M. Bestvina - Translation Lengths in FF_N
(joint work in progress, w/ C. Horbez & R. Wade)

<u>Free factor complex</u> $(N \geq 3)$: FF_N = poset of conjugacy classes of free factors in F_N.

FF_N is hyperbolic.

$f \in Out(F_N)$ is loxodromic $\Leftrightarrow$ iwip

$$\tau_{FF_N}(f) = \lim_{k\to\infty} \frac{d(x, f^k(x))}{k} > 0 \iff f \text{ iwip}$$

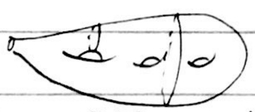
<u>Conjecture</u>: $\exists\, \varepsilon_N > 0$ s.t. if f is iwip, then $\tau_{FF_N}(f) \geq \varepsilon_N$
Further, $\varepsilon_N \asymp 1/N^2$ (in 1-skeleton)

<u>Evidence</u>: 1. Cannot do better.

$$a_1 \to a_2 \to a_3 \to \cdots \to a_N \to a_1 a_2$$

giving an explicit lower bound on this seems hard

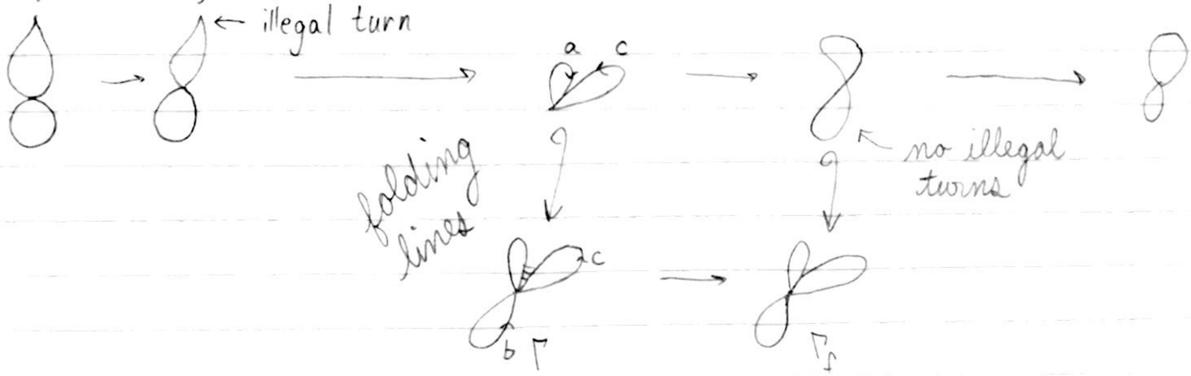
2. $Mod(\Sigma) \curvearrowright \mathcal{C}(\Sigma)$, corresponding $\varepsilon_g \asymp 1/g^2$



(BHW) $\mathcal{C} \to FF_N$ natural map, uniformly q.i.-embedding

<u>Thm</u> (BHW) $f \in Out(F_N)$ iwip, $f: \Gamma \to \Gamma$ train track map for f. Assume there exists a greedy folding line through Γ which is f-periodic. Then $\tau_{FF_N}(f) \geq \varepsilon_N > 0$.
(something like $\varepsilon_N \sim 1/N^5$?)

$\langle ba\bar{c}, a\rangle \to \langle c\bar{a}, b\rangle \longrightarrow \langle a, c\rangle \longrightarrow \langle b, ab\rangle = \langle a, b\rangle \longrightarrow \langle b, c\rangle$



Outer Space $\quad a \to b \to c \to ab$

$CV_N = \{(\Gamma, \varphi) \mid \Gamma \text{ metric graph, volume } 1,\ \varphi : R_N \to \Gamma\}/\sim$

If $\Gamma \in CV_N$, B free factor, $B|\Gamma$ = core of the B-cover of Γ

<u>Coarse projection:</u> $\pi : CV_N \longrightarrow FF_N$ ← 1-Lipschitz

$\Gamma \mapsto \langle \text{embedded loop} \rangle$

<u>Facts:</u> 1. If $vol(B|\Gamma) \leq k$, then $d_{FF_N}(\pi(\Gamma), B) \leq 10k + 10$

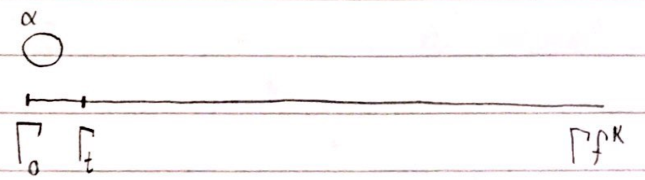
2. Suppose $\{\Gamma_t\}$ is a folding path, $d_{FF_N}(\Gamma_0, \Gamma_T) \leq 10$. Then the interval $\{\Gamma_t\}_{t\in[0,1]}$ can be subdivided into a bounded number of subintervals, and on each subinterval there is a free factor B whose volume stays bounded by V_N

<u>Clay-Mangahas-Pettet</u> If f is <u>not</u> an iwip, $f : \Gamma \to \Gamma$ train track map, then $\exists$ free factor B which is f-periodic and $vol(B|\Gamma) \leq V_N$.

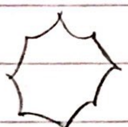
<u>To summarize:</u> We need to show that there is a free factor B that stays of bounded volume along many (N^5?) fund. domains

First, think about the case $rk\, B = 1$. ← this is the hard case.

What does this rank 1 free factor look like?



If α is legal, then $f^{N^2}(\alpha)$ crosses every edge in Γ, so its length is ≥ 1.
then grows exponentially...

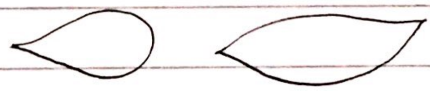
So α is a k-gon  $L = L(\alpha) \leq V_N$ (length; volume)

pretend there is only 1 illegal turn in Γ

$L_t = \dfrac{L - 2kt}{1-t}$, $\dfrac{d}{dt}\Big|_{t=0} L_t = L - 2k$

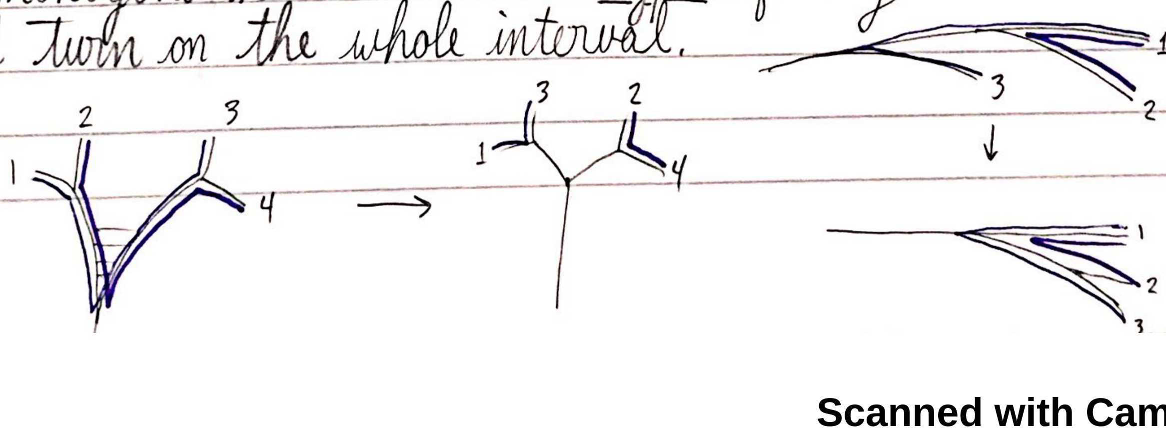
If $k \gg V_N$, then folding α loses a lot of illegal turns in a short amount of time

This implies: Step 1: We can assume that α is a k-gon, with k-uniformly bounded and (after subdividing the interval into a bounded # of segments) constant on the whole path.

Step 2: Think about $k = 1, 2$ 

Strategy: Divide all 1 and 2-gons into a bounded number of "types" and argue there is at most one of each type ($\rightsquigarrow \exists f$-periodic free factor $*$)

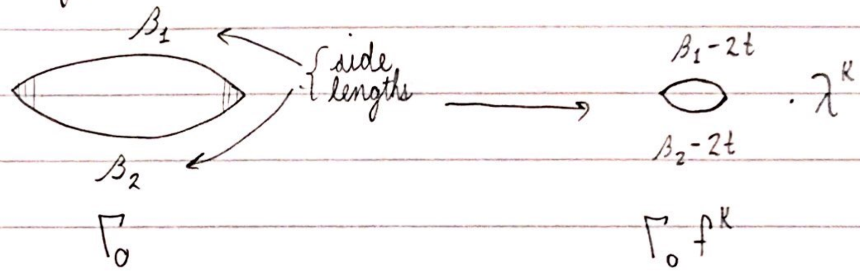
Two monogons have the same type if they cross the same illegal turn on the whole interval.



Under our assumptions, if two monogons cross the same illegal turns at the end of the path, they have the same type.

<u>Step 3</u> There is at most one monogon/bigon of a fixed type.

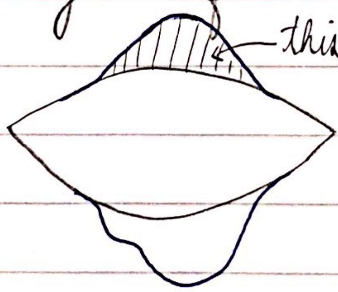
<u>Calculation</u>:



$\beta_i - 2t \leq 3/\lambda^K$, so $\beta_1 - \beta_2 \leq 3/\lambda^K$

Note that the shortest egde in Γ is $\sim 1/\lambda^{N^2} >> 3/\lambda^K$

Superimpose 2 bigons of the same type



this bigon is too short!
(a contradiction)