Tree Substitutions for Parageometric Iwips
Warwick, Thursday April 19th 2018
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I Self similarity

$\sigma : \Gamma \to \Gamma$ irreducible train-track representative of an iwip automorph

Attracting lamination [Bestvina-Feighn-Handel]

$L_\sigma = \{$bi-infinite paths in Γ with subpaths in $\sigma^n(e)\}$

$[e]$ cylinder of L_σ : paths w reads e at postition 0

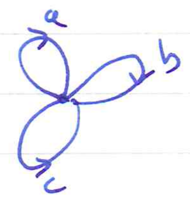
$$\boxed{\text{Prop: } [e] = \biguplus_{\substack{e' \in E(\Gamma) \\ \sigma(e') = p.e.s}} \underbrace{p^{-1}}_{\text{shifting } |p| \text{ times}} \sigma([e'])}$$

known as MOSSE's recognizability theorem.

proof: $\sigma(L_\sigma) = L_\sigma$ and all leaves are legal : no cancellation $\square$

ex. σ: $a \to ab$, $b \to ac$, $c \to a$

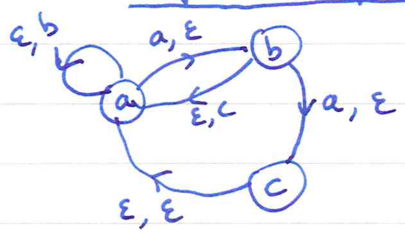
Γ:



$[a] = \sigma([a]) \uplus \sigma([b]) \uplus \sigma([c])$
$[b] = a^{-1}\sigma([a])$
$[c] = a^{-1}\sigma([b])$

We can iterate this desubstitution

Prefix-suffix automaton



vertices : edges of Γ

edge : $e' \xrightarrow{p, s} e$ if $\sigma(e') = p.e.s$

$L_\sigma \xrightarrow{\Gamma}$ infinite backward paths in the automaton

$w \longmapsto a_0 \xleftarrow{p_0, s_0} a_1 \xleftarrow{p_1, s_1} a_2 \leftarrow \cdots$

Remark that $w = \cdots \sigma^2(p_2)\,\sigma(p_1)\,p_0 . a_0\, s_0\, \sigma(s_1)\,\sigma^2(s_2) \cdots$

Γ is one-to-one except if $p_i = \varepsilon\ \forall i \gg 1$ or $s_i = \varepsilon\ \forall i \gg 1$

Self similar decomposition:

$$\begin{cases} L_\sigma = \biguplus_{|\gamma|=n} [\gamma] \\ [\gamma] = p(\gamma)^{-1}\,\sigma^n([a_n]) \end{cases}$$

$[\gamma]$ cylinders of $w \in L_\sigma$ with desubstution $\Gamma(w)$ ending with γ.

II Fractals

1) RAUZY fractal

σ substitution M_σ abelianization matrix

σ irreducible PISOT: PERRON-FROBENIUS expansion factor $\lambda_\sigma >$ all other eigenvalues $|\mu| < 1$.

$\mathbb{R}^A = \mathbb{R}u \oplus E_c$ $\quad \pi_c : \mathbb{R}^A \longrightarrow E_c$ contracting space

$w \in L_\sigma$ define $\varphi(w) = \varphi(\gamma) = \sum_{n=0}^{+\infty} M_\sigma^n \cdot \pi_c(p_n)$

RAUZY fractal: $R_\sigma = \varphi(L_\sigma)$ converging series.

$$\begin{cases} R_\gamma = \varphi([\gamma]) = \pi_c(p(\gamma)) + M_\sigma^n R_{a_n} \\ R_\sigma = \bigcup_{|\gamma|=n} R_\gamma \end{cases}$$

See [ARNOUX-BERTHÉ-HILIO SIEGEL] for RAUZY fractals and IWIPs and train-tracks.

2) Repelling tree

$T_{\sigma^{-1}} \in \partial CV_N \quad F_N \curvearrowright T_{\sigma^{-1}}$

σ iwip automorphism $\quad H$ contracting homothety $\frac{1}{\lambda_{\sigma^{-1}}}$

$\forall u \in F_N \ \forall P \in T_{\sigma^{-1}} \quad H(uP) = \sigma(u)\,H(P)$

$w \in L_\sigma$ define $Q(w) = Q(\gamma) = \lim_{n \to +\infty} p_0^{-1} H(p_1^{-1} H(\cdots\ p_n^{-1}P)\cdots)$

CAUCHY-sequence converges in $T_{\sigma^{-1}}$.

$\Omega_\sigma = Q(L_\sigma)$ limit set.

$$\begin{cases} \Omega_\gamma = Q([\gamma]) = p(\gamma)^{-1} H^n(\Omega_{a_n}) \\ \Omega_\sigma = \bigcup_{|\gamma|=n} \Omega_\gamma \end{cases}$$

<u>Fractal limit set</u>

<u>One more hypothesis</u> : σ parageometric :

Ω_σ and Ω_γ are connected (= compact subtrees)

$\Updownarrow$ Botany [C-H]

T_σ is geometric : dual to a foliation on a finite CW complex.

3) <u>Lack of injectivity</u>

$Q : L_\sigma \longrightarrow \Omega_\sigma$ is one-to-one except at singular leaves

$L_\sigma \twoheadrightarrow L_\sigma/\sim \xrightarrow{\cong} \Omega_\sigma$

[CH-Reynolds, Kapovich-Lustig, CHL]

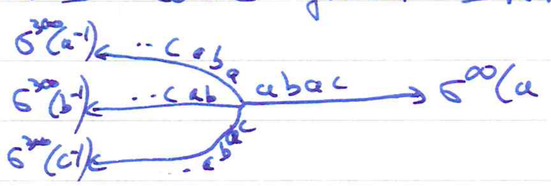
$\sim$ [bi-infinite words that share a half

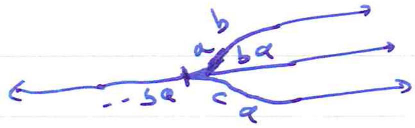
<u>Global picture</u>

$$\begin{tikzcd} L_\sigma \arrow[rr, "\Gamma"] \arrow[dr, "Q"] \arrow[ddr, "\varphi"'] & & \mathcal{P} \arrow[dl, "Q"'] \arrow[ddl, bend left, "\varphi"] \\ & \Omega_\sigma \subseteq \overline{T_{\sigma^{-1}}} \arrow[d, "\psi"] & \\ & R_\sigma \subseteq E_c & \end{tikzcd}$$

<u>Moreover</u>:
- There are finitely many singular leaves [QUEFFELEC
- The index is bounded above by $2N-2$ [GJLL]
- parageometric $\Longleftrightarrow$ index $= 2N-2$ [GL]
- Singular leaves come from INPs.

<u>ex</u> σ : $a \longrightarrow ab$, $b \longrightarrow ac$, $c \longrightarrow a$





σ' : $a \longrightarrow ba$, $b \longrightarrow ca$, $c \longrightarrow a$.

<u>Consequences</u> : a) $\forall \gamma \neq \gamma'$ $|\gamma| = |\gamma'| = n$

Ω_γ and $\Omega_{\gamma'}$ have at most one point in common

σ parageometric : a <u>gluing-point</u> or <u>singular point</u>

b) There exists a map $\Omega_\sigma \xrightarrow{\psi} R_\sigma$ — PEANO-tree inside the RAUZY fractal.

σ PISOT.

III Tree substitutions [BRESSAUD-JULLIAN]

Prop σ substitution and parageometric iwip.
$\forall a \in \mathcal{A}$ there exists finite set $\mathrm{Sing}(\Omega_a)$ of gluing points:
- $\{P'\} = \Omega_\gamma \cap \Omega_{\gamma'} \Rightarrow P' = p(\gamma)^{-1} H^n(P)$ with $P \in \mathrm{Sing}(\Omega_{a_n})$
- Eventually periodic desubstitution path, for $P \in \mathrm{Sing}(\Omega_a)$

We provide an algorithm to compute them.

Tree substitution goes back to [BRESSAUD-JULLIAN]

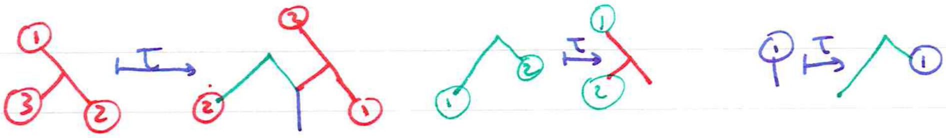
Prototiles W_a span of $\mathrm{Sing}(\Omega_a)$ (or any tree with these points.)

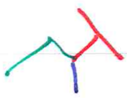
Substitution rule $W_a \xmapsto{\tau} \bigcup_{a \xleftarrow{p,s} b} W_b$ with gluings

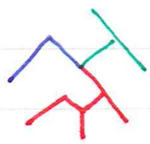
Initial pattern $W = \bigcup_{a \in \mathcal{A}} W_a$ with gluings

Map for gluing points: $P \in \mathrm{Sing}(\Omega_a)$ $\tau(P) \in \mathrm{Sing}(\Omega_b)$ with $a \xleftarrow{p,s} b$

ex $a \to ab$
$b \to ac$
$c \to a$



W: 

$\tau(W)$: 

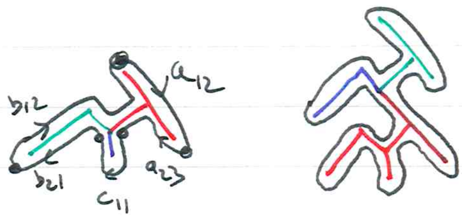
exercise: $\tau^2(W)$?

Thm 1 [C-MINERVINO] σ substitution and parageometric iwip
There exists a tree substitution $(\tau, (W_a)_{a \in \mathcal{A}}, W)$
such that $\frac{1}{\lambda_{\sigma^{-1}}^n} \tau^n(W) \xrightarrow[n \to +\infty]{} \Omega_\sigma$.

We provide an algorithm to design it.
In all PISOT examples we realize it injectively inside the contracting space, yielding PEANO trees

2) Contour substitution

Contours of W and $\tau(W)$
are circles
τ induces a map $S^1 \longrightarrow S^1$
Subdiving the contour at singular
points yields a contour substitution



$$\chi : \begin{aligned} a_{31} &\longrightarrow a_{12} \\ a_{12} &\longrightarrow a_{23}\, c_{11}\, b_{21} \\ a_{23} &\longrightarrow b_{12}\, a_{31} \\ c_{11} &\longrightarrow b_{21}\, b_{12} \\ b_{21} &\longrightarrow a_{31} \\ b_{12} &\longrightarrow a_{12}\, a_{23} \end{aligned}$$

This is a self-similar
interval exchange transformation
known as ARNOUX-YOCCOZ.

There exists a dual contour substitution
forget map $f : \tilde{A} \longrightarrow A$
$\tilde{A} = \{a_{12}, a_{23}, a_{31}, b_{12}, b_{21}, c_{11}\}$

$$\chi^* : \begin{aligned} a_{31} &\longrightarrow a_{23}\, b_{21} \\ a_{12} &\longrightarrow a_{31}\, b_{12} \\ a_{23} &\longrightarrow a_{12}\, b_{12} \\ b_{21} &\longrightarrow a_{12}\, c_{11} \\ b_{12} &\longrightarrow a_{23}\, c_{11} \\ c_{11} &\longrightarrow a_{12} \end{aligned}$$

> Thm 2 [C-MINERVINO] σ substitution and
> parageometric iwip.
> The contour substitution provides $\tilde{A} \xrightarrow{f} A$
> and $\chi^* : \tilde{A} \longrightarrow \tilde{A}^*$ $\quad f\chi^* = \sigma f$
> χ^* is a pseudo Anosov iwip automorphism,
> We provide an algorithm to compute χ^*.

IV PERSPECTIVES

- Remove parageometric hypothesis
- Get more pictures
- Better understand the relationship σ and χ^*