Relative outer automorphism groups of RAAGs and restriction maps

Matthew Day

University of Arkansas

matthewd@uark.edu

On joint work with Ric Wade, University of Oxford

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What are we interested in?

- Let Γ be a finite graph and let A_Γ be the *right-angled Artin group (RAAG)* determined by Γ. So A_Γ has the following presentation:
  - The presentation has a generator for each vertex of Γ
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- These include $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{Out}(F_n)$ and many other examples.
- We are interested in the finiteness properties and structure of $\mathrm{Out}(A_\Gamma)$.

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- G is of type F $\implies \mathrm{cd}(G) < \infty$ and G is finitely presentable.

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- $\mathrm{vcd}(G)$ is well defined by a theorem of Serre.

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- Culler and Vogtmann build a space with an action of $\mathrm{Out}(F_n)$, *outer space*, in order to show these things.

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Theorem (Day 2008)

For any Γ, $\mathrm{Out}(A_\Gamma)$ *is finitely presented.*

Theorem (Charney–Vogtmann 2009)

For any Γ, $\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) < \infty$.

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There is an outer space for the untwisted subgroup $U(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$, and $U(A_\Gamma)$ is of type VF.

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- $U(A_\Gamma) = \mathrm{Out}(A_\Gamma)$ iff there are no pairs $u, v \in \Gamma$ with u adjacent to v and $\mathrm{lk}(v) \subset \mathrm{st}(u)$.

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Theorem (Duncan–Remeslennikov 2017)

The subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by transvections and inversions has the structure of an iterated semidirect product. The factors in this product are copies of $\mathsf{GL}_n(\mathbb{Z})$, free abelian groups, and a third kind of group that is hard to describe. These factors are all finitely presentable.

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Theorem (D–W 2017)

For every graph Γ*, the group* $\mathrm{Out}(A_\Gamma)$ *is of type VF.*

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- We didn't prove this by finding an outer space for $\mathrm{Out}(A_\Gamma)$.
- Instead, we proved this using restriction maps (more on this later).

Preliminaries for Main Theorem 2

- Suppose G is a group with a free product decomposition

$$G = G_1 * G_2 * \ldots * G_r * F_m,$$

(not necessarily a Grushko decomposition; any G_i may be freely decomposible, or infinite cyclic). Let $\mathcal{G} = \{G_1, \ldots, G_r\}$.

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- The *Fouxe-Rabinovitch group* $\mathsf{FR}(G; \mathcal{G})$ is the subgroup of $\mathrm{Out}(G)$ with $[\phi] \in \mathsf{FR}(G; \mathcal{G})$ if for each G_i, there is $\phi_i \in [\phi]$ with $\phi_i|_{G_i} = \mathrm{id}_{G_i}$.

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- Now let A_Γ be a RAAG. A *special subgroup* H of A_Γ is $H = \langle \Delta \rangle = A_\Delta$, for some subgraph Δ of Γ.
- $\mathrm{Out}^0(A_\Gamma)$ is the subgroup of $\mathrm{Out}(A_\Gamma)$ "without graph symmetries". $\mathrm{Out}^0(A_\Gamma)$ is normal, $[\mathrm{Out}(A_\Gamma) : \mathrm{Out}^0(A_\Gamma)] < \infty$, and $\mathrm{Out}(A_\Gamma)/\mathrm{Out}^0(A_\Gamma)$ is a quotient of $\mathrm{Aut}(\Gamma)$.

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Let Γ *be a graph. Then* $\mathrm{Out}^0(A_\Gamma)$ *has a subnormal series*

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Notes: Often $\mathrm{GL}_1(\mathbb{Z})$ shows up. If A_Δ is edgeless and $\mathcal{H} = \varnothing$, then $\mathrm{FR}(A_\Delta; \mathcal{H})$ is $\mathrm{Out}(F_m)$.

Charney–Crisp–Vogtmann restriction and projection maps

- The following theorem is the motivation for our technique.

Theorem (Charney–Crisp–Vogtmann 2007)

Suppose Γ *is connected and not a cone on another graph. Then there are proper subgraphs* $\Delta_1, \dots, \Delta_k$*, such that for each* i*, restriction to* A_{Δ_i} *induces a homomorphism*

$$R_i \colon \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}(A_{\Delta_i}),$$

and the product $R = \prod_i R_i$ *has a free abelian kernel:*

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- *Projection maps* are another part of this story.
- Hurdle: the image of R is difficult to describe.

Relative outer automorphism groups

Definition

Let G be a group, and H a subgroup of G.

- $[\phi] \in \mathrm{Out}(G)$ *preserves* H if there is $\phi \in [\phi]$ with $\phi(H) = H$.
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Definition

Let $\mathcal{G}$ and $\mathcal{H}$ be collections of subgroups of G. The *relative outer automorphism group of G with respect to* $\mathcal{G}, \mathcal{H}$, denoted $\mathrm{Out}(G; \mathcal{G}, \mathcal{H}^t)$, is the subgroup of $\mathrm{Out}(G)$ consisting of maps that preserve every group in $\mathcal{G}$ and act trivially on every group in $\mathcal{H}$.

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- $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H})$ is a *relative outer automorphism group of a RAAG (ROAR)* if A_Γ is a RAAG and $\mathcal{G}$ and $\mathcal{H}$ are collections of special subgroups.

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- OARs are ROARs, and many well-studied non-OARs are also ROARs.

Preliminaries to another theorem

- If $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ is a ROAR and $A_\Delta \in \mathcal{G}$, then there is a restriction map $R_\Delta \colon \mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t) \to \mathrm{Out}(A_\Delta)$.

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- Given a special subgroup A_Δ, define $\mathcal{G}_\Delta$ to be $\{A_{\Lambda\cap\Delta} \mid A_\Lambda \in \mathcal{G}\}$, and define $\mathcal{H}_\Delta$ similarly.

Preliminaries to another theorem

- If $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ is a ROAR and $A_\Delta \in \mathcal{G}$, then there is a restriction map $R_\Delta \colon \mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t) \to \mathrm{Out}(A_\Delta)$.
- For technical reasons, we usually consider $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$, which is $\mathrm{Out}^0(A_\Gamma) \cap \mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$.
- Given a special subgroup A_Δ, define $\mathcal{G}_\Delta$ to be $\{A_{\Lambda\cap\Delta} \mid A_\Lambda \in \mathcal{G}\}$, and define $\mathcal{H}_\Delta$ similarly.
- Relative sets $\mathcal{G}, \mathcal{H}$ are *saturated* if they are as full as they can possibly be without changing $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H})$.

Main technical theorem

Theorem (D–W)

Let $\mathrm{Out}(A_\Gamma;\mathcal{G},\mathcal{H}^t)$ *be a ROAR, and suppose* $\mathcal{G}$ *is saturated.*

1. *Suppose* $A_\Delta \in \mathcal{G}$. *Then the restriction map* R_Δ *fits in an exact sequence*

$$1 \to \mathrm{Out}^0(A_\Gamma;\mathcal{G},(\mathcal{H}\cup\{A_\Delta\})^t) \to \mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t) \xrightarrow{R_\Delta} \mathrm{Out}^0(A_\Delta;\mathcal{G}_\Delta,\mathcal{H}^t_\Delta) \to 1.$$

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2. *Suppose $A_\Lambda \leq Z(A_\Gamma)$, and suppose $\Lambda \subset \bigcup\mathcal{H}$. Let $\Delta = \Gamma \setminus \Lambda$. Then there is a projection map fitting in an exact sequence*

$$1 \to \mathbb{Z}^m \to \mathrm{Out}^0(A_\Gamma;\mathcal{G},\mathcal{H}^t) \xrightarrow{P_\Delta} \mathrm{Out}^0(A_\Delta;\mathcal{G}_\Delta,\mathcal{H}^t_\Delta) \to 1.$$

Here the $\mathbb{Z}^m$ is generated by twists with multipliers in Λ.

Remarks

- Finding the saturation of $\mathcal{G}, \mathcal{H}$ is tedious. However, we have a procedure for directly finding a smaller $\mathcal{G}', \mathcal{H}'$ such that $\mathrm{Out}^0(A_\Delta; \mathcal{G}', \mathcal{H}'^t)$ is the image of R_Δ.

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- The hard part of the technical theorem is surjectivity of the restriction map.
- This helps:

Theorem (D–W)

Let $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ be a ROAR. Then $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ is generated by the inversions, transvections, and extended partial conjugations that it contains.

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Suppose $\mathrm{Out}(A_\Gamma; \mathcal{G}, \mathcal{H})$ *is a ROAR and* $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ *has no nontrivial projections or restrictions. Then* $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ *is*

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- $\mathrm{GL}_n(\mathbb{Z})$ *where* n *is vertex count of* Γ*, or*
- $\mathrm{FR}(A_\Gamma, \mathcal{K})$ *for some free decomposition* $\mathcal{K}$*.*

An example

(Switch to the other document.)

Sketch of VF theorem, generalities

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- In particular, this implies that type F is preserved under taking group extensions.
- We use the level-3 subgroups of $\mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t)$ at each step.

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Theorem (Guirardel–Levitt 2007)

The outer space of a free product is contractible.

- RAAGs are of type F because Salvetti complexes are finite $K(A_\Gamma, 1)$-complexes.

Induction details: Invariant special subgroups

- A special subgroup A_Δ admits a restriction map iff
  - for all $v \in \Delta$ and $w \in \Gamma$, if $\mathrm{lk}(v) \subset \mathrm{st}(w)$, then $w \in \Delta$.
  - for all $w \in \Gamma$, if $\mathrm{st}(w)$ separates Δ, then $w \in \Delta$.

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- This quickly implies previously known examples, such as maximal equivalence classes and maximal stars.

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- $\mathcal{K}$-paths, $\mathcal{K}$-connectedness, and the $\mathcal{K}$-neighborhood $N_\mathcal{K}$ of a set are defined using this.
- A subgraph of Γ is *relatively connected* if it is $\mathcal{G}$-connected.

Induction details: the peripheral structure

- Let $A_\Delta \in \mathcal{G}$ and $\mathcal{K} \subset \mathcal{G}$. If Θ is a $\mathcal{K}$-connected subset of $\Gamma \setminus \Delta$, then the intersection of the $\mathcal{K}$-neighborhood of Θ with Δ generates an invariant special subgroup. In symbols:

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- Let $\mathcal{H}^*$ be the union of $\mathcal{H}$ with the one-vertex-complements:

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- Let $\mathcal{P}_\Delta$ contain all the groups $P_{\mathcal{K},\Theta}$, where $\mathcal{K}$ is $\varnothing$ or some $\mathcal{G}^v$, and Θ is a $\mathcal{K}$-connected subgraph of $\Gamma \setminus \Delta$. Then

$$R_\Delta \colon \mathrm{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}^t) \to \mathrm{Out}^0(A_\Delta; \mathcal{G}_\Delta \cup P_\Delta, \mathcal{H}^t_\Delta)$$

  is surjective, even if $\mathcal{G}$ is not saturated.

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- The relatively connected case was a surprise; it can arise from breaking down absolute OARs.
- In each of the cases in the case analysis, we find subgraphs admitting restrictions or projections, or we are in a base case.

Thank you!