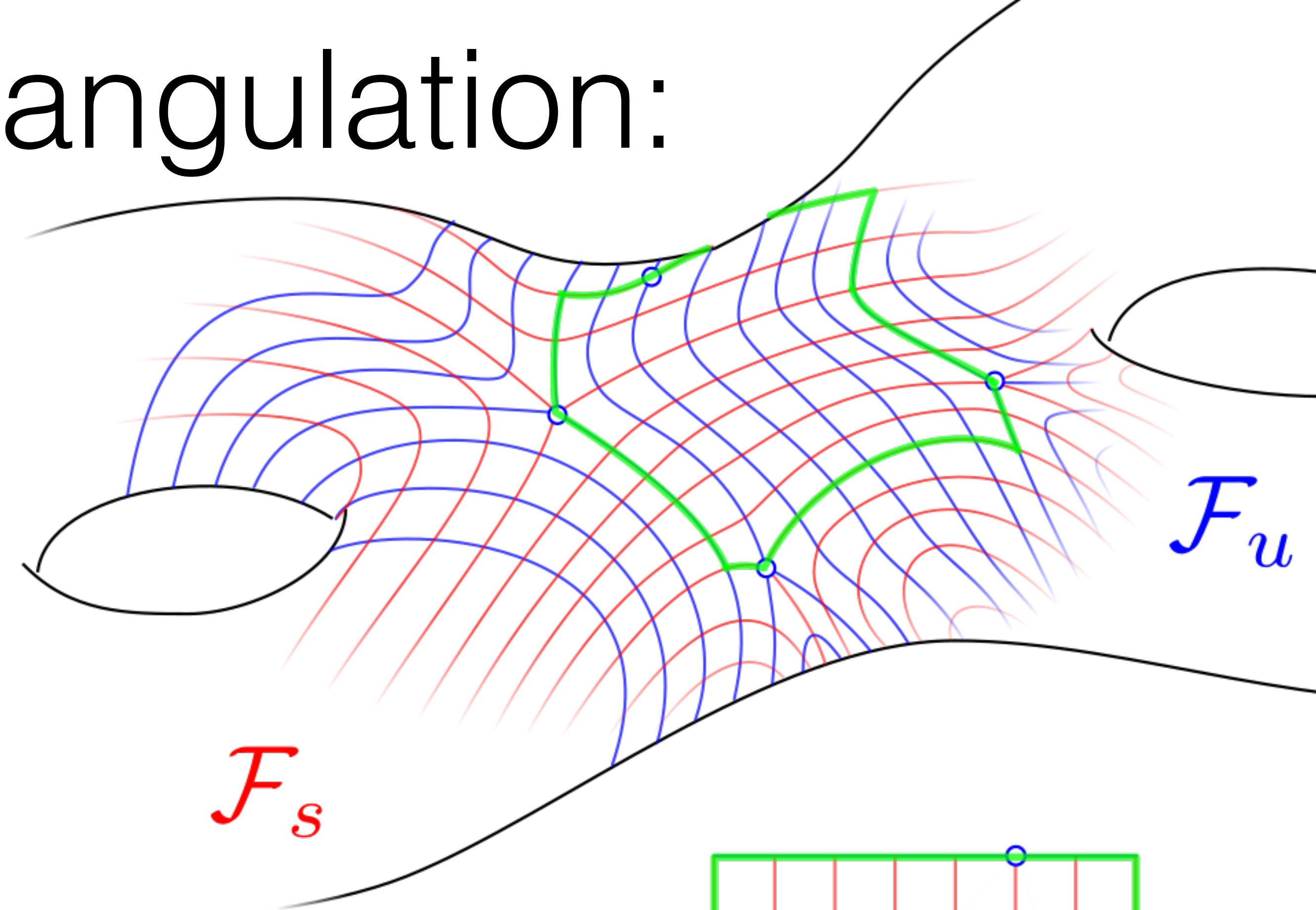
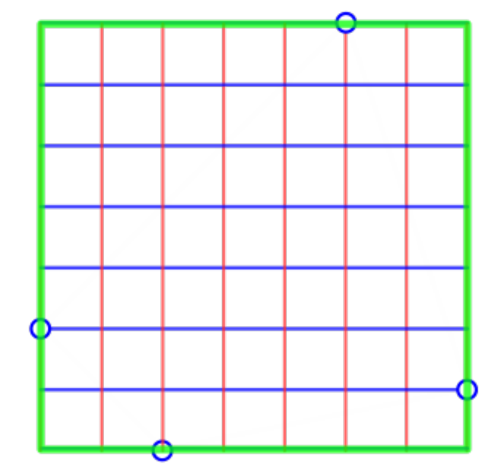
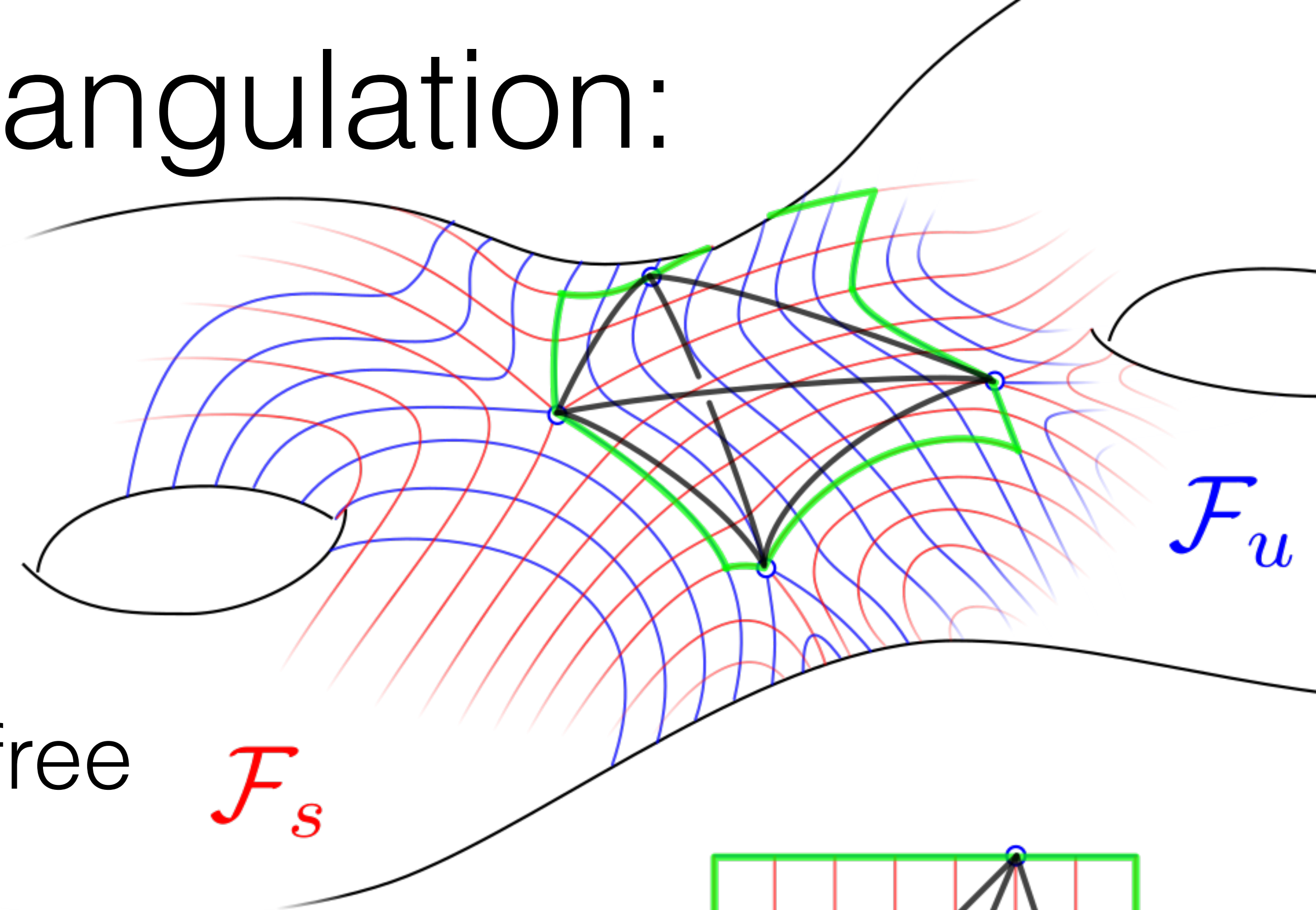
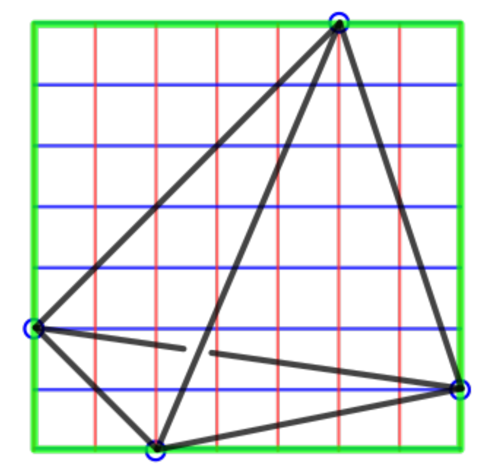




the layered veering triangulation:

- first, puncture Σ at every singularity of the invariant foliations, to obtain Σ°
- for every maximal singularity-free rectangle we layer on a flat tetrahedron, with vertices at the four boundary punctures





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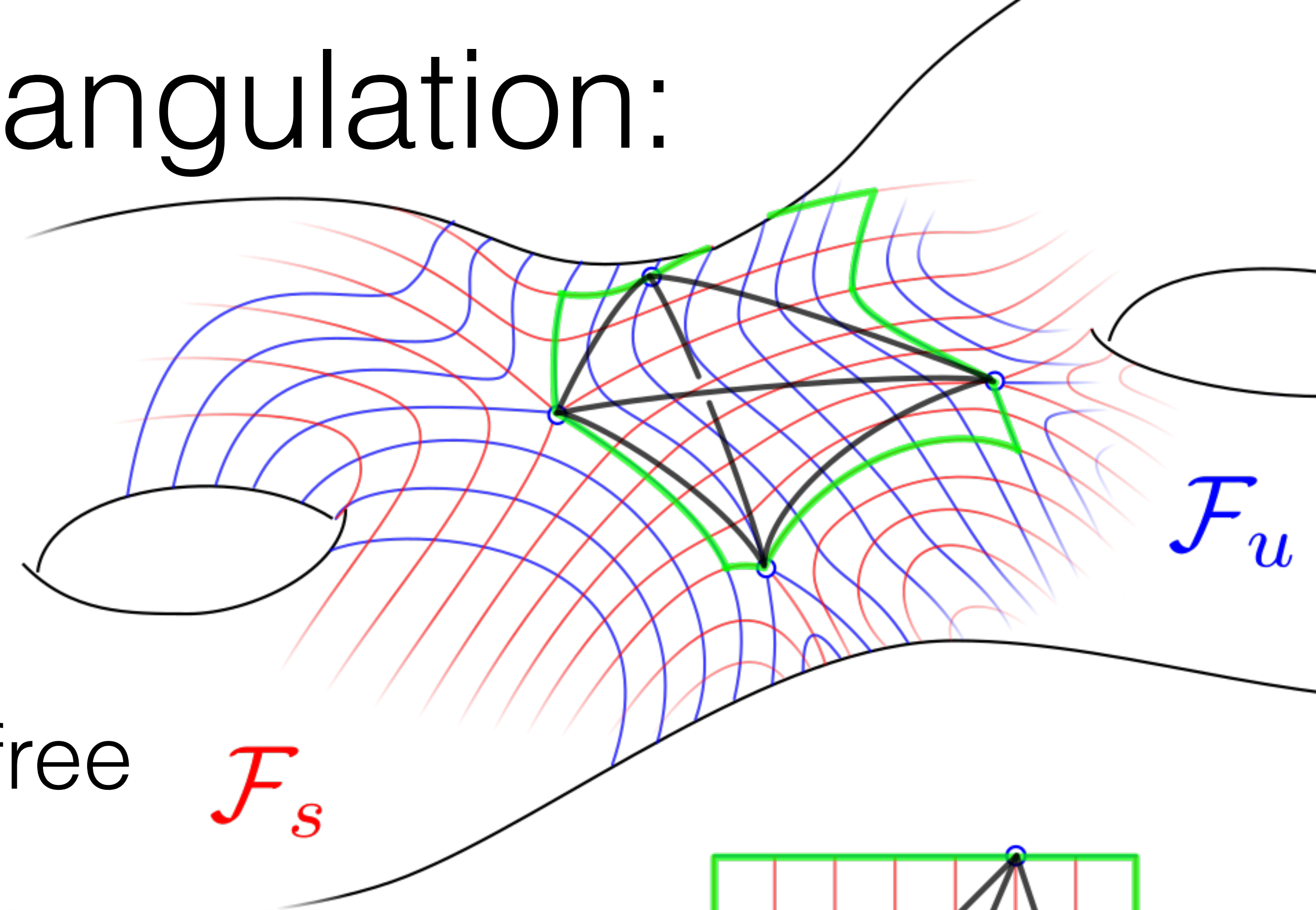

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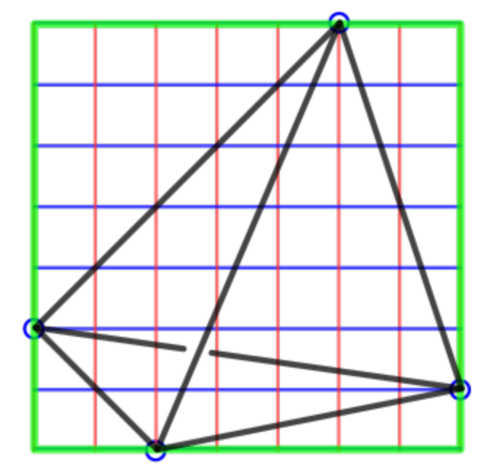

- the diagonal with greatest slope (in absolute value) goes "on top"

- Remark: we should really be working in the universal cover of Σ°, endowed with the lifted incomplete singular flat metric.

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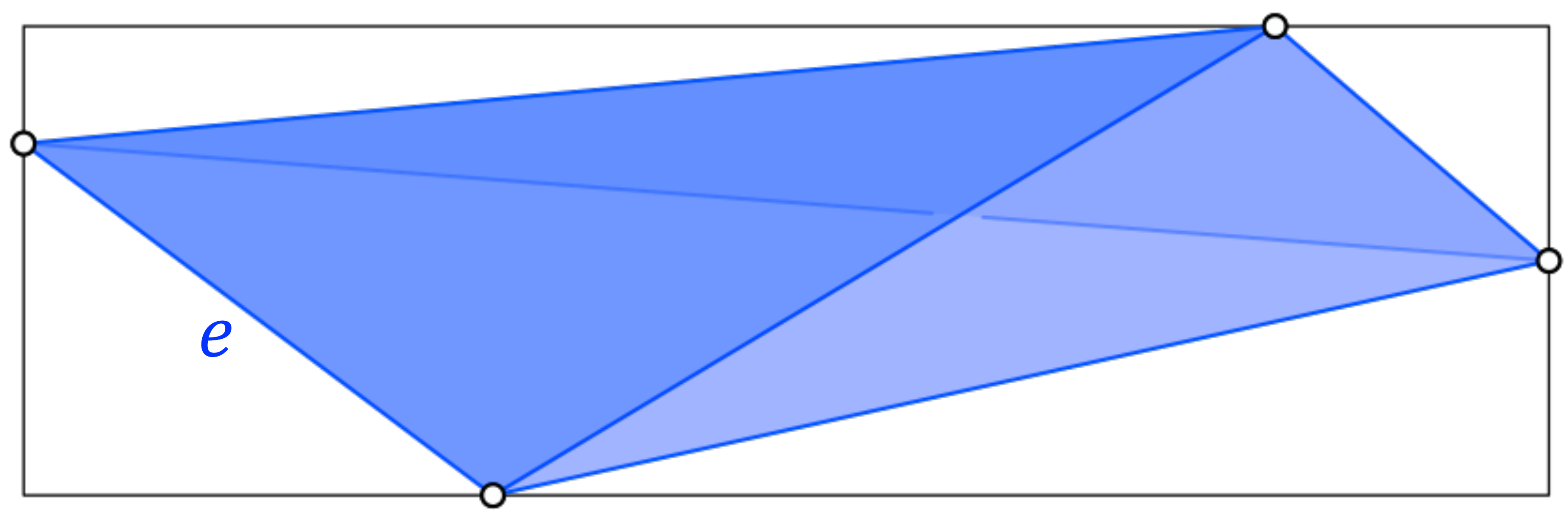
- let’s see what happens around an edge e:



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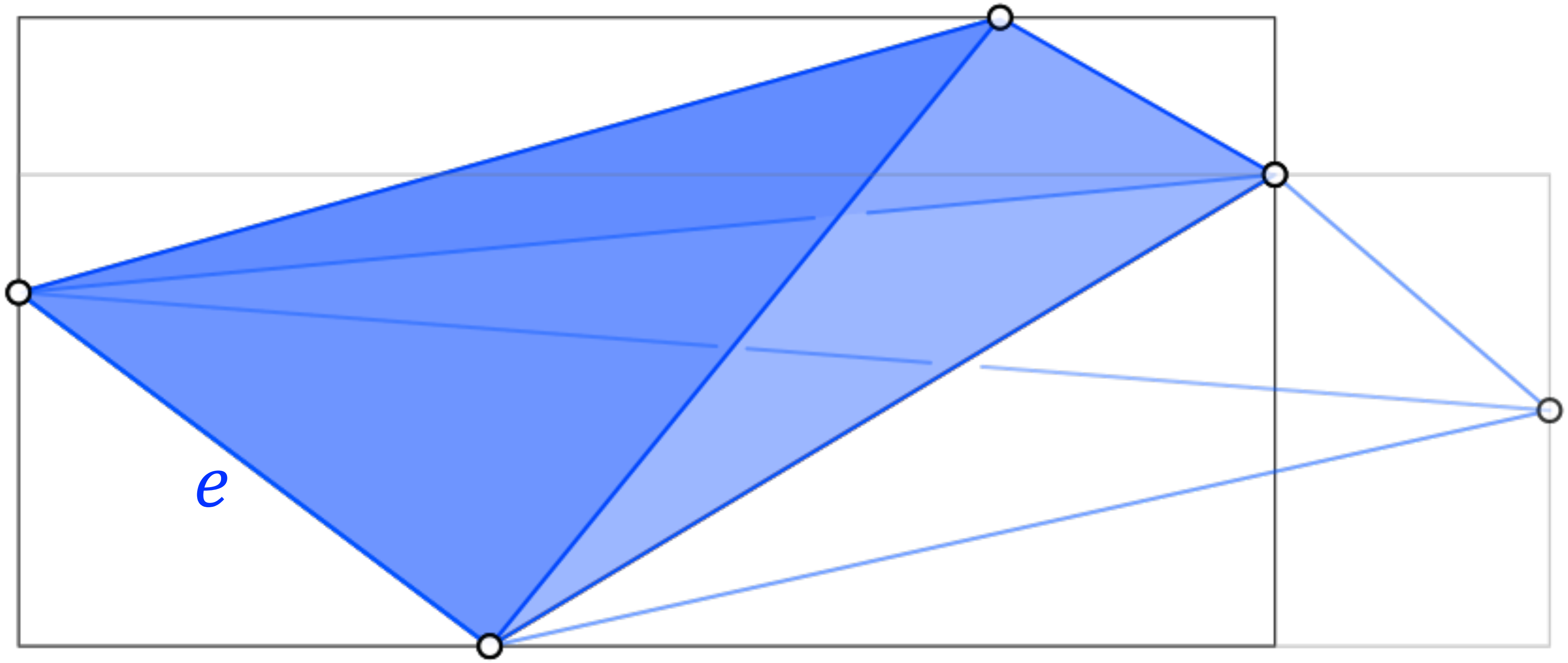

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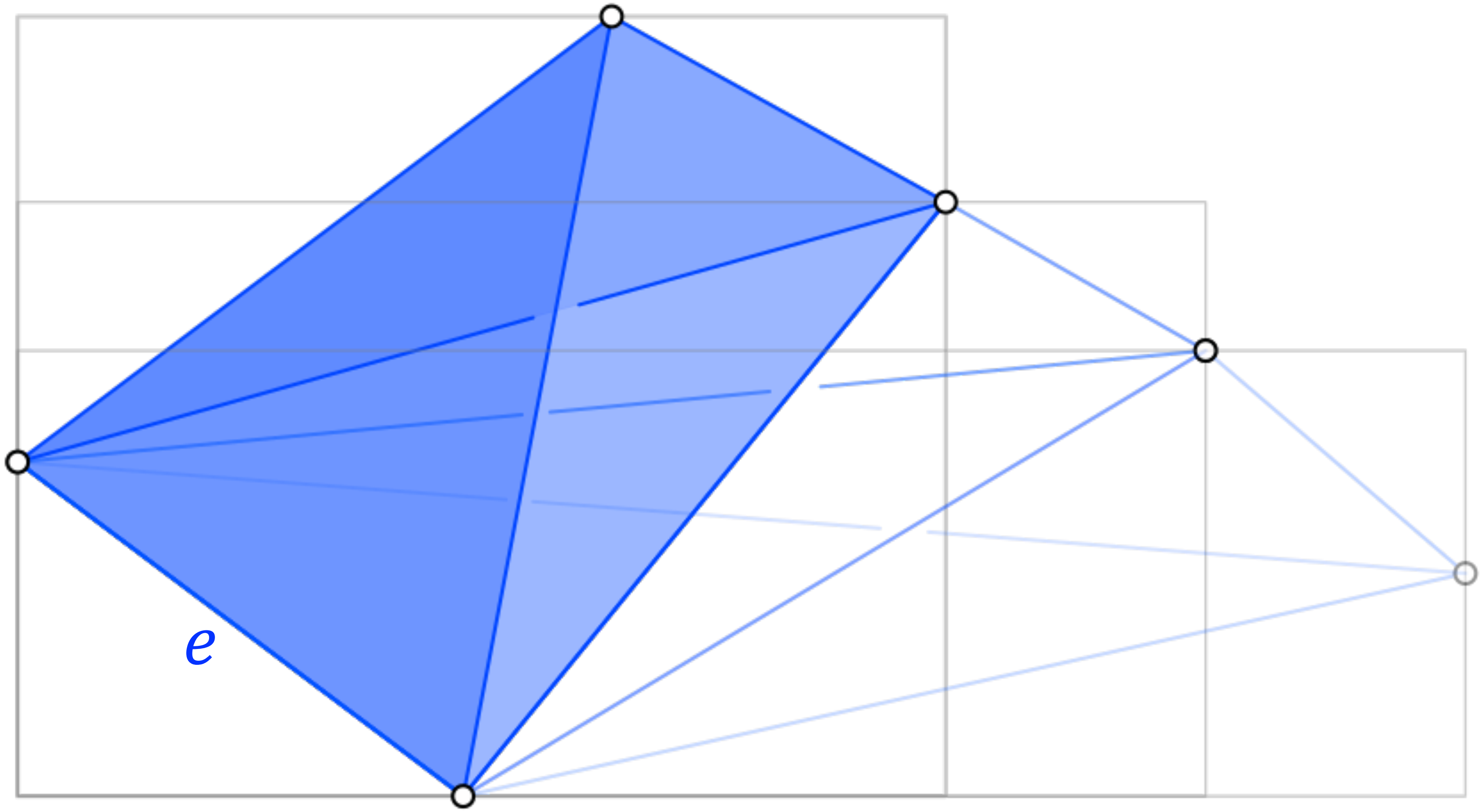

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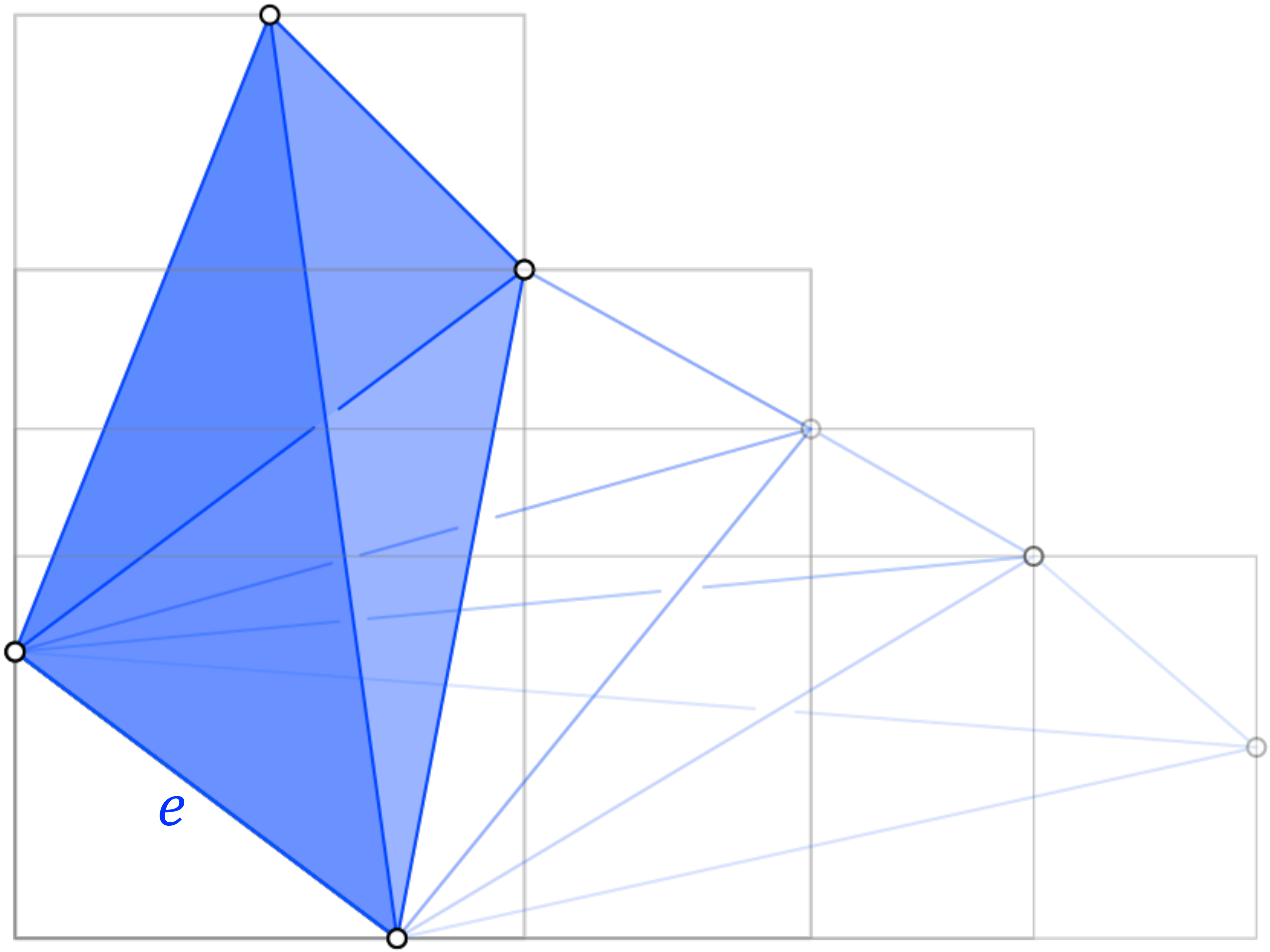

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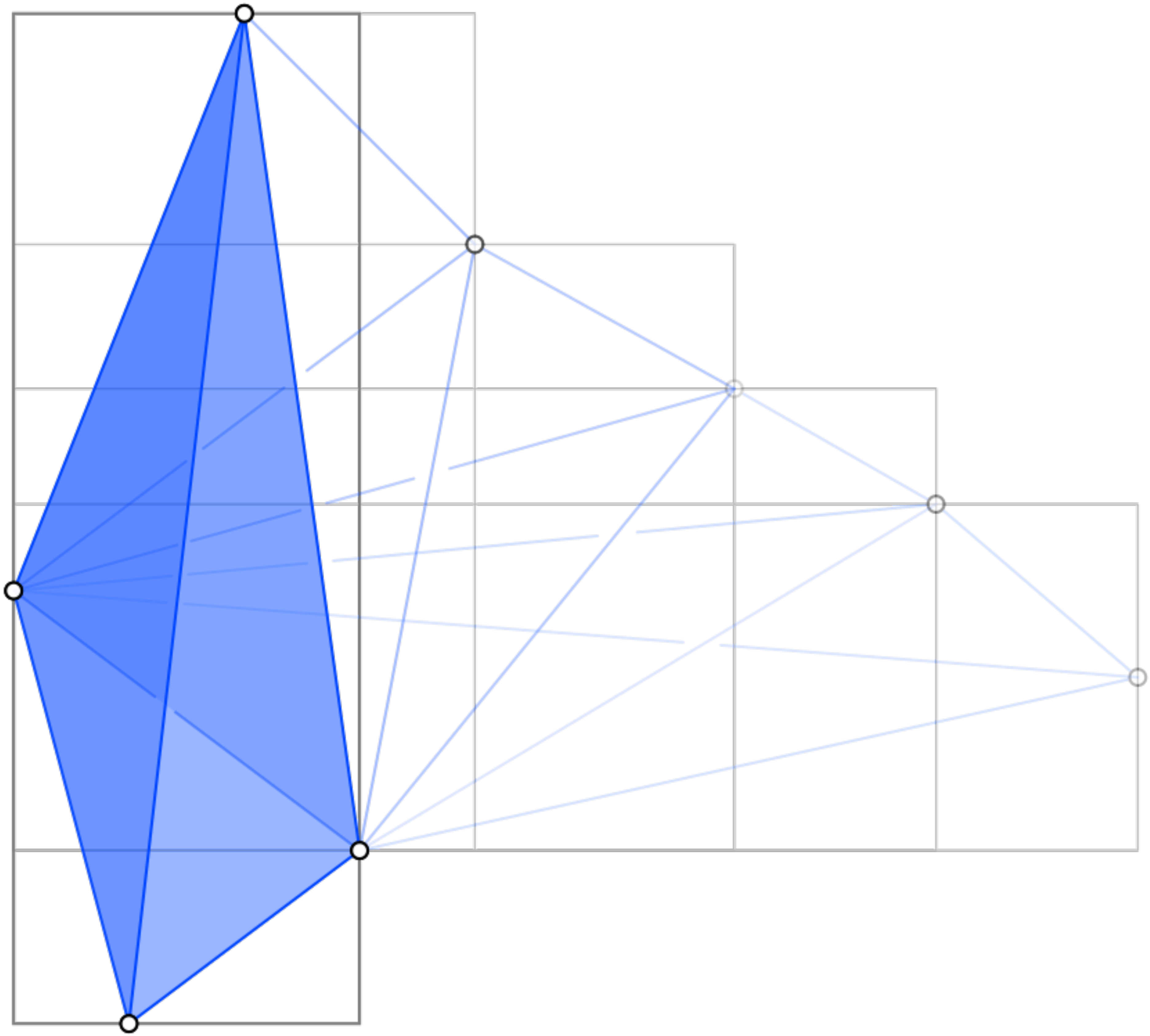

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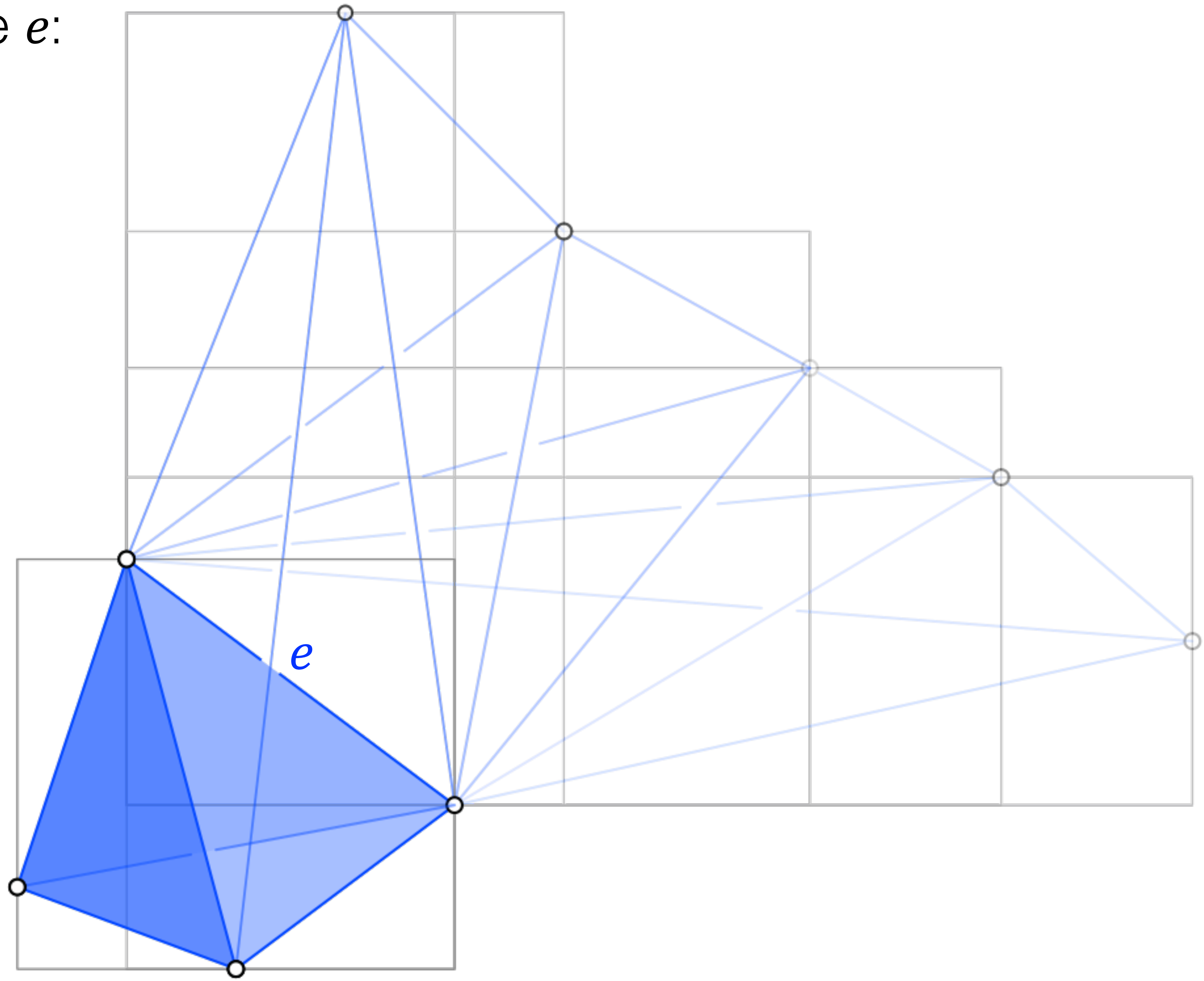

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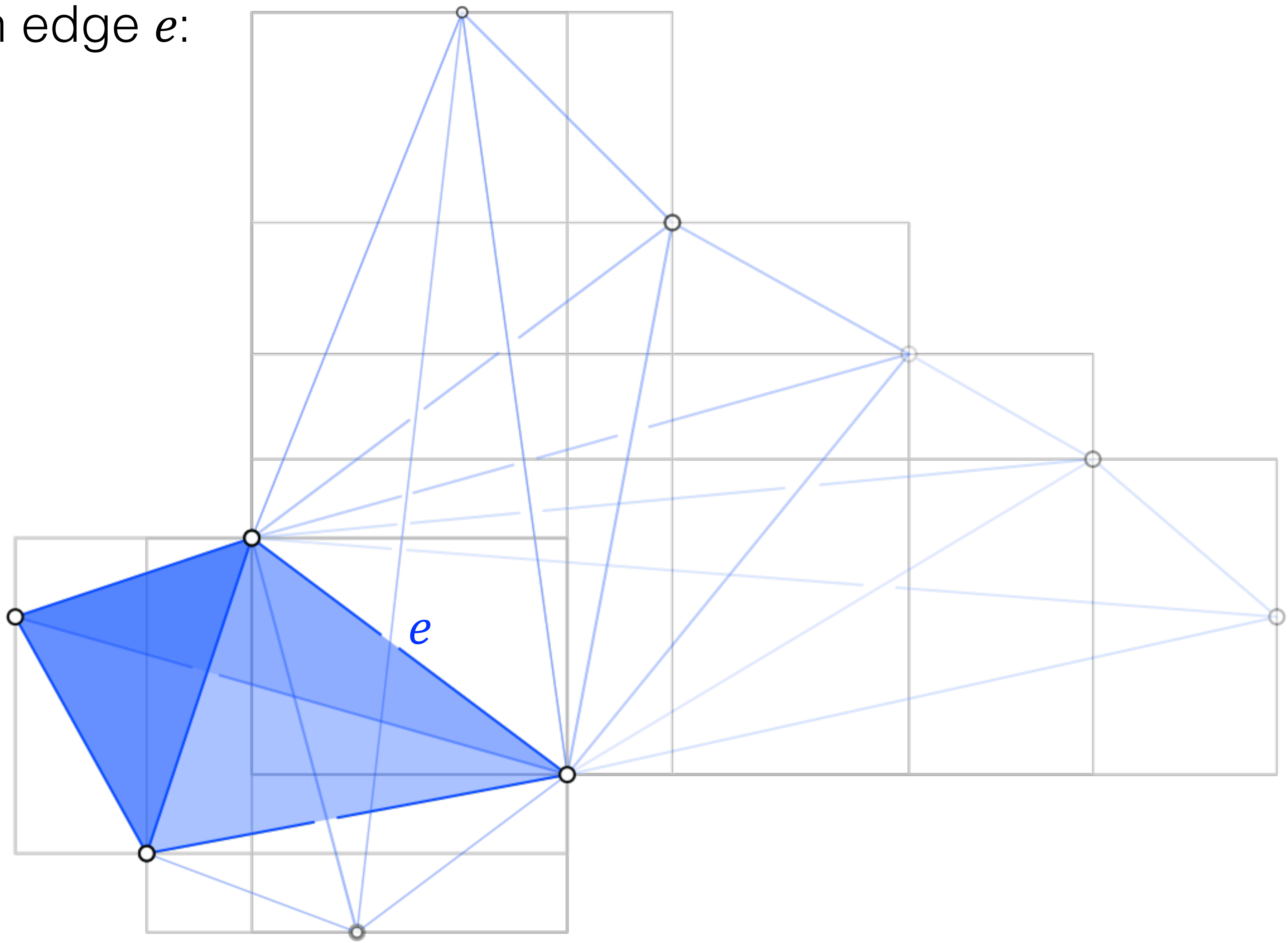
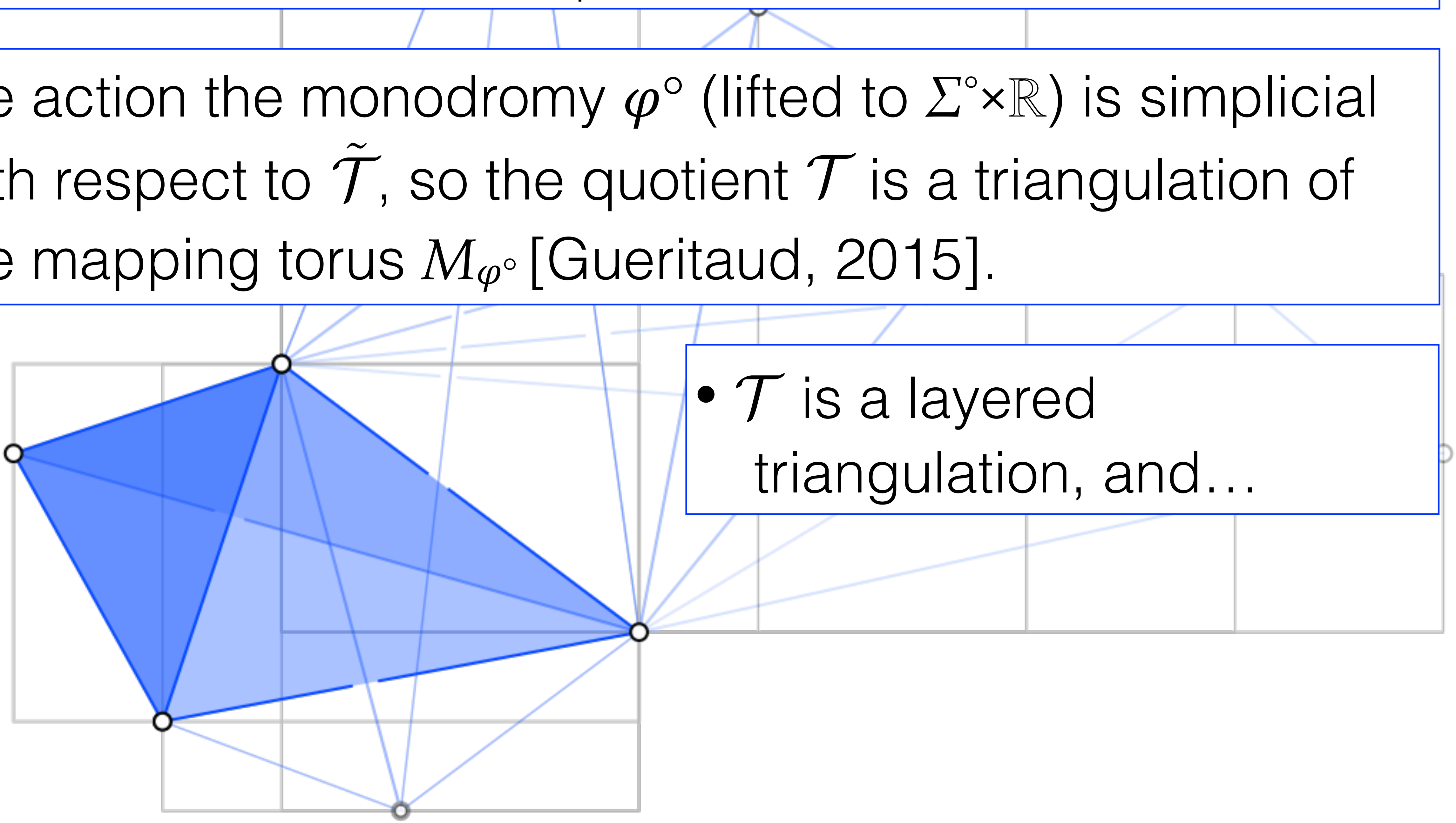


the layered veering triangulation:

- upshot: after layering on every tetrahedron corresponding to a maximal singularity-free rectangle (there are infinitely many!), we get a triangulation $\tilde{\mathcal{T}}$ of the infinite cyclic cover $\Sigma^{\circ}\times\mathbb{R}$ of $M_{\varphi^{\circ}}$, where $\varphi^{\circ}=\varphi|_{\Sigma^{\circ}}$ is the restriction.

- the action the monodromy φ° (lifted to $\Sigma^{\circ}\times\mathbb{R}$) is simplicial with respect to $\tilde{\mathcal{T}}$, so the quotient $\mathcal{T}$ is a triangulation of the mapping torus $M_{\varphi^{\circ}}$ [Gueritaud, 2015].

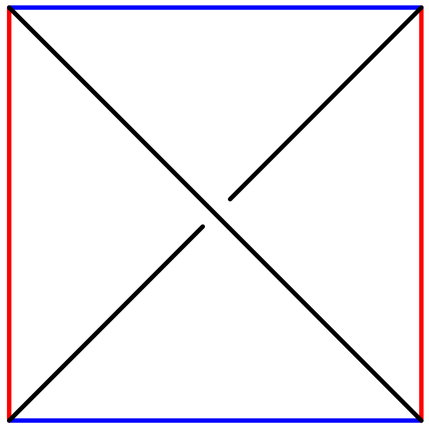
- $\mathcal{T}$ is a layered triangulation, and…



the layered ***veering*** triangulation:

- [Agol 2011, Gueritaud 2015]: the triangulation that we constructed is *veering,* ***i.e.:***

    - tetrahedra look like this:



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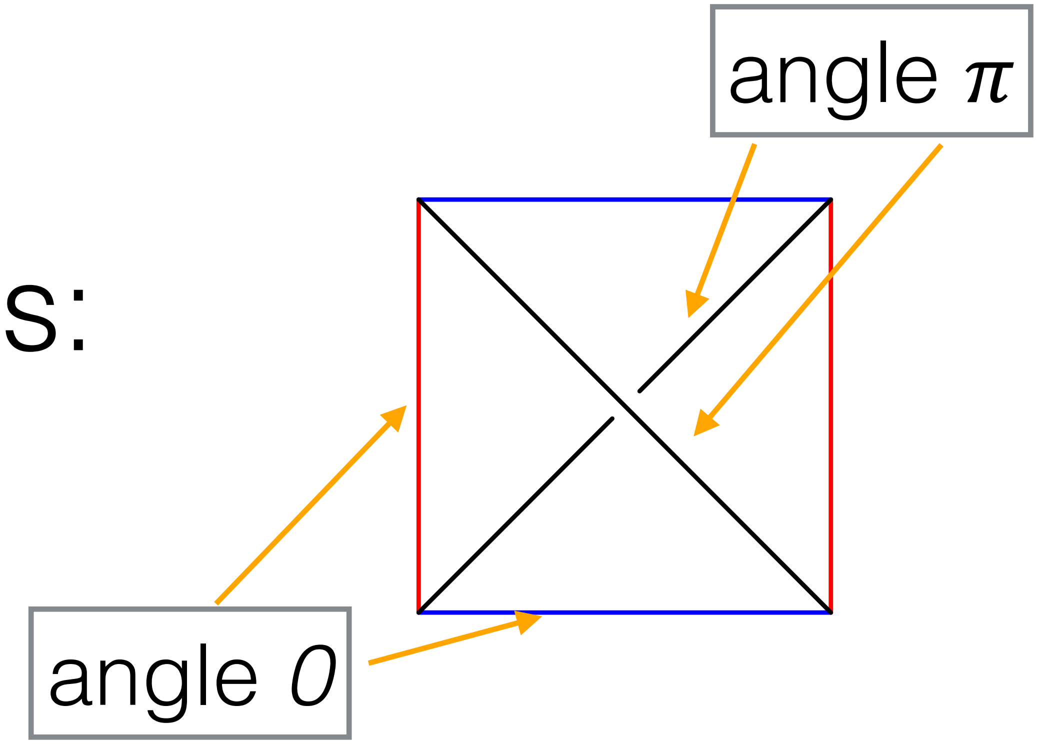

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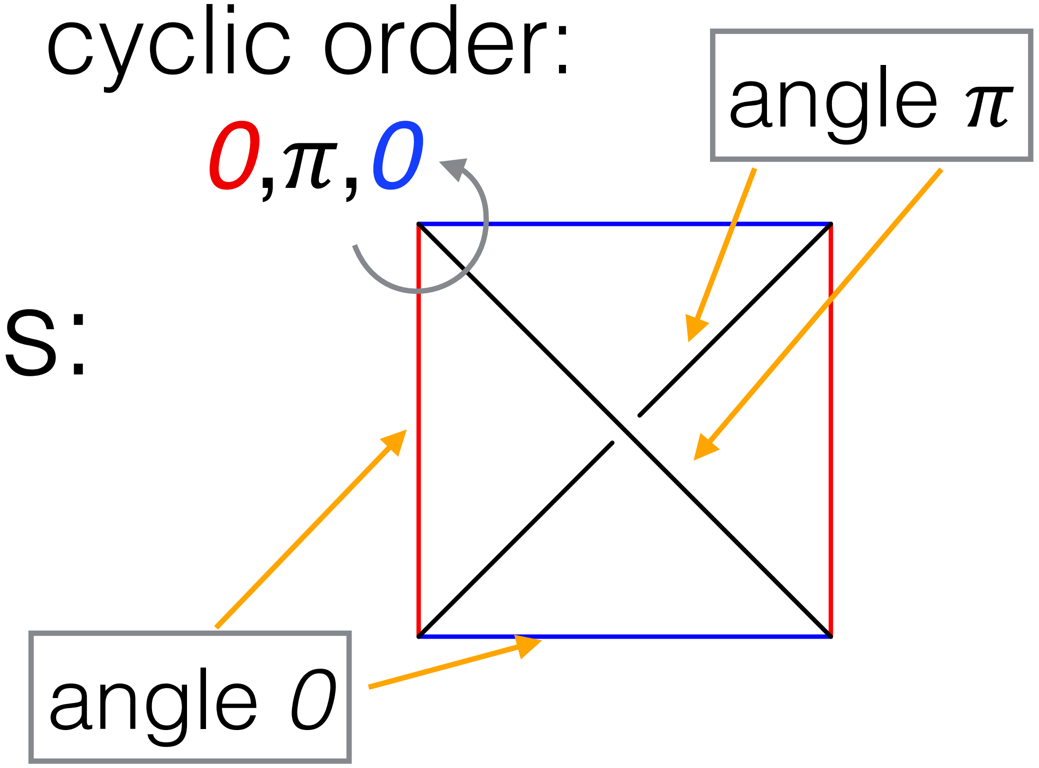

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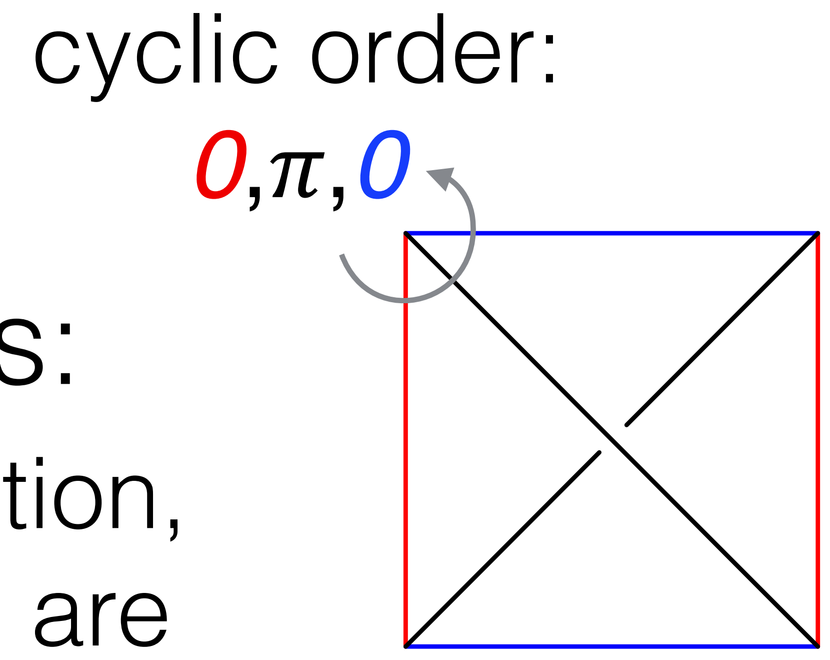

  - tetrahedra look like this:
    - with Gueritaud's construction, this is easy to see: edges are geodesics in the singular flat metric, with well-defined slope. Just make positive slope edges red, and negative slope edges blue.

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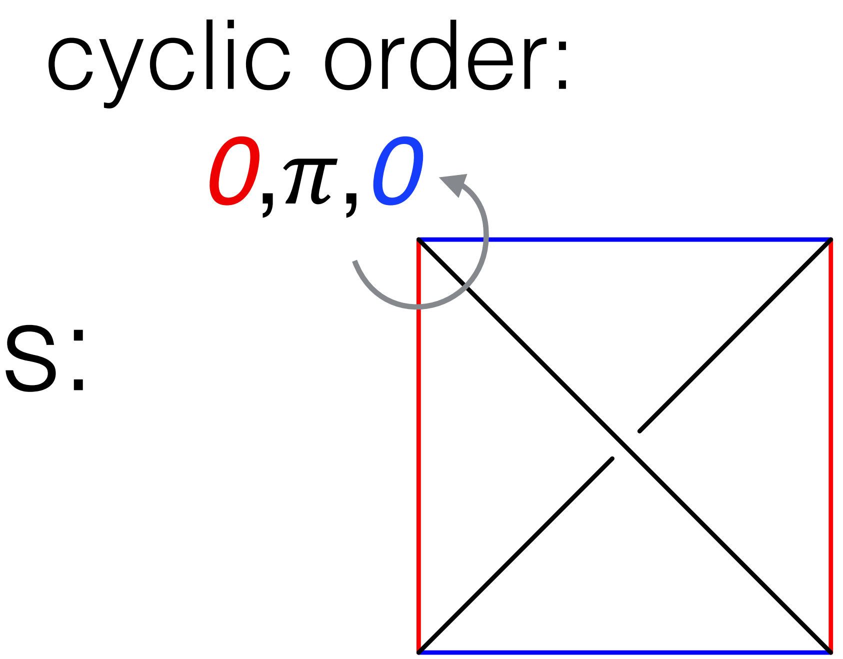
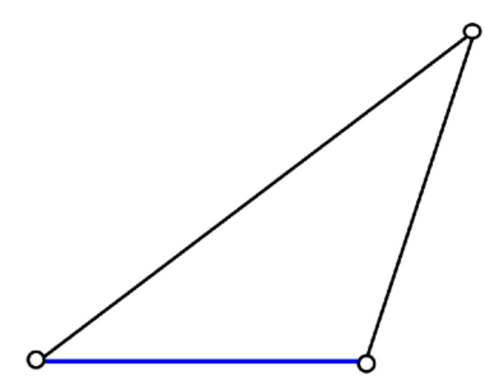


  …or equivalently:

- faces attached to an edge veer right or <u>left</u>:



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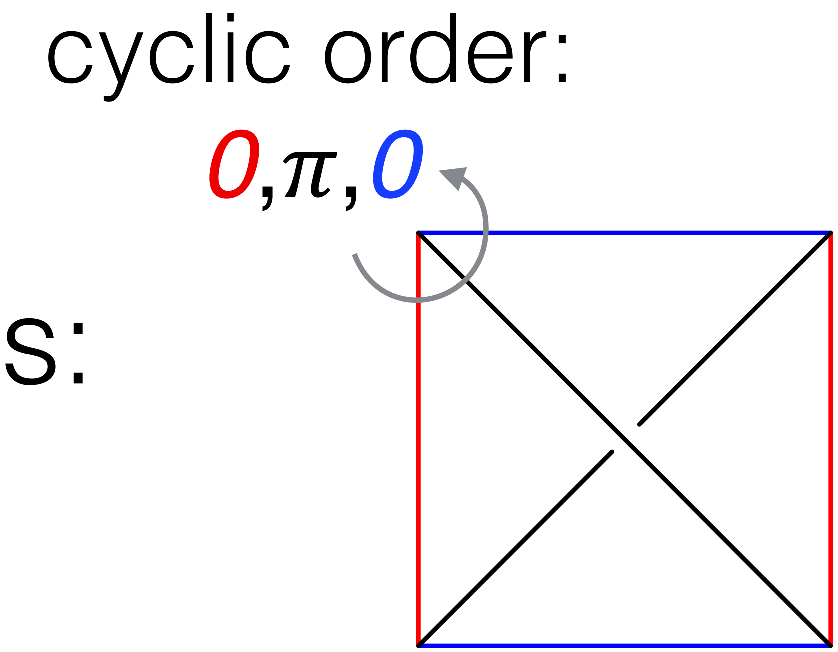

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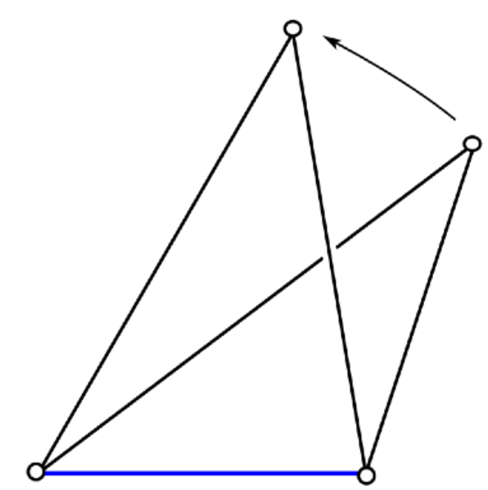

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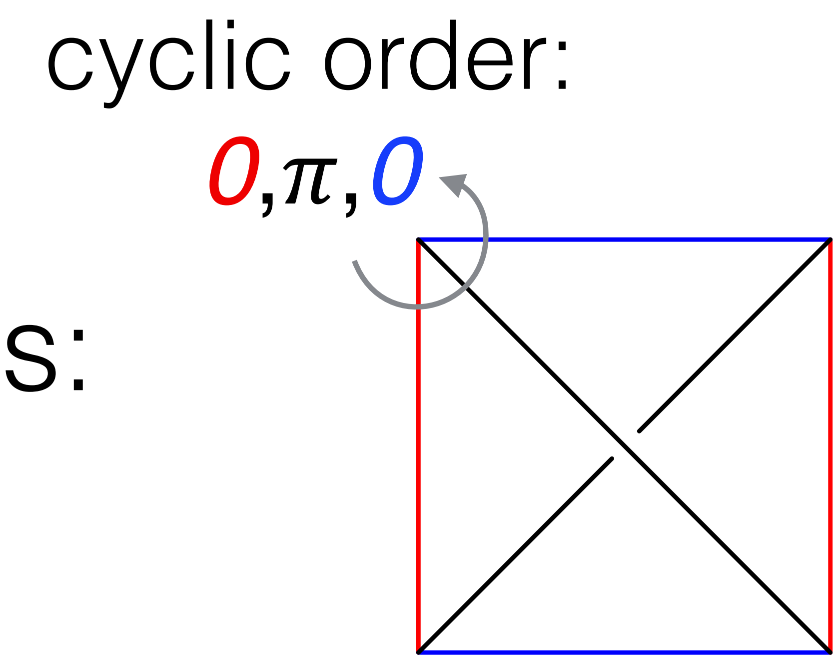

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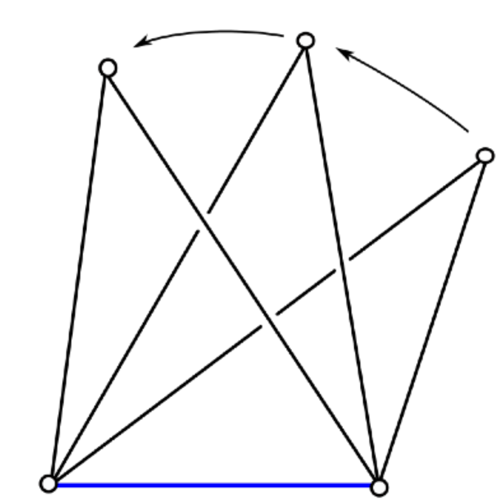

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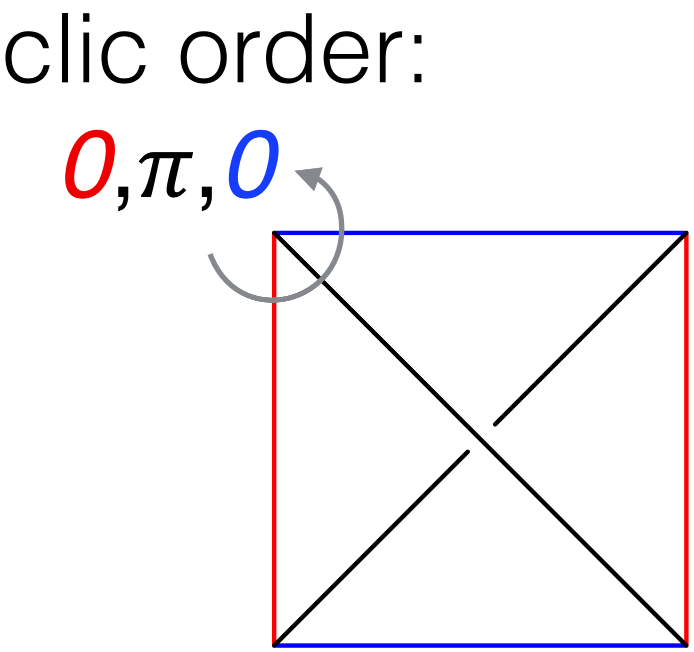

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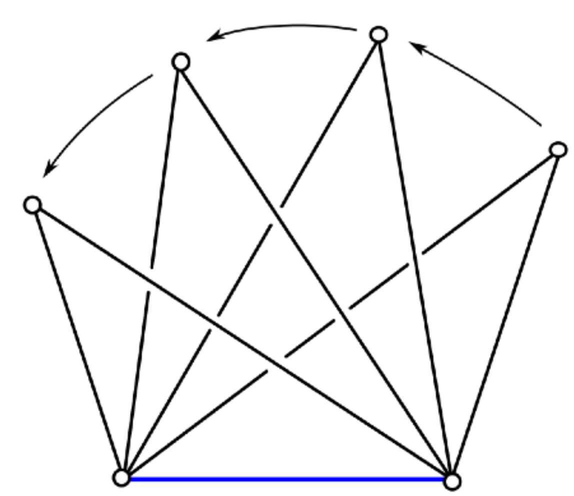
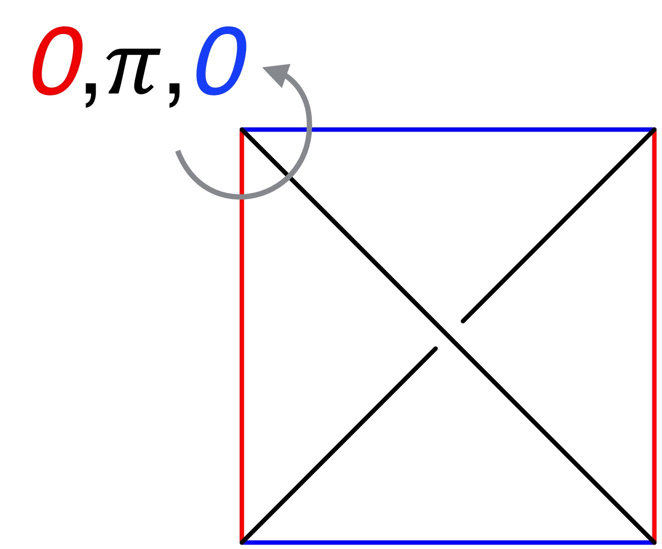


the layered ***veering*** triangulation:

- [Agol 2011, Gueritaud 2015]: the triangulation that we constructed is *veering,* ***i.e.:***

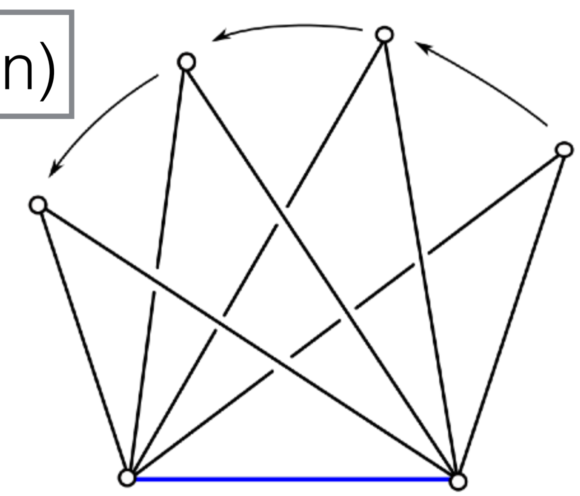
cyclic order:



  - tetrahedra look like this:

    …or equivalently:

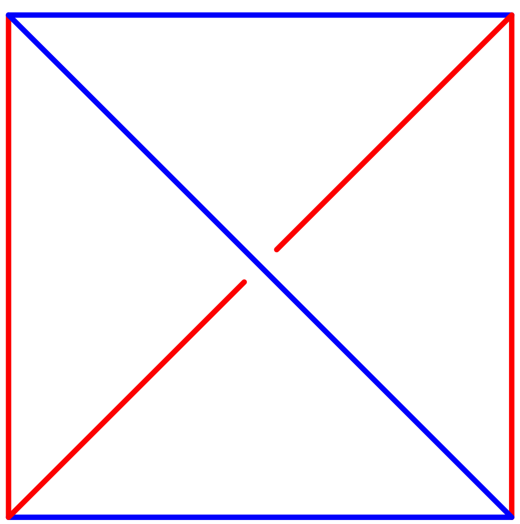
(Hodgson-Rubinstein-Segerman-Tillmann)



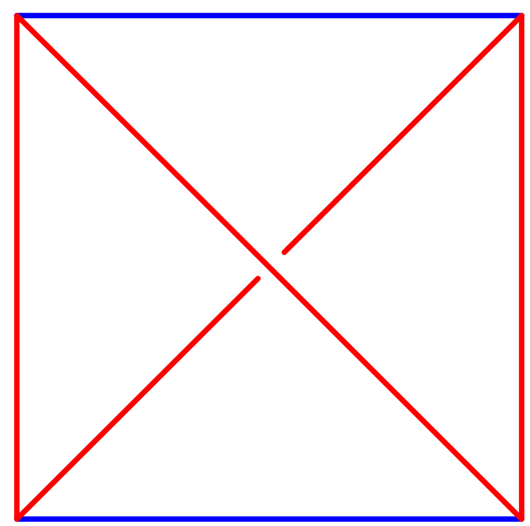
  - faces attached to an edge veer right or left:

the layered *veering* triangulation:

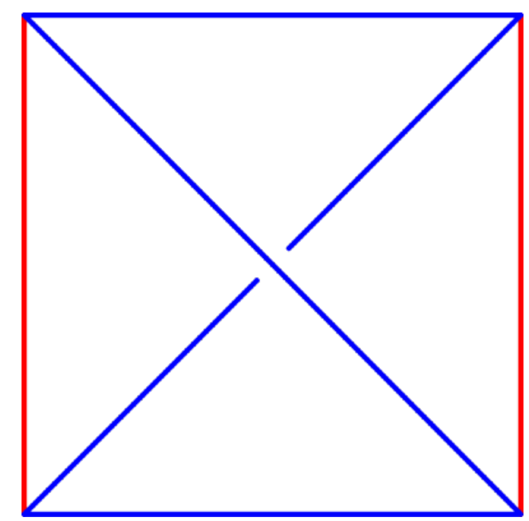
- two types of tetrahedra: *hinges* and *non-hinges*



hinge



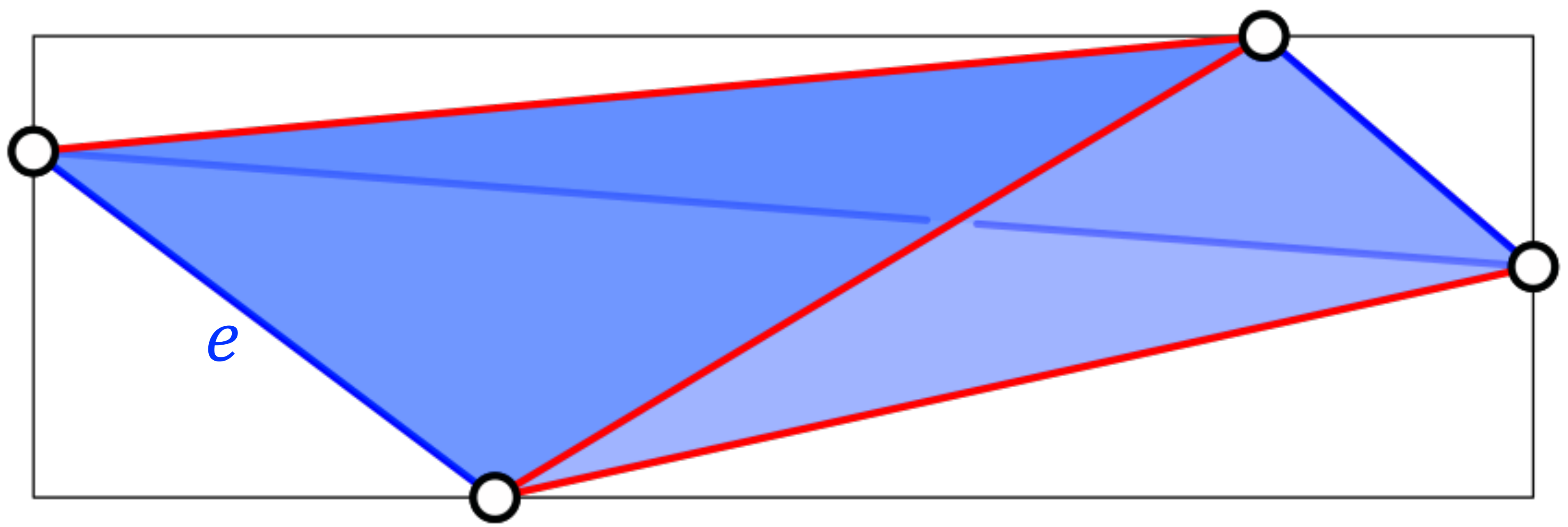
or



non-hinge (a.k.a. fan tetrahedron)

- **(very rough) guiding principle**: non-hinges layer onto annuli where Dehn-twisting happens (forming solid tori), hinges are the glue that holds it all together.

the tetrahedra around e, in color…



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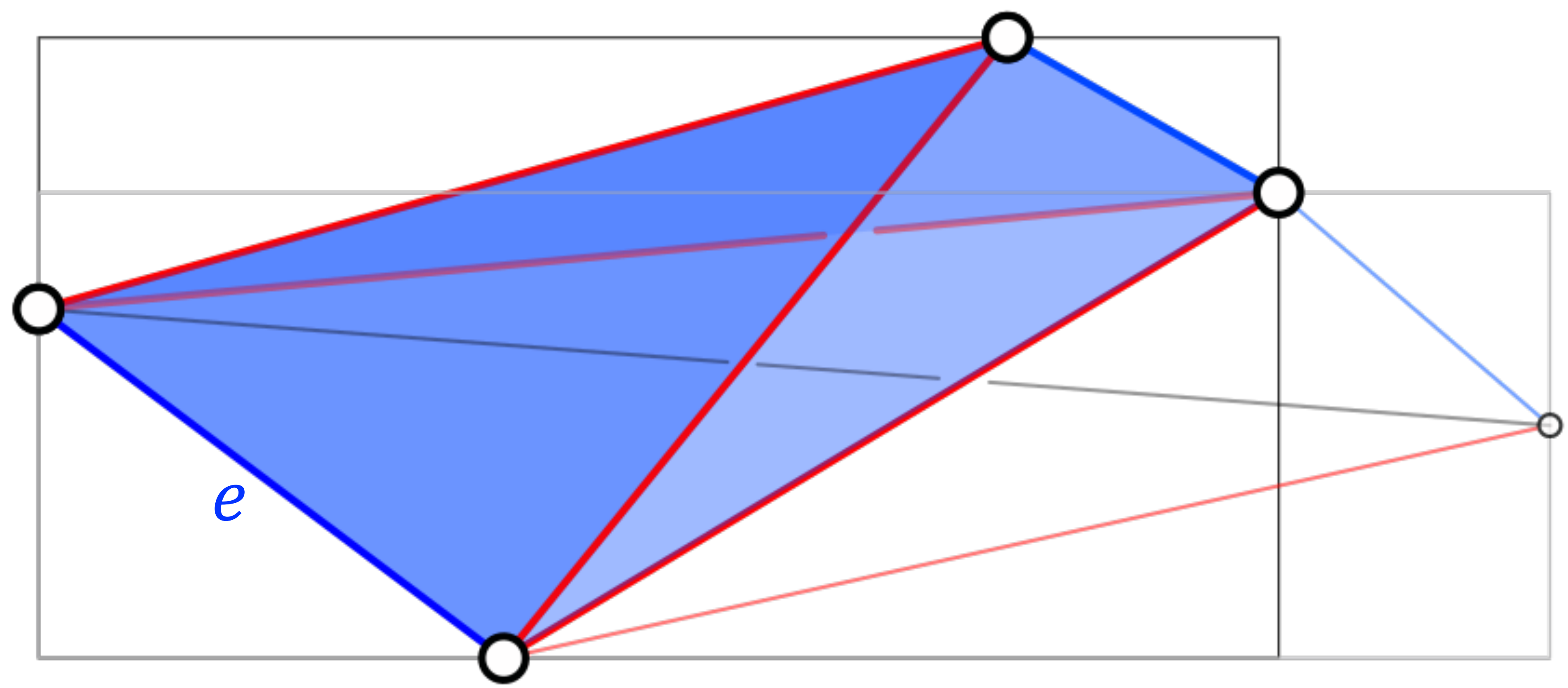

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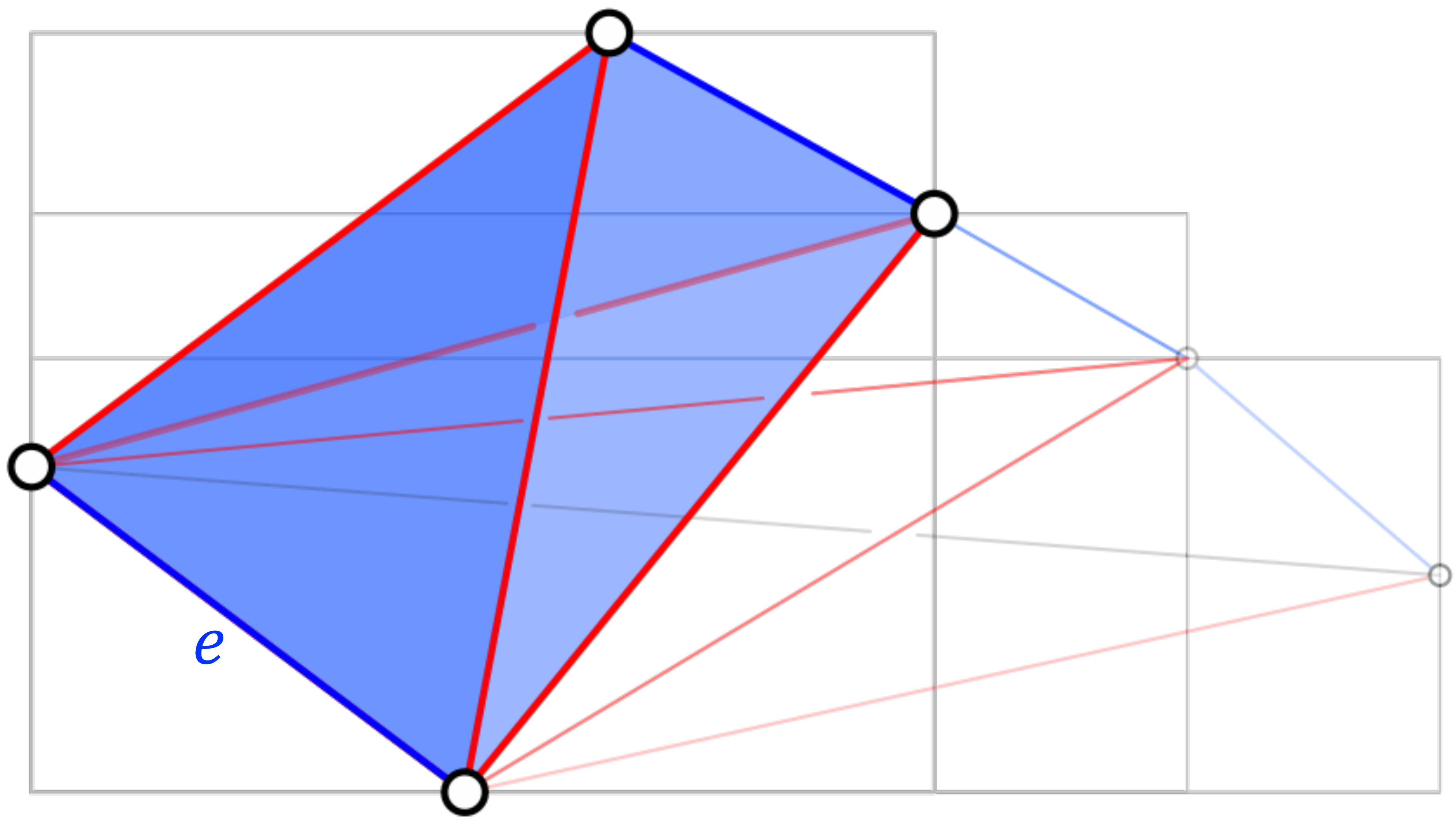

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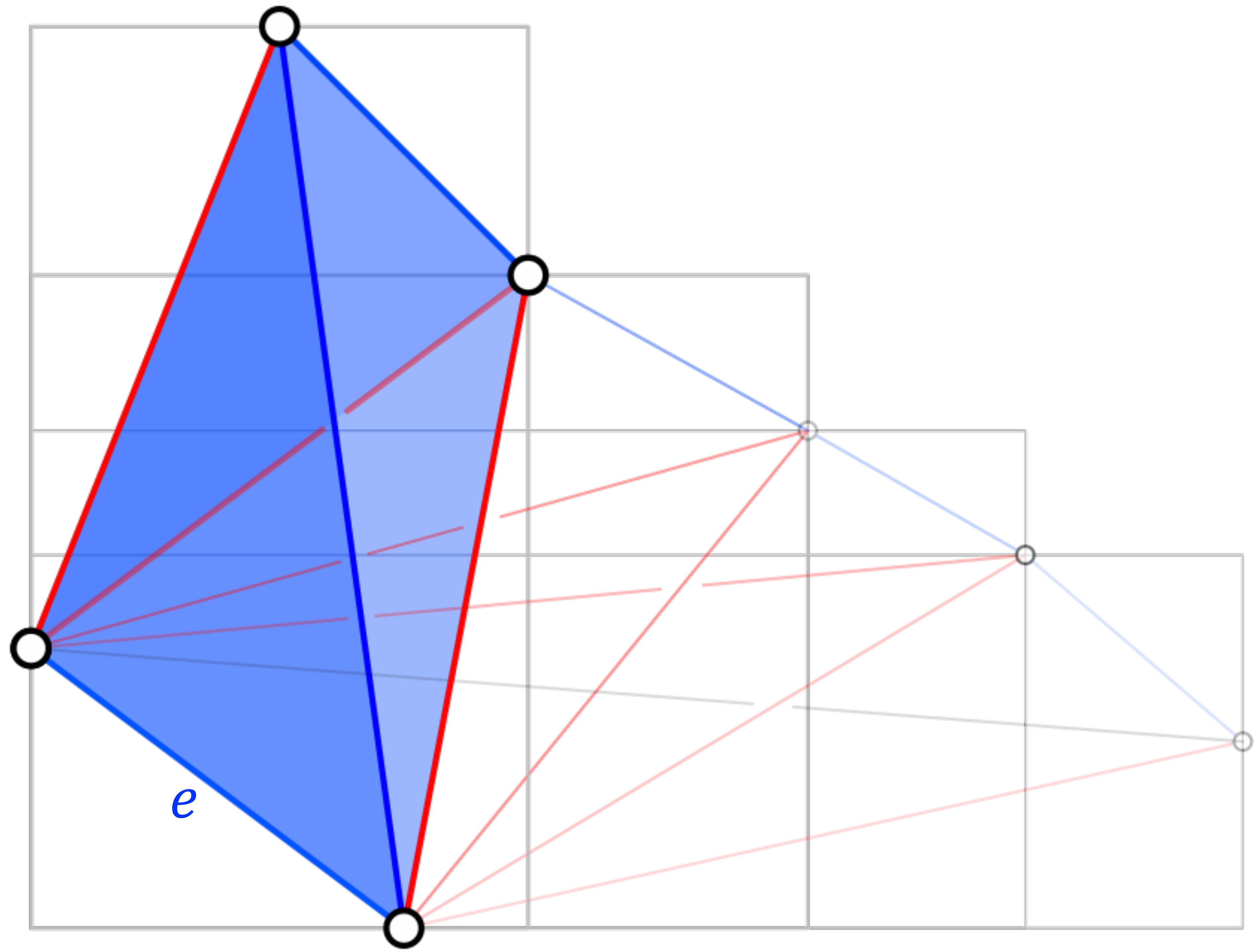

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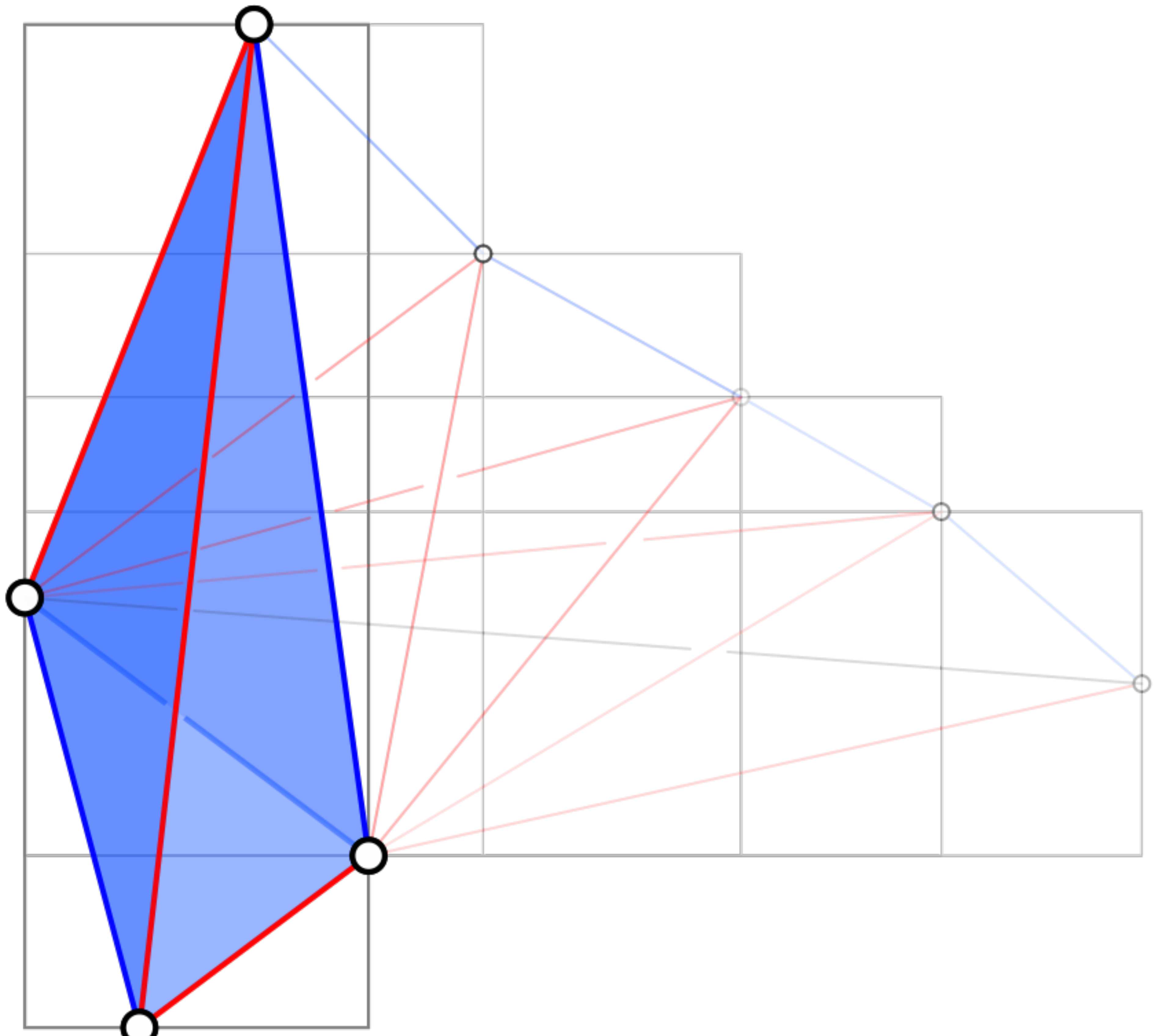

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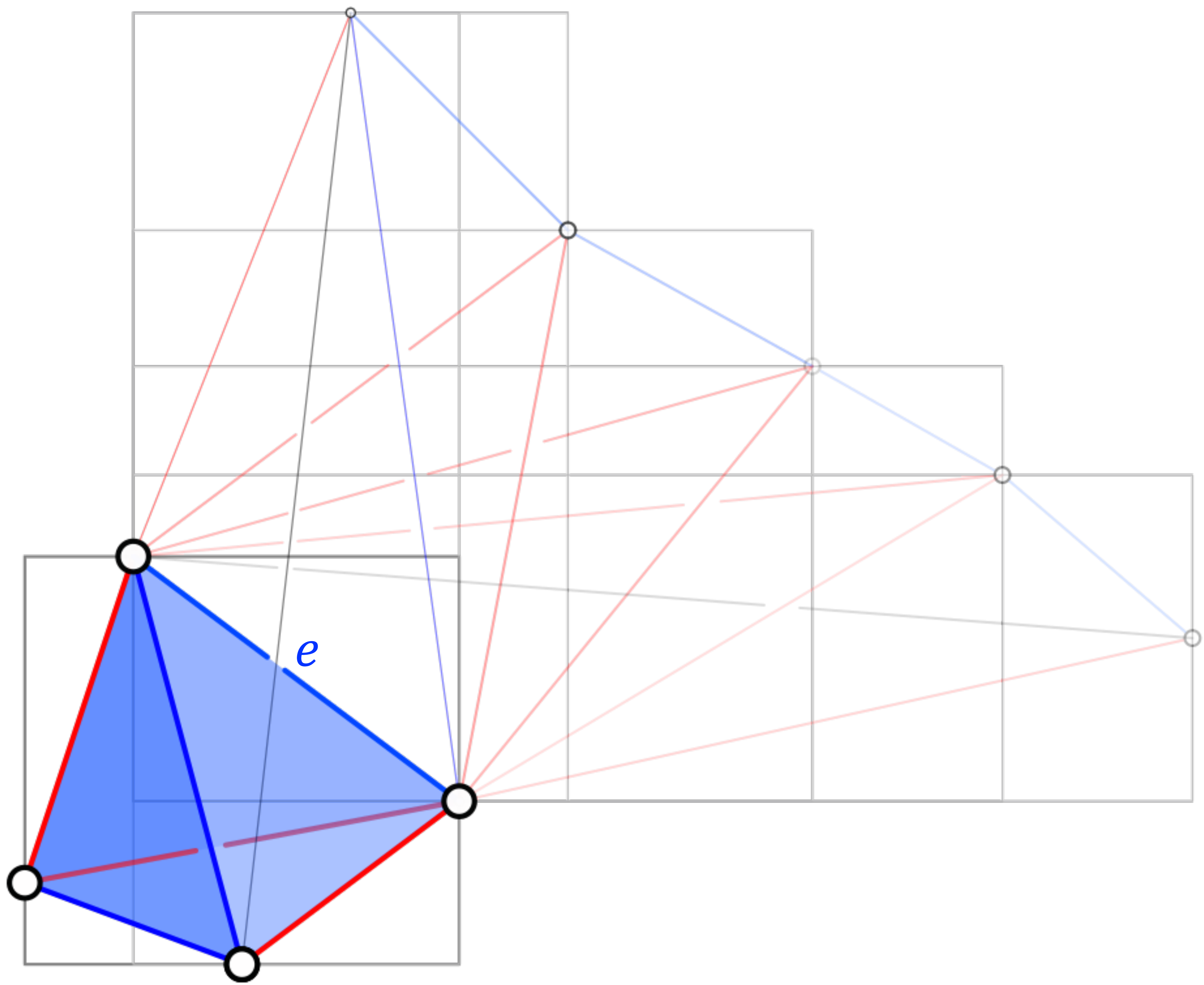

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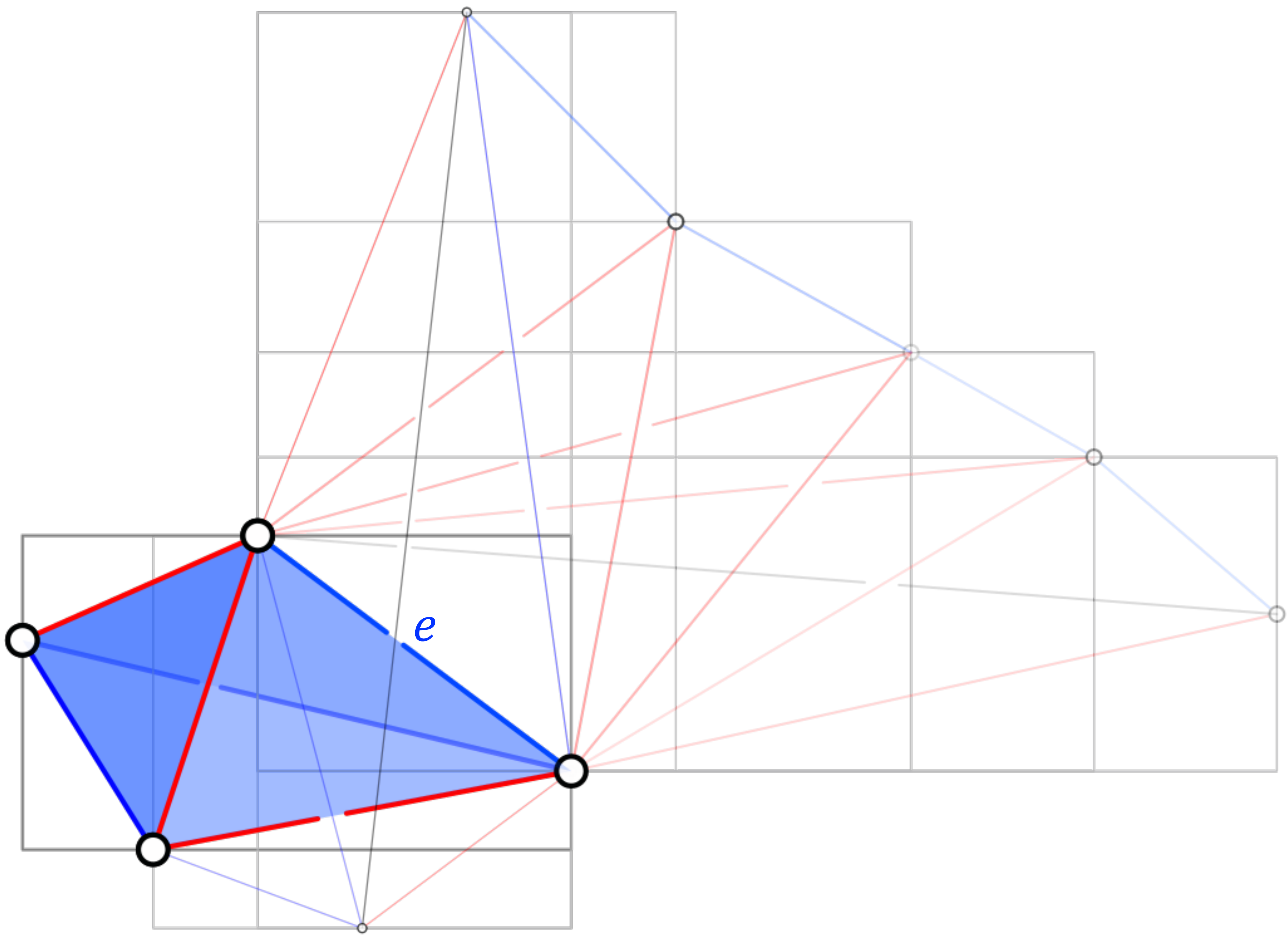

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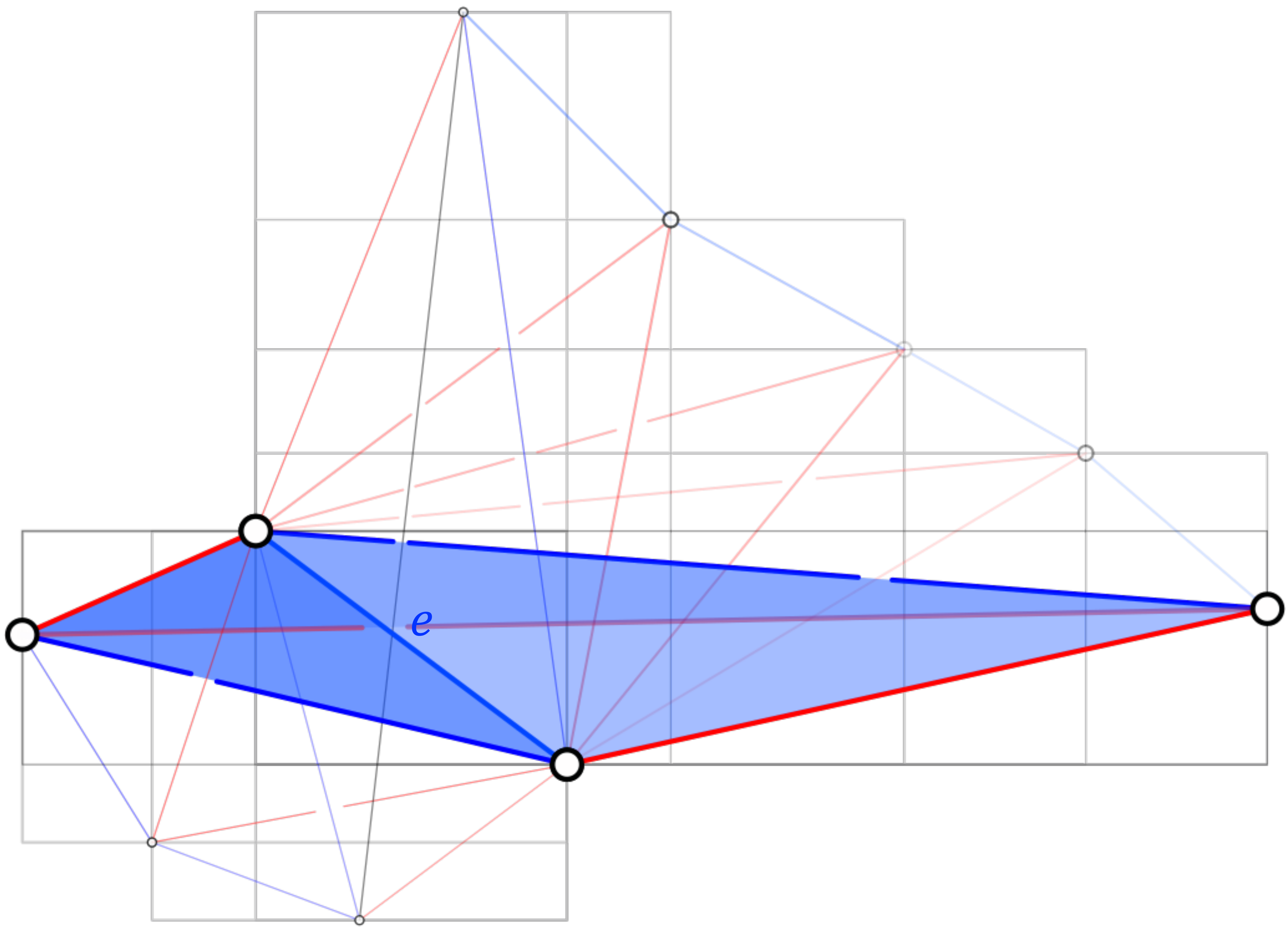
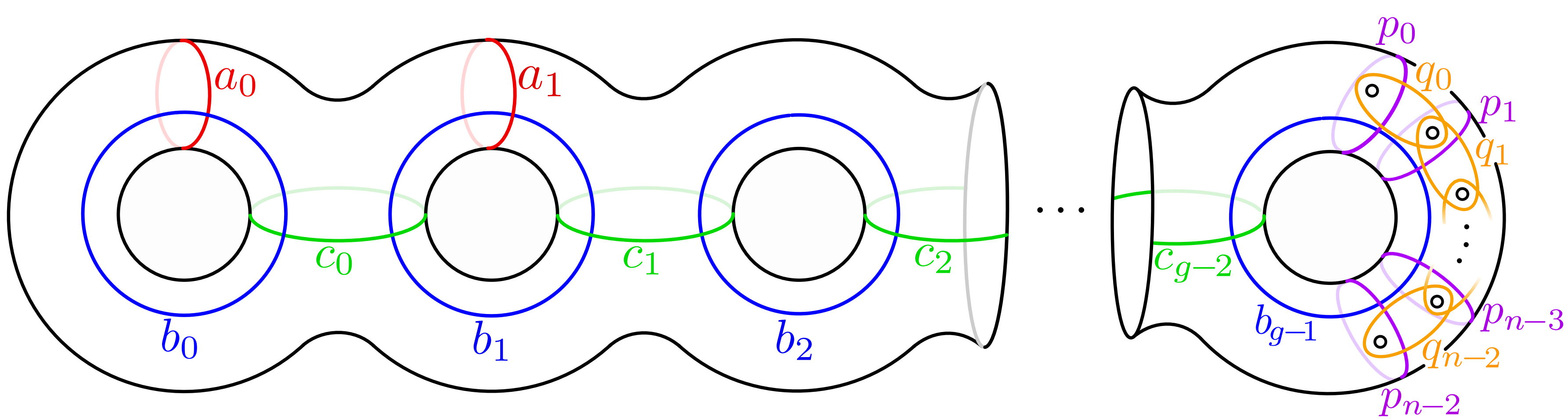


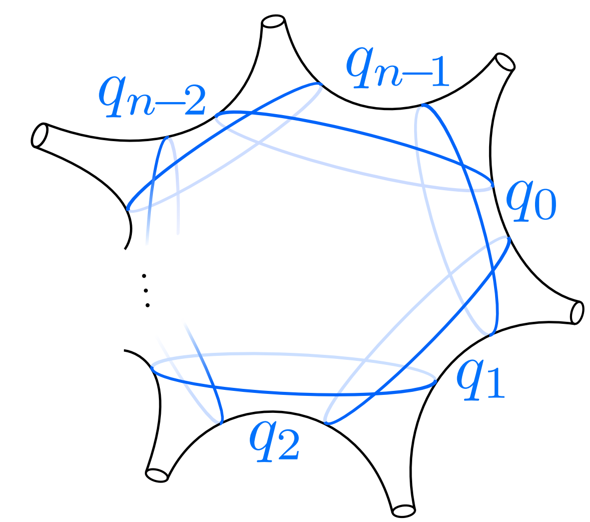
Veering Triangulations: context

- (Agol, 2011): introduced veering triangulations.
- (Hodgson-Rubinstein-Segerman-Tillmann, Futer-Gueritaud 2013): veering triangulations admit positive angle structures, but…
- (Hodgson-Issa-Segerman, 2016): they are not always geometric.
- (Gueritaud, 2016): new construction via quadratic differential, relation to Cannon-Thurston maps.
- (Minsky-Taylor, 2017): subsurface distance between $\mathcal{L}_s$ and $\mathcal{L}_u$ bounded in terms of number of tetrahedra in $\mathcal{T}$.
- (Bell, 2013-2017): developed `flipper`, a computer program which constructs veering triangulations (and more).

randomly sampling pseudo-Anosov maps:

1. to sample from $\mathrm{Mod}(\Sigma)$ we take a random walk in its Cayley graph (backtracking is allowed), with respect to some generating set.





- for sufficiently long words, we will almost surely get a pseudo-Anosov map (Rivin, Maher 2008).

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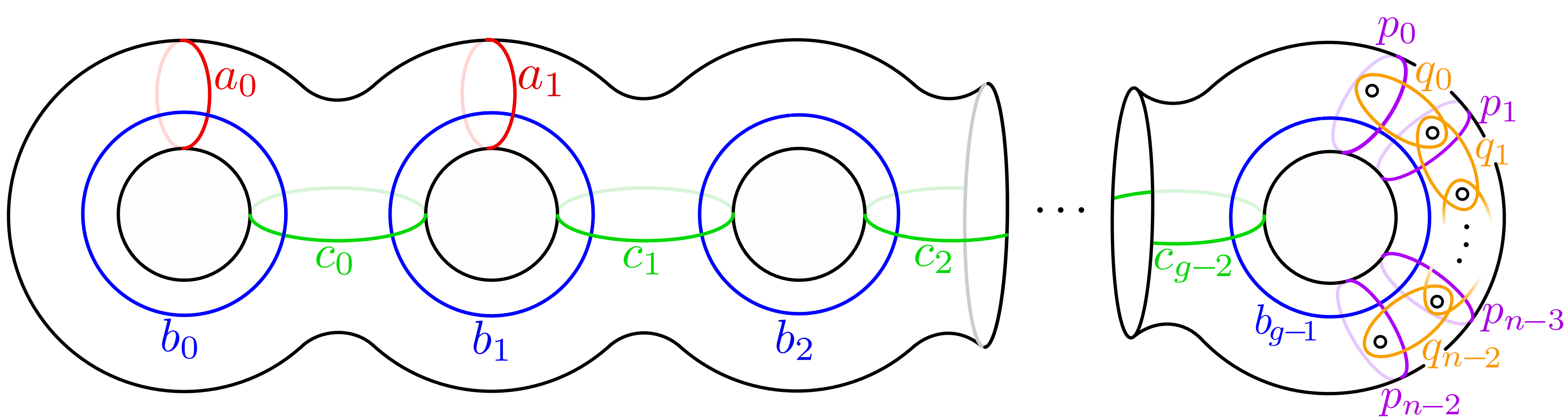

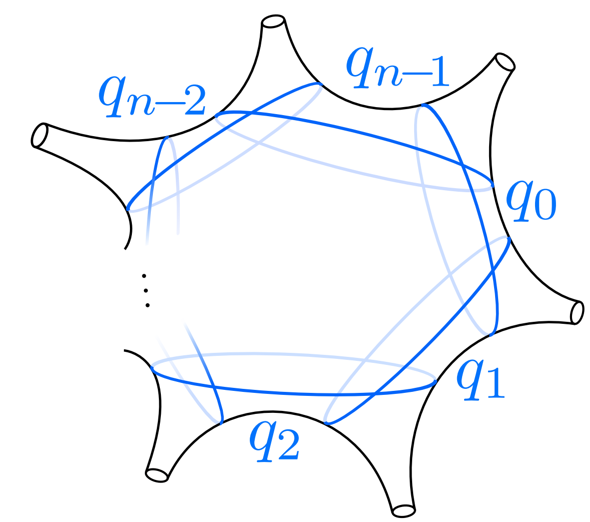

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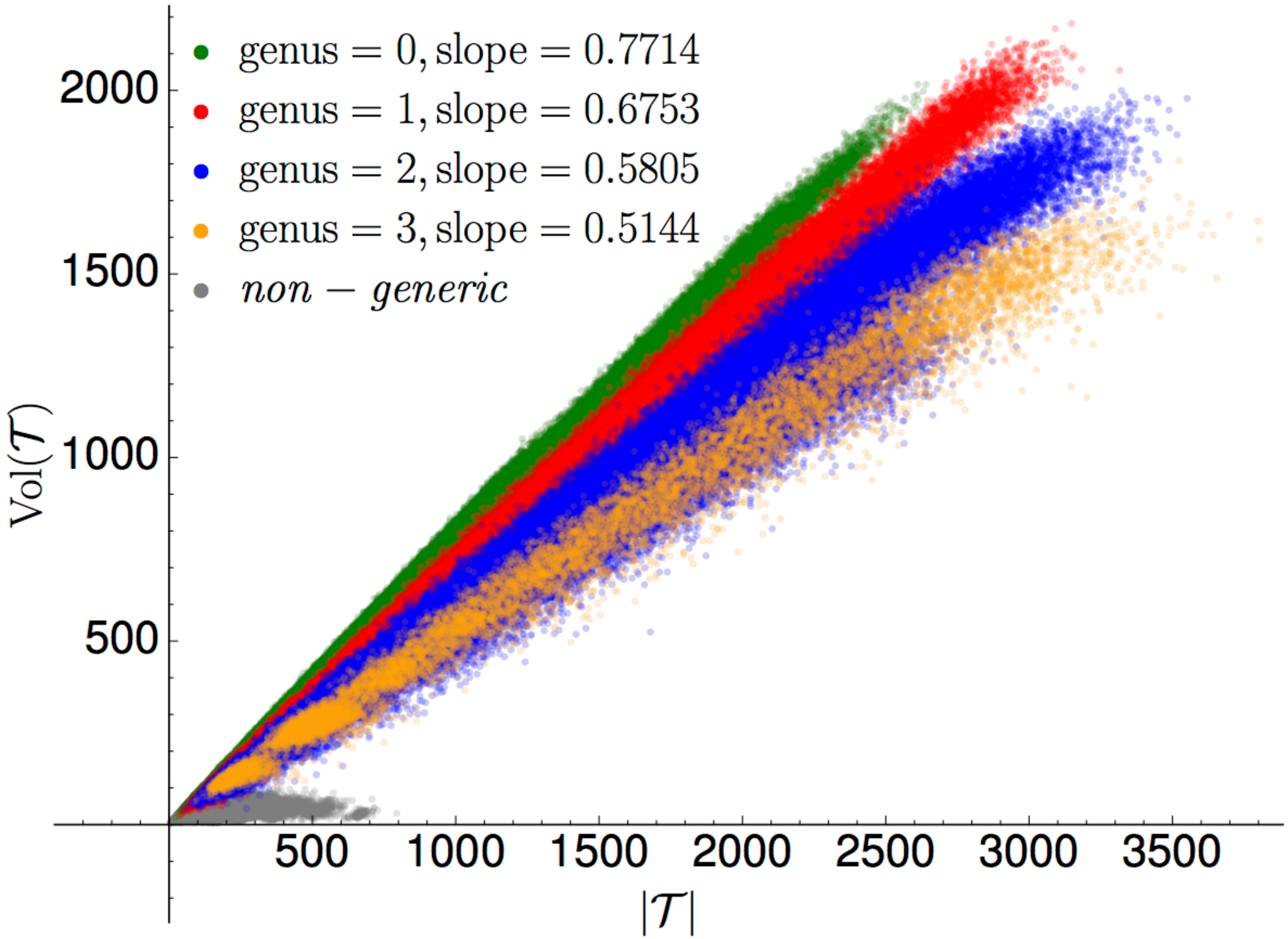
2. we then construct the veering triangulation as above, using `flipper`.

3. for geometric information, we feed the resulting triangulation into `SnapPy` [Culler, Dunfield, Goerner, Weeks].

Q1: Can the combinatorial information of the veering triangulation tell us anything about the geometry of M_{φ°?

Q2: How frequently are veering triangulations realized geometrically?

Volume:



- (Rivin): generically, volume is coarsely linear in the number of tetrahedra in $\mathcal{T}$.

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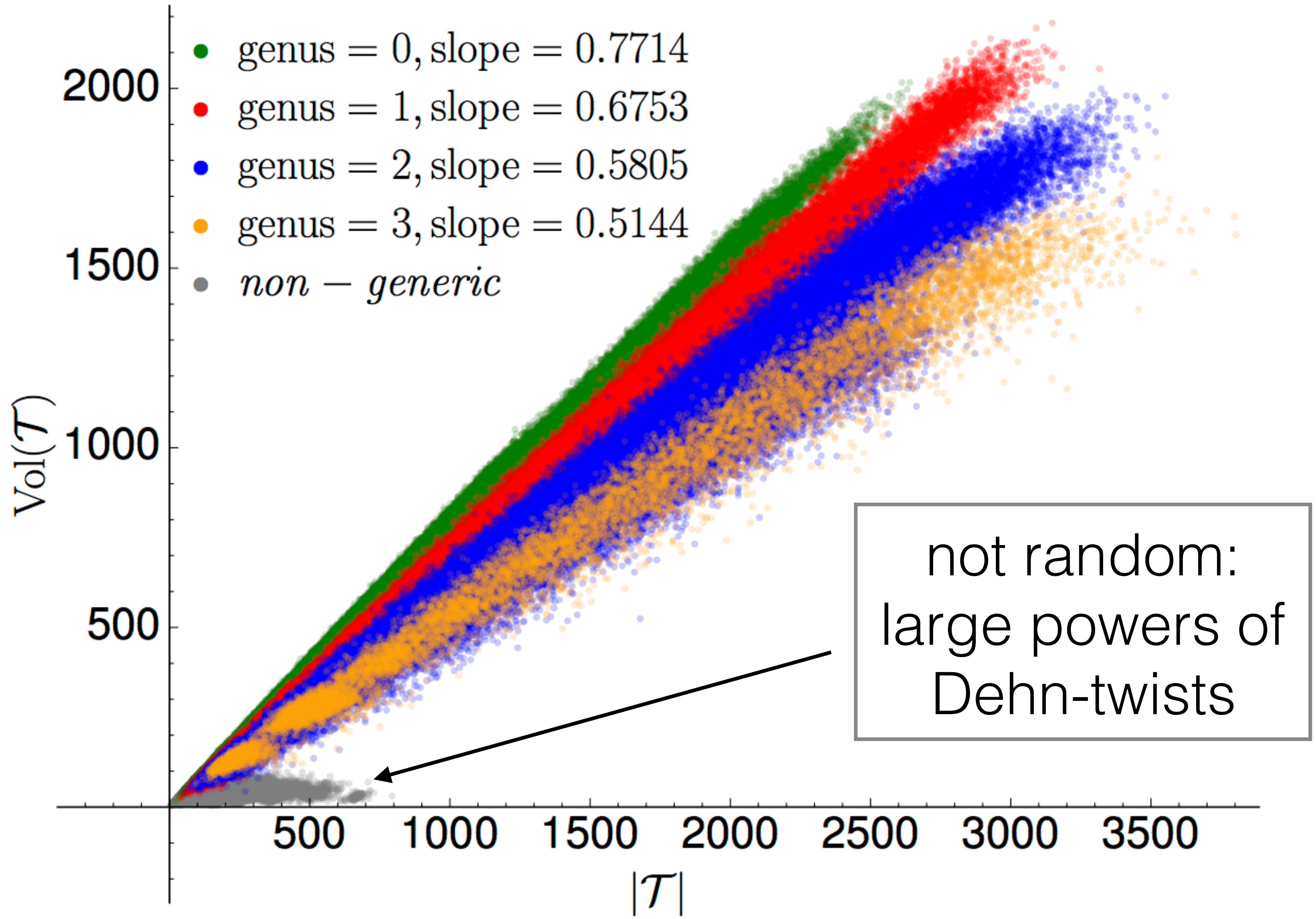

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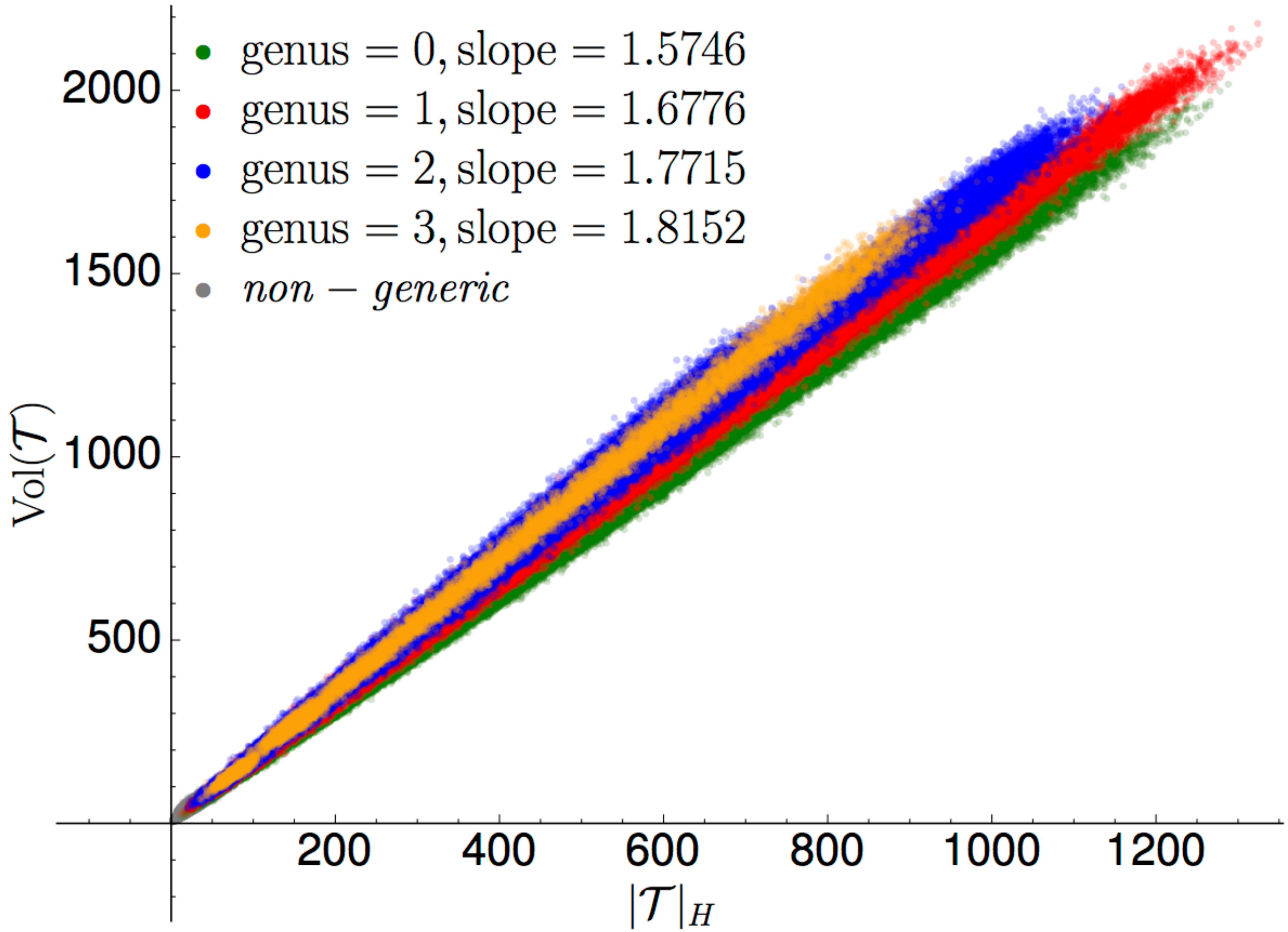
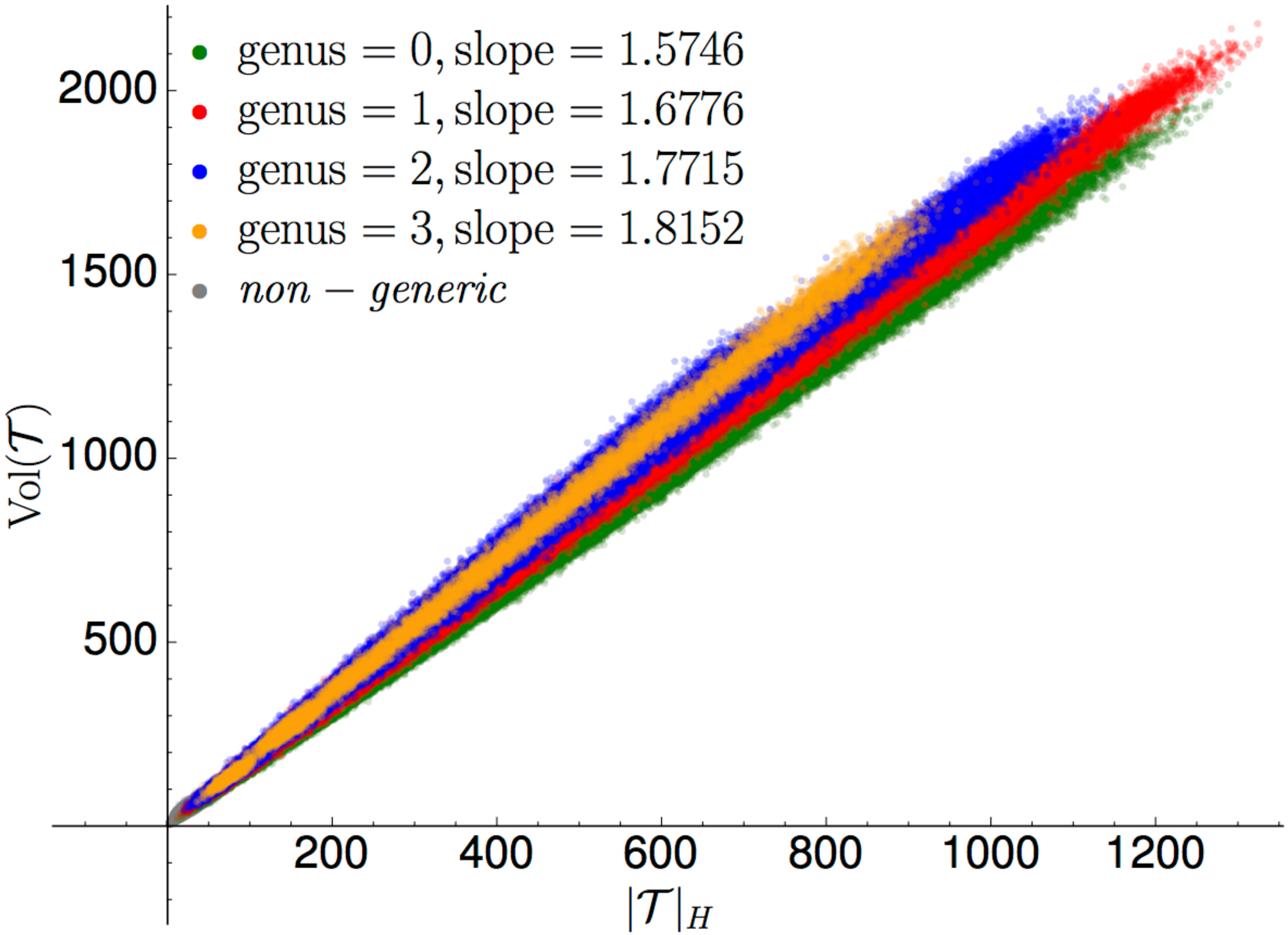


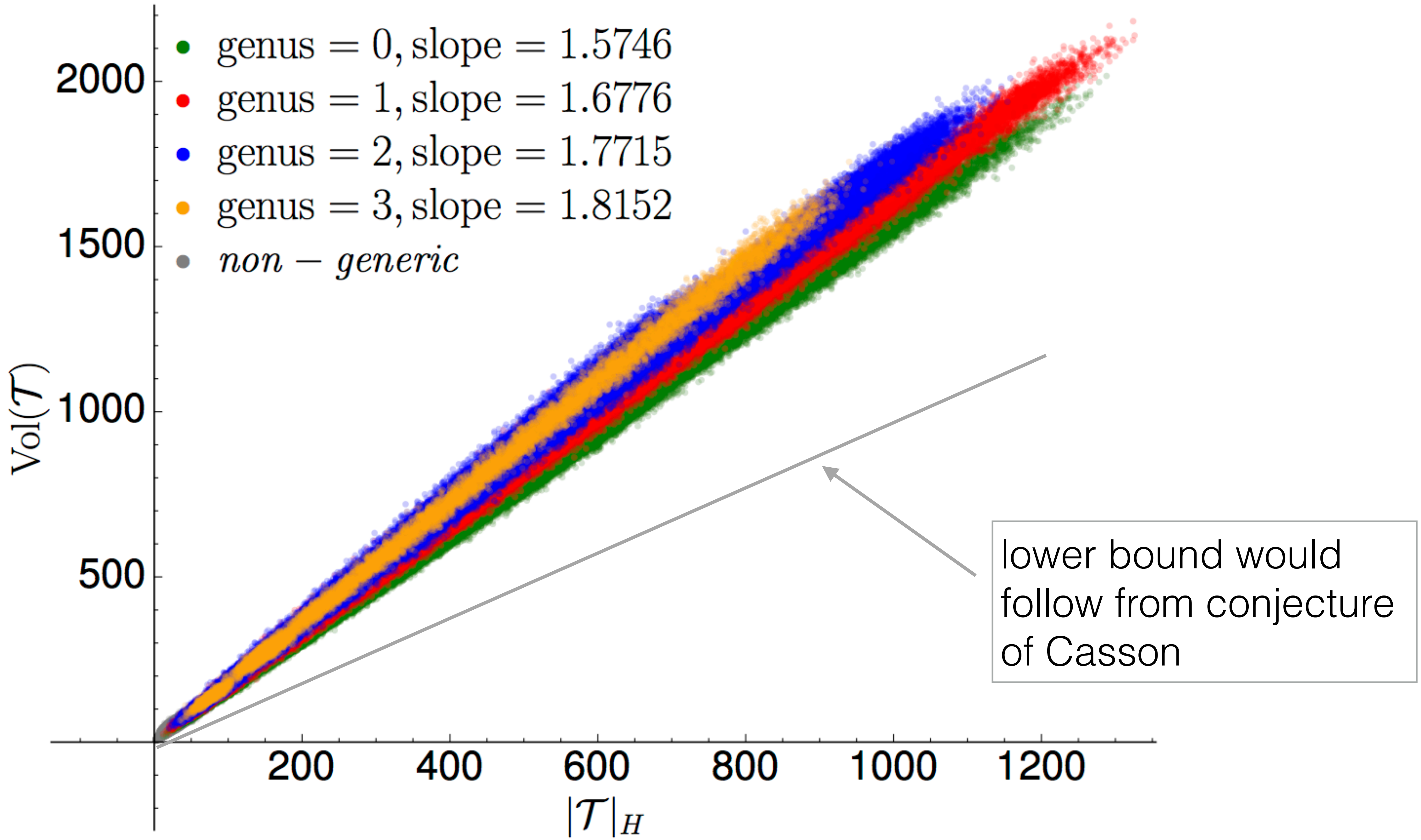
- if we plot volume against the number of ***hinge*** tetrahedra, the linear relation appears to improve.

Volume:



- the slope $\mathrm{Vol}(\mathcal{T})/|\mathcal{T}|_H$ appears to depend on the surface Σ.

Volume:



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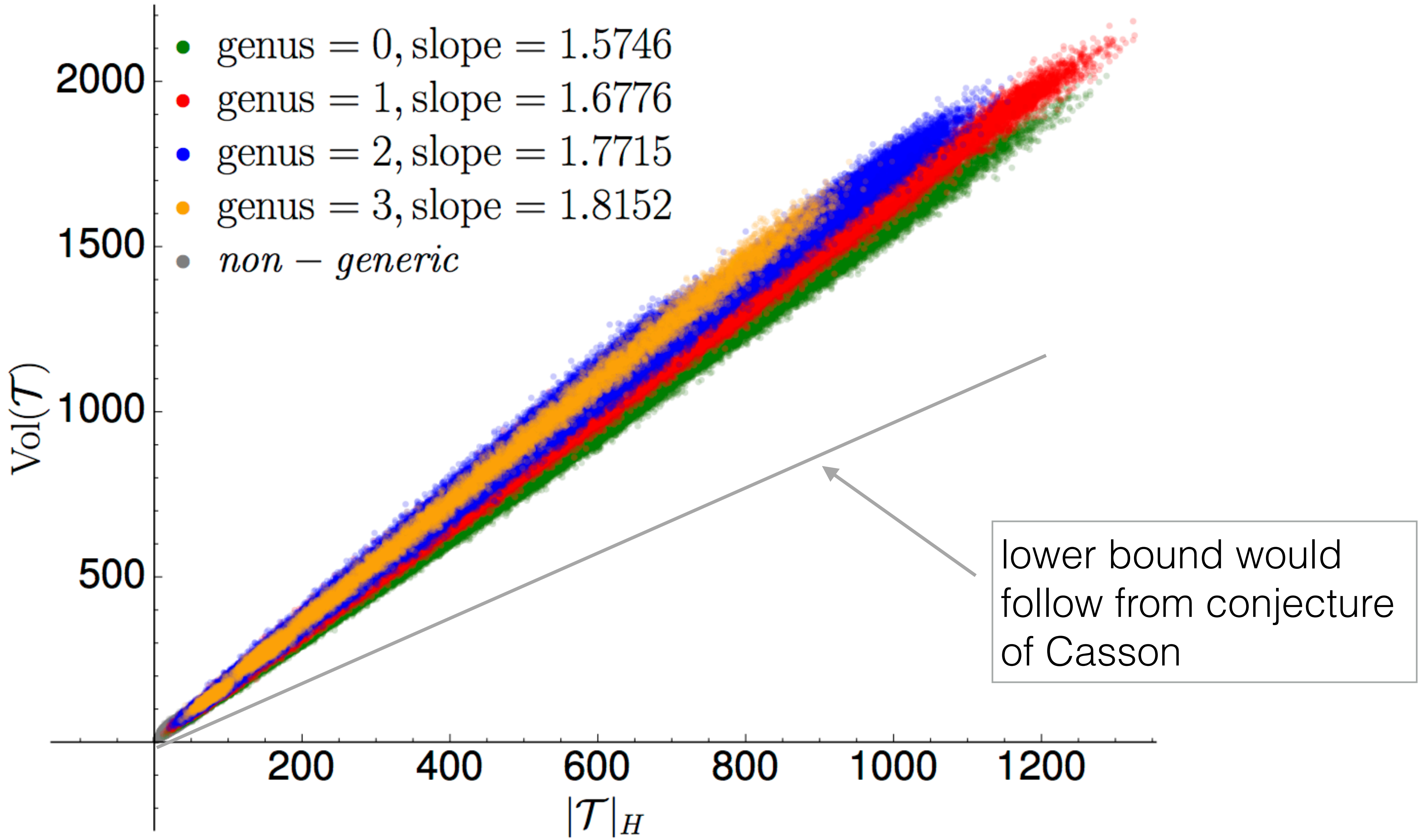
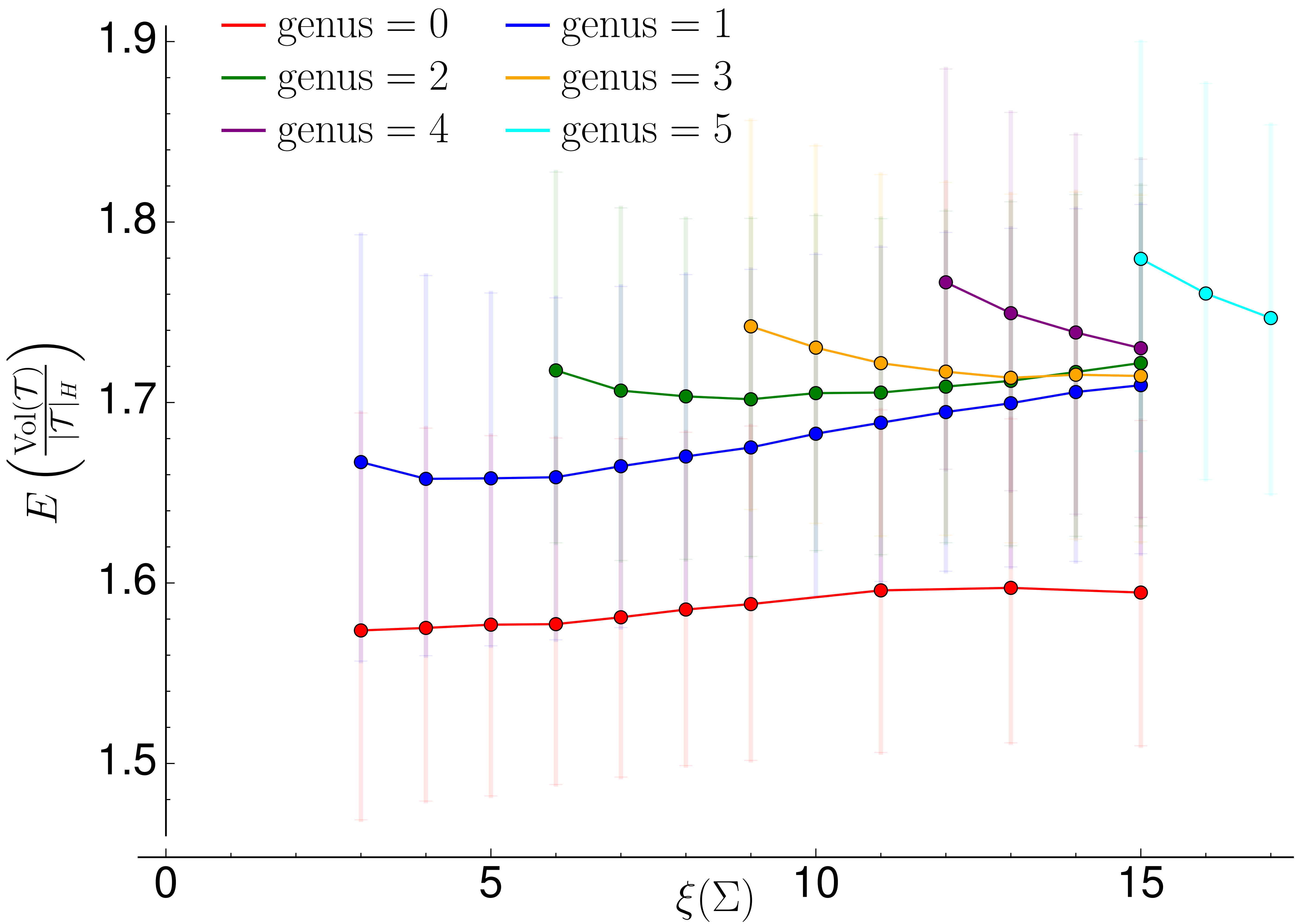


- Aougab-Brock-Futer-Minsky-Taylor: the slope is bounded above by a linear function of $\chi(\Sigma)$.

how do (the means of) these histograms depend on the complexity $\xi(\Sigma) = (3g - 3 + n)$ of Σ ?



Q1: Can the combinatorial information of the veering triangulation tell us anything about the geometry of M_{φ°?

Q2: How frequently are veering triangulations realized geometrically?

Are generic veering triangulations geometric?

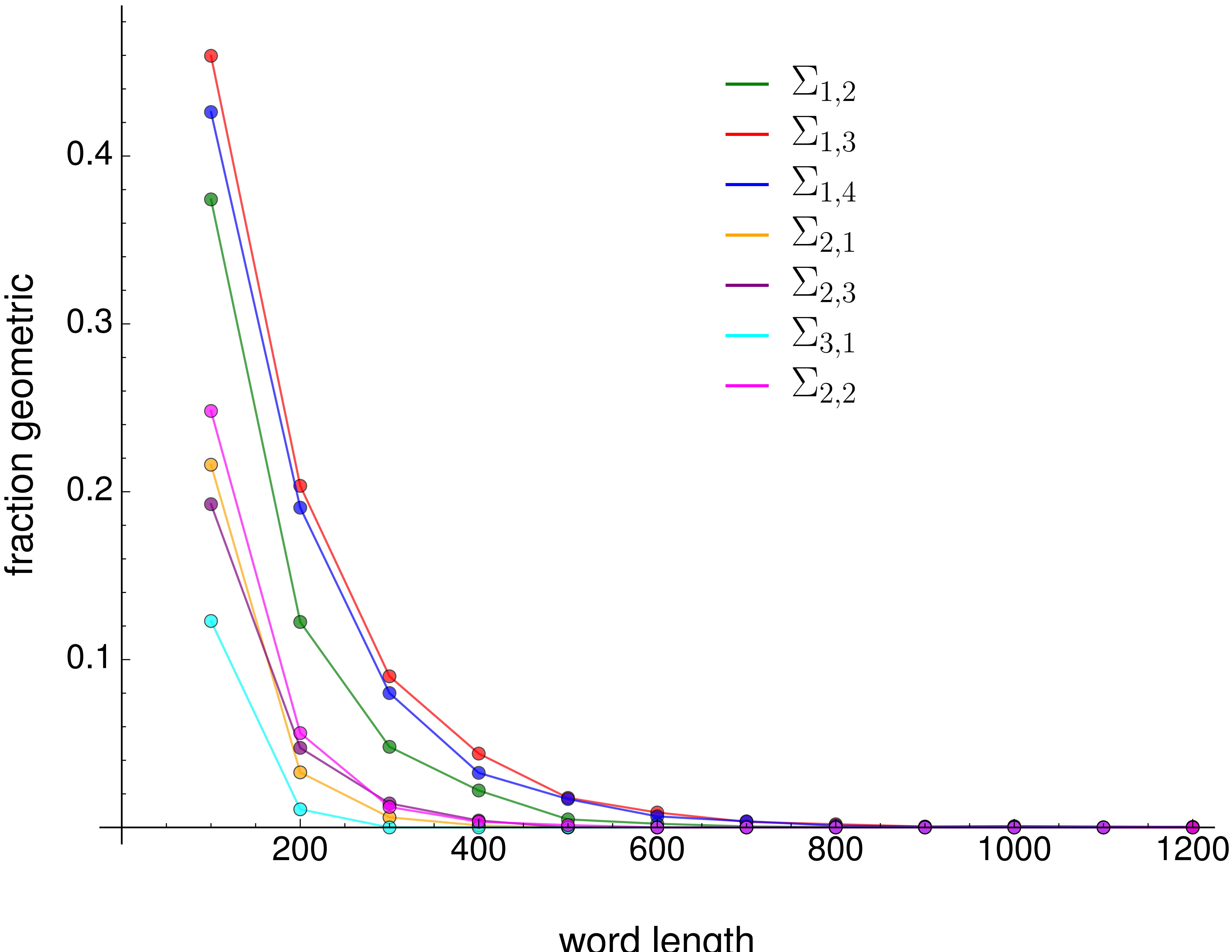
- for the case of the once-punctured torus $\Sigma_{1,1}$ and the four-punctured sphere $\Sigma_{0,4}$, $\mathcal{T}$ is always geometric (in fact, canonical).

(Akioshi, Lackenby, Gueritaud)

- Hodgson-Issa-Segerman gave first examples of non-geometric veering triangulations.

- but what will a *random* veering triangulation look like?

Are generic veering triangulations geometric?



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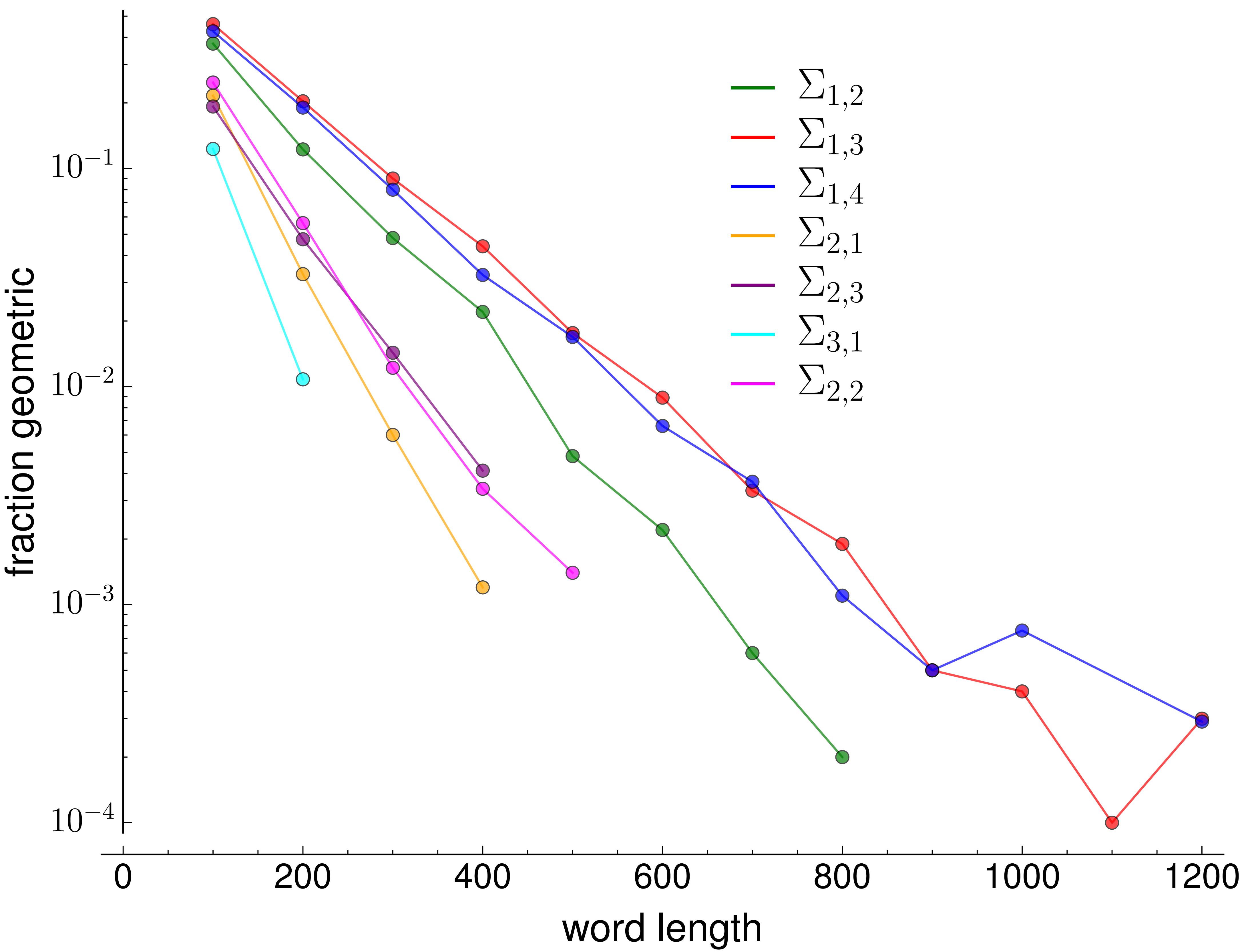
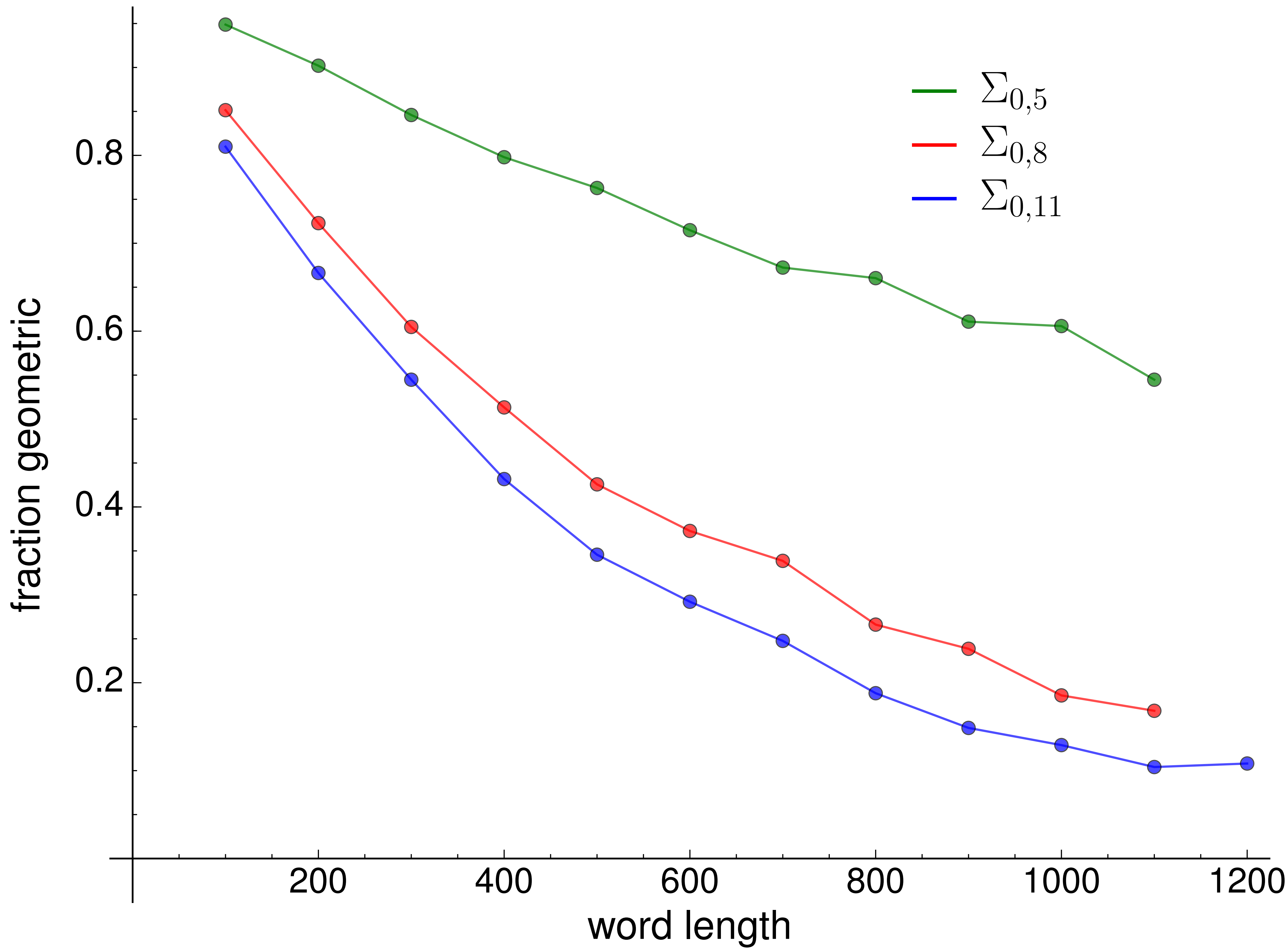
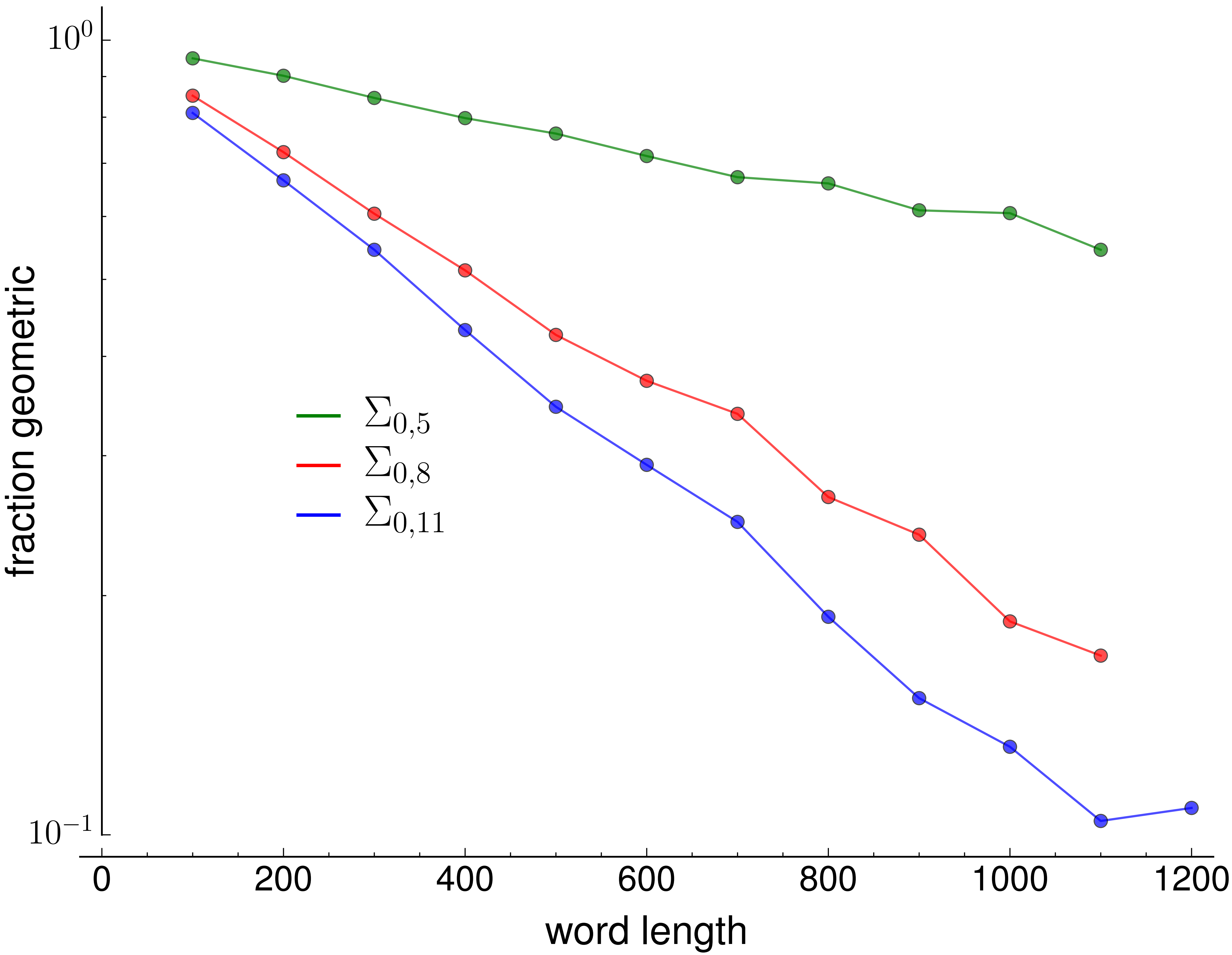


for genus *0*, decay is not nearly as fast…



…but still appears to be exponential.



Conjecture:

Let $\Sigma = \Sigma_{g,n}$ be a surface of complexity $\xi(\Sigma) \geq 2$, and let $\mathrm{P}_{k,\Sigma}$ be the probability that a simple random walk of length k, on a set of generators for $\mathrm{Mod}(\Sigma)$, yields a pseudo-Anosov mapping class for which the layered veering triangulation of the mapping torus M_{φ° is geometric. Then there exists a constant $c = c(\Sigma) > 0$ such that

$$\lim_{k\to\infty} \frac{\mathrm{P}_{k,\Sigma}}{e^{-k\cdot c}} = 0$$

Theorem (Futer-Taylor-W):

As above, but without exponential decay, i.e.:

$$\lim_{k\to\infty} \mathrm{P}_{k,\Sigma} = 0$$

Proof outline:

1. **Lemma**: For every Σ with $\xi(\Sigma) \geq \mathbf{2}$, there exists a pseudo-Anosov $\varphi{:}\Sigma \to \Sigma$ (in the principal stratum) such that the associated veering triangulation of M_{φ° is non-geometric.

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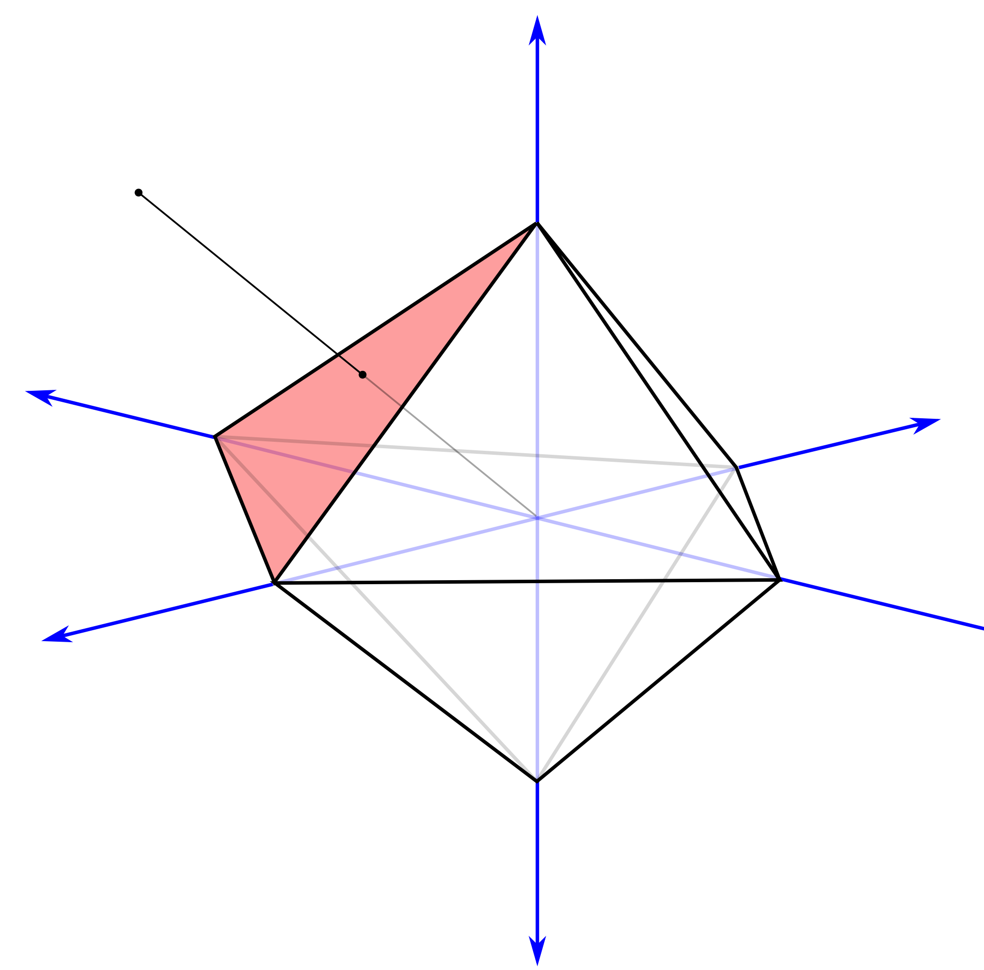
veering triangulations are *somewhat* rigid:

- a typical mapping torus M fibers in infinitely many ways, and each of these fibers appear in fibered faces of the Thurston norm ball, a convex polytope in $H_2(M,\partial M;\mathbb{R})$.
- every integer homology class contained in a ray passing through a fibered face, is a fiber for some fibration of M.



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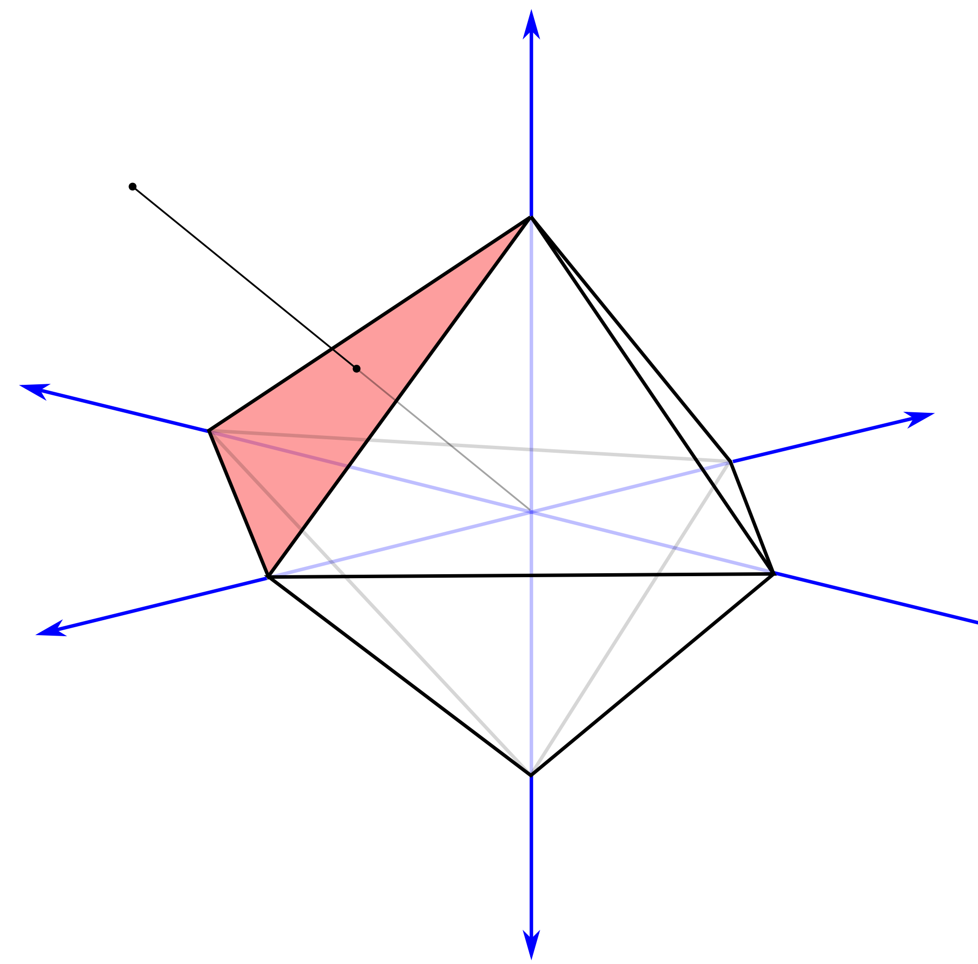
- Fact: all fibrations on a given face induce the same *2*-dimensional foliation of $M \implies$ the veering triangulation is an invariant of the fibered face.



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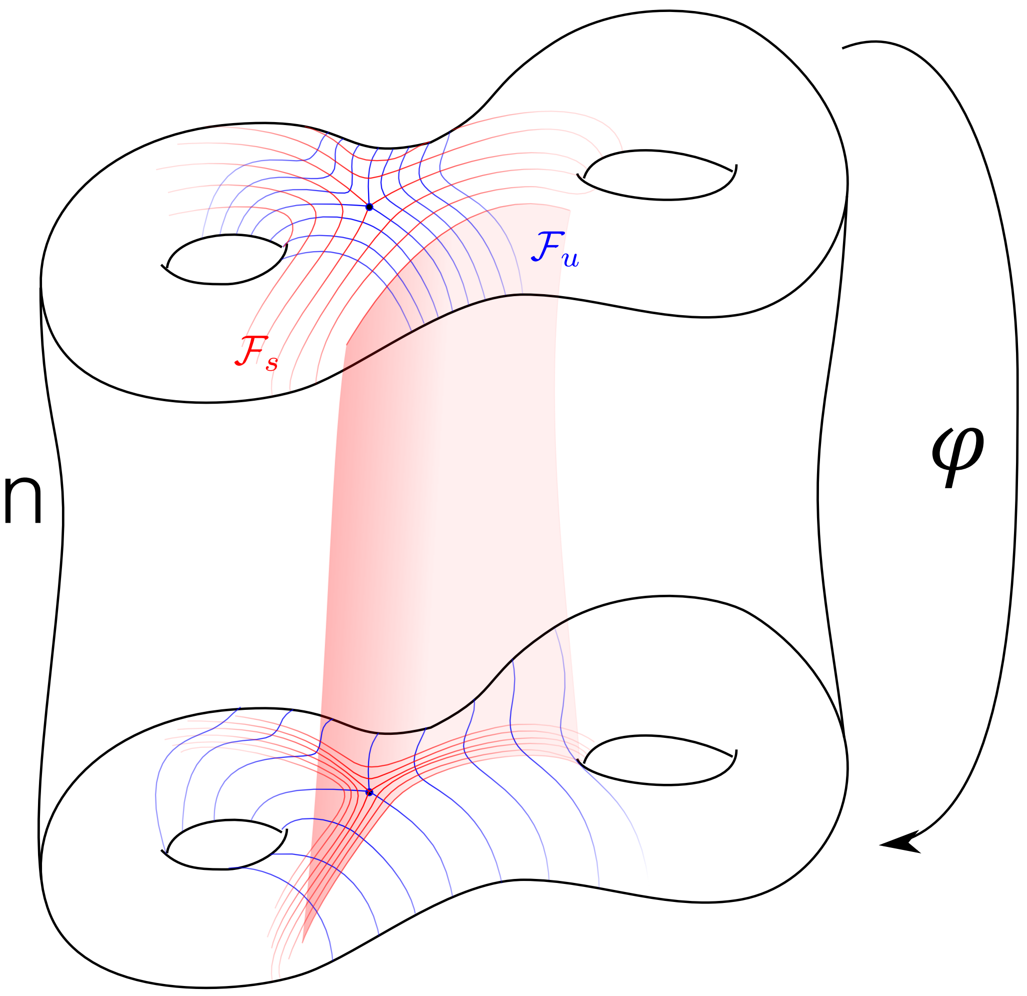

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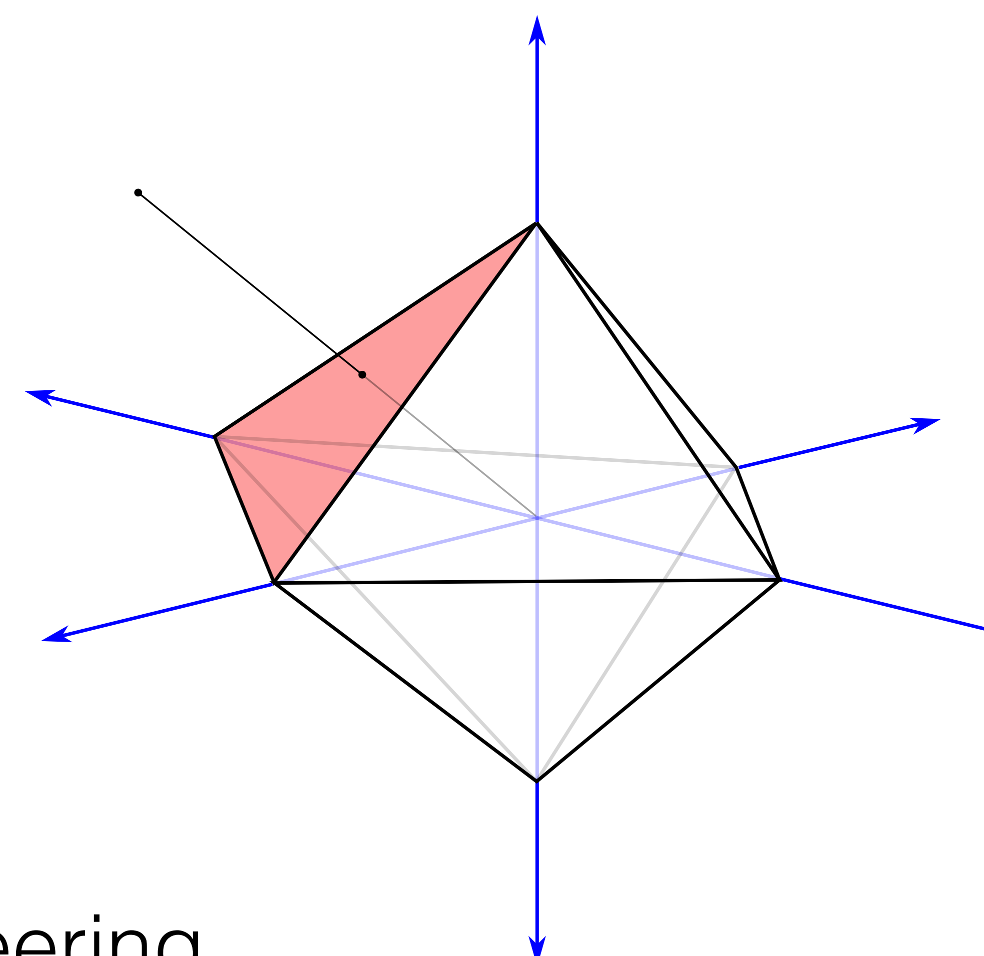

- Fact: all fibrations on a given face induce the same *2*-dimensional foliation of M $\implies$ the veering triangulation is an invariant of the fibered face.
- $\implies$ if one fibration has non-geometric veering triangulation, then so do all other fibrations of the same face.

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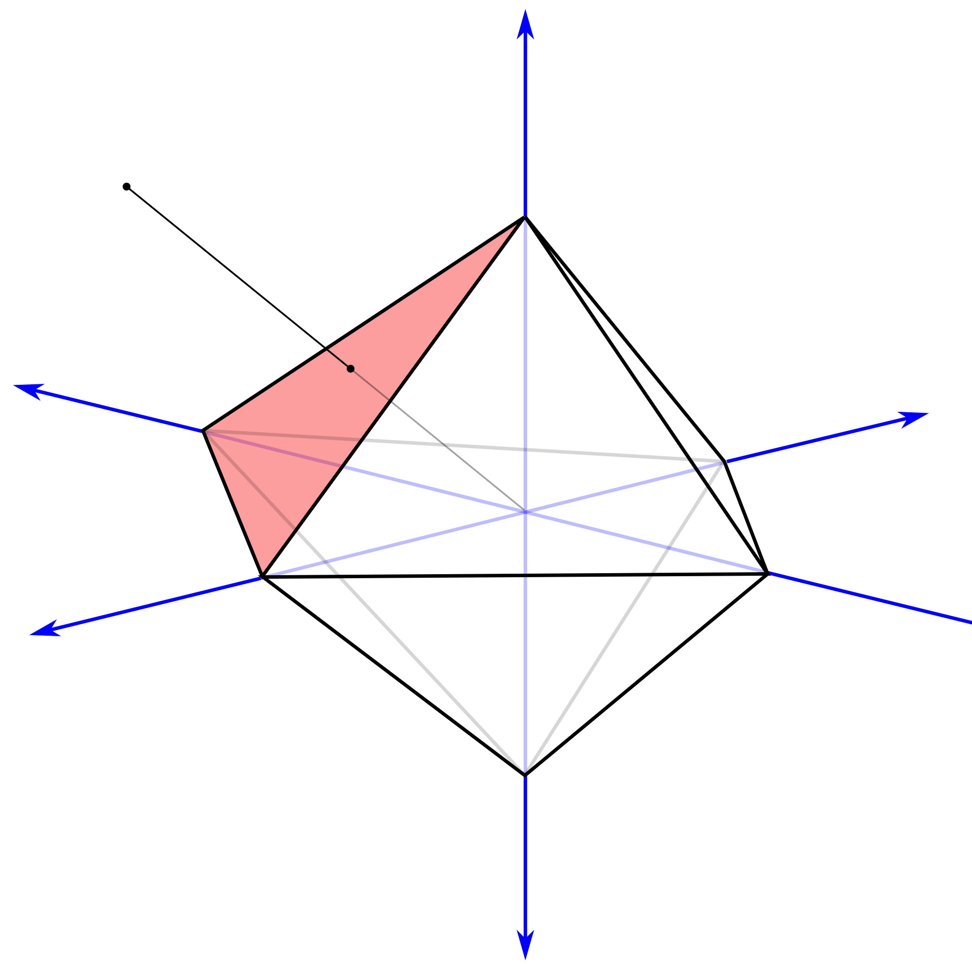

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   - we prove the lemma by finding fibered faces (with non-geometric veering triangulations) containing many fibers:
     1. a face containing fibers $\Sigma_{1,n}$ for all $n \geq 2$
     2. a face containing fibers $\Sigma_{g,n}$ for all $g \geq 2$, $n \geq 1$
     3. a face containing fibers $\Sigma_{0,n}$ for all $n \geq 7$
   - … and by using covering space tricks for closed surfaces (every closed surface covers $\Sigma_{2,0}$).
   - In the above we use `Regina` [Burton] to help compute fibered faces.

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…so we get a continuous map from quadratic differentials on Σ° to doubly degenerate representations of $\pi_1(\Sigma^\circ)$.

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2. **Theorem** [Gadre-Maher]: Fix $g \in \mathrm{Mod}(\Sigma)$ with associated quadratic differential q in the principal stratum of $\mathrm{Teich}(\Sigma)$, ω_{n} a bi-infinite simple random walk on $\mathrm{Mod}(\Sigma)$. Then there exists ρ such that, for any D, $\gamma_{\omega\mathrm{n}}$ ρ-fellow travels with a translate of γ_g for distance D, if n is sufficiently large.

3. [G-M] $\implies$ $q_n \longrightarrow q$ in $\mathrm{Teich}(\Sigma)$ $\implies$ $q_n{}^\circ \longrightarrow q^\circ$ in $\mathrm{Teich}(\Sigma^\circ)$

4. **Lemma**: $q_n{}^\circ \longrightarrow q^\circ$ in $\mathrm{Teich}(\Sigma^\circ)$ $\implies$ $\tilde{\tau}_n \rightarrow \tilde{\tau}$, based at edges of $\tilde{\tau}$ of length less than some fixed L.
    - i.e., for fixed L: if e is an edge of $\tilde{\tau}$ of length < L, then e is an edge of $\tilde{\tau}_n$ for n large.

5. **Theorem** [Leininger-Schleimer]: There exists a homeomorphism

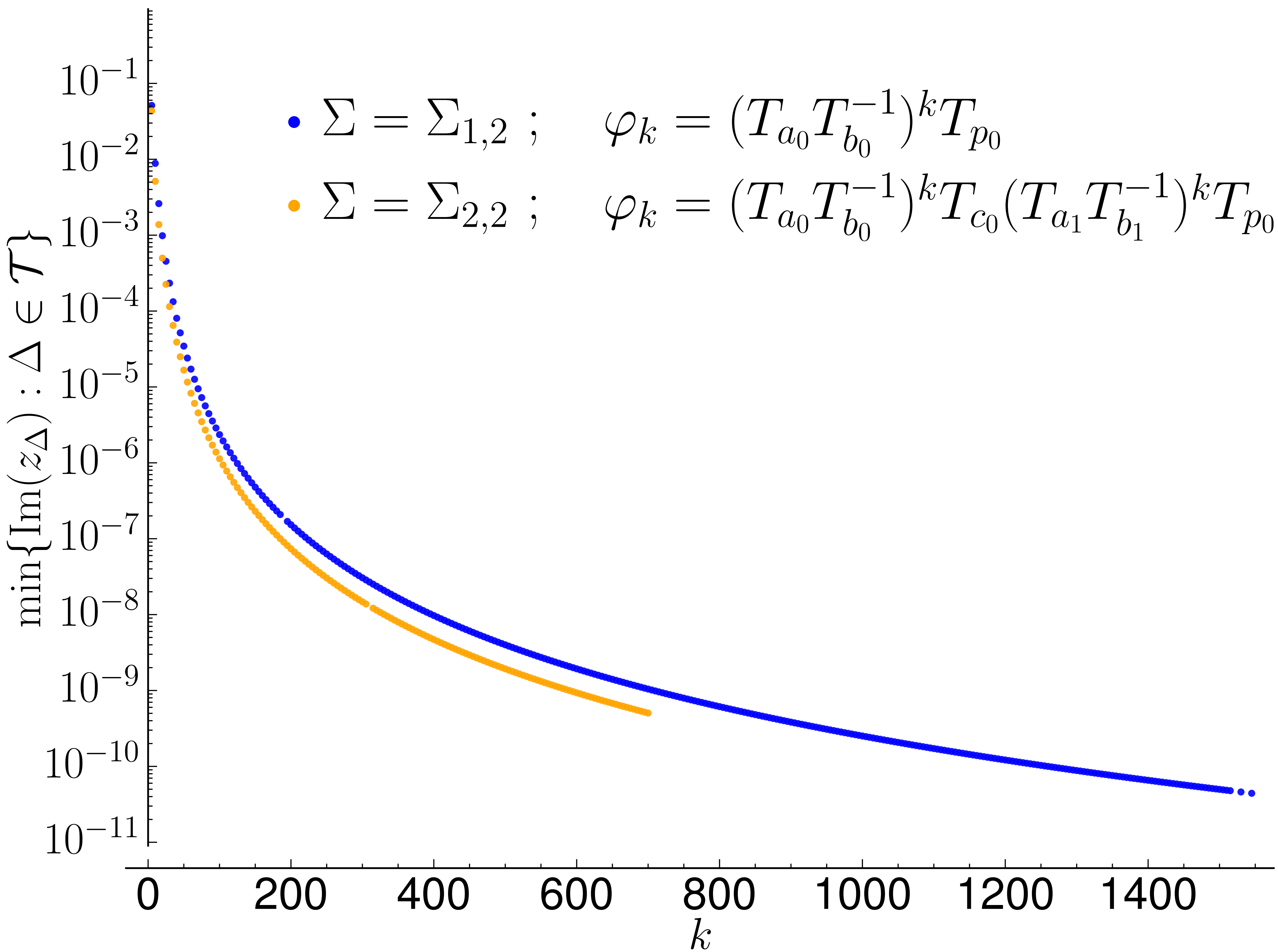
$$\mathcal{E} : DD(\Sigma^\circ) \to (\mathcal{EL}(\Sigma^\circ) \times \mathcal{EL}(\Sigma^\circ)) \setminus \Delta$$

$$QD(\Sigma^\circ)$$

…so we get a continuous map from quadratic differentials on Σ° to doubly degenerate representations of $\pi_1(\Sigma^\circ)$.

- this will give us convergence of parabolic fixed points corresponding to the vertices of our negatively oriented tetrahedron, so for n large $\tilde{\tau}_n$ will have a negative tetrahedron

Q3: we now know that generic veering triangulations are non-geometric, but can we still find infinite families of geometric veering triangulations?



Thank You!