Take $f \in \mathrm{Aut}(F_n)$. How to understand it?

Consider $F_n \rtimes_f \mathbb{Z} = \langle a_1, \dots, a_n, t \mid a_i^t = f(a_i) \rangle$

free-by-cyclic group.

Often we can write $G = F_m \rtimes_g \mathbb{Z}$,

$g \in \mathrm{Aut}(F_m)$,

relating automorphisms of different free groups.

<u>Aim</u>: understand the different ways G "fibres" ($G \cong F_m \rtimes_g \mathbb{Z}$).

<u>Def</u> Let G be a group.

$G*_f = \langle G, t \mid t^{-1} g t = f(g) \rangle$ with $f: G \hookrightarrow G$

is an <u>ascending HNN extension</u>.

The natural map $G*_f \twoheadrightarrow \mathbb{Z}$ is the <u>induced character</u>.

$G \longmapsto 0$

$t \longmapsto 1$

Def (Bieri–Neumann–Strebel = BNS)

G f.g.

$\Sigma(G) \subseteq H^1(G;\mathbb{R}) \setminus \{0\}$ st.

$\forall \varphi : G \twoheadrightarrow \mathbb{Z}$; $\varphi \in \Sigma(G)$ iff

$G \cong H *_f$, H f.g., $f : H \hookrightarrow H$,

φ is the induced character

Facts • $\varphi, -\varphi \in \Sigma(G) \iff G \cong H \rtimes \mathbb{Z}$, H f.g.

• If $G \cong F_n *_f$ asc. HNN, then

$\varphi, -\varphi \in \Sigma(G) \iff G \cong F_m \rtimes \mathbb{Z}$.

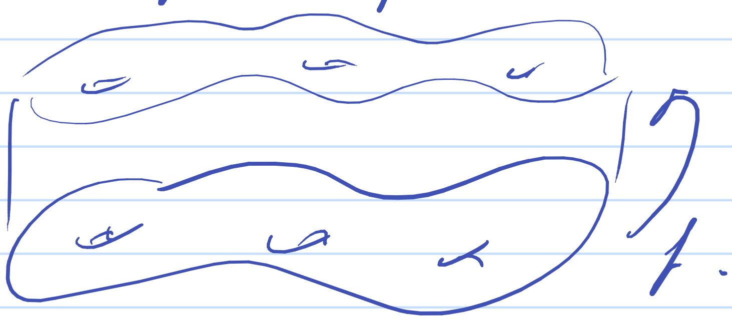
What is the structure of $\Sigma(F_n *_f)$?

It is understood for $F_n \rtimes_f \mathbb{Z}$ with f geometric (i.e. in $MCG(\Sigma_{g,1}) \leq Out(F_g)$).

Thanks to Thurston.

Def A 3-mfld M (cpct oriented) **fibers** (over the circle)

iff M is homeomorphic to a mapping torus of a homeomorphism of a cpct oriented surface.



$f \in MCG(\Sigma_g)$.

On the group level:

$$\pi_1(\Sigma_g) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z}$$

$$\therefore \quad \pi_1(M) \cong \pi_1(\Sigma_g) \rtimes_f \mathbb{Z} \xrightarrow{\text{induced character}} \mathbb{Z}.$$

Facts [Stallings]

M fibers inducing $\varphi : \pi_1(M) \twoheadrightarrow \mathbb{Z}$ iff $\pi_1(M)$ can be written as $\pi_1(\Sigma_g) \rtimes \mathbb{Z}$ inducing φ.

• (BNS) $\Sigma(\pi_1(M)) = -\Sigma(\pi_1(M))$

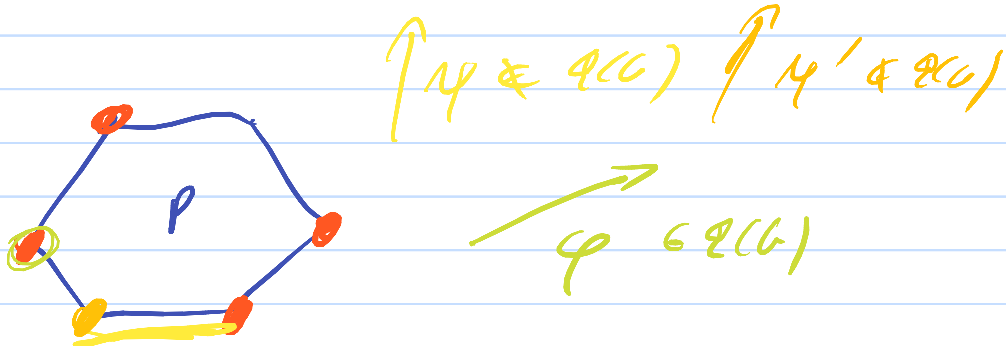
i.e. $\forall\, \varphi \in \Sigma(\pi_1(M))$, $\varphi : \pi_1(M) \twoheadrightarrow \mathbb{Z}$ we have $\pi_1(M) \cong \pi_1(\Sigma_g) \rtimes \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$.

Theorem [Thurston, K]

Let M be a cpt oriented connected manifold, $G = \pi_1(M)$. or $G = F_n *_f$ asc HNN.

$\exists$ cpt convex polytope $P \subseteq H_1(G;\mathbb{R})$ with vertices in $H_1(G;\mathbb{Z})$ and some vertices marked s.t.

$\forall \varphi : G \twoheadrightarrow \mathbb{Z}$: $\varphi \in \Sigma(G)$ iff φ attains its min on P exactly at a marked vertex.



Let's prove it! (focus on $G = F_n *_f$ case).

Thm [Sikorav]

G f.g. $\varphi \in H^1(G;\mathbb{R}) \setminus \{0\}$.

$$\varphi \in \Sigma(G) \iff H_1(G; \widehat{\mathbb{Z}G}^{\varphi}) = 0$$

Novikov ring

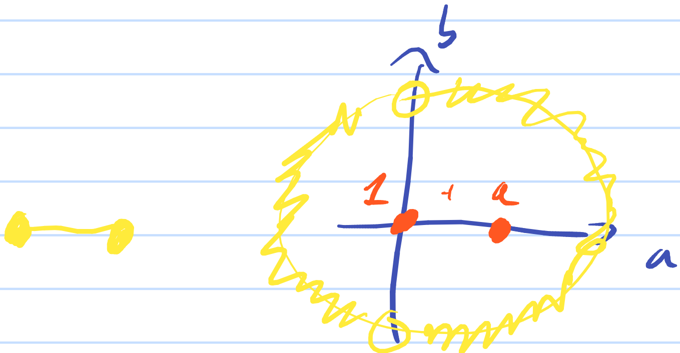
<u>Def</u> (Novikov ring). $\varphi \in H^1(G;\mathbb{R})$.

$$\widehat{\mathbb{Z}G}^{\varphi} = \left\{ x : G \to \mathbb{Z} \;\middle|\; \operatorname{supp} x \cap \varphi^{-1}(-\infty, c] \text{ is finite } \forall c \in \mathbb{R} \right\}$$

What is important: what is invertible in $\widehat{\mathbb{Z}G}^{\varphi}$.

<u>Ex</u> $G = \mathbb{Z}^2 = \langle a, b \rangle$

$1 - a \in \mathbb{Z}G$



$\varphi \in H^1(G;\mathbb{R})$. If $\varphi(a) > 0$ then

$$\sum_{i=0}^{\infty} a^i \in \widehat{\mathbb{Z}G}^{\varphi}, \quad (1-a)\left(\sum a^i\right) = 1$$

$1 - a$ invertible!

If $\varphi(a) < 0$. Then $(-a^{-1}) \sum_{i=0}^{\infty} a^{-i} \in \widehat{\mathbb{Z}G}^{\varphi}$

$$\underbrace{(1-a)\cdot(-a^{-1})}_{= 1 - a^{-1}} \cdot \sum a^{-i} = 1$$

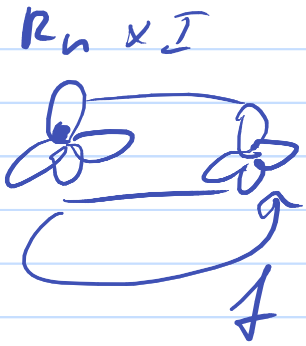
$1 - a$ invertible! (in the other direction).

$\varphi(a) = 0 \Rightarrow 1 - a$ is <u>not</u> invertible!

Back to the proof: need to compute

$$H_1(G; \widehat{\mathbb{Z}G}^q), \quad G = F_n *_f .$$

Take

$X =$ 

$X : K(G,1)$.

$H_1(G; R)$ is computed by C,

the cellular chain complex of $\hat{X}$,

tensored with R.

$$H_1(G; \widehat{\mathbb{Z}G}^q) = H_1(C \otimes \widehat{\mathbb{Z}G}^q).$$

C:

$$\begin{array}{ccccc} C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ \mathbb{Z}G^n & \xrightarrow{\partial_2} & \mathbb{Z}G^{n+1} & \xrightarrow{\partial_1} & \mathbb{Z}G \end{array}$$

For every cell of X pick a lift in $\hat{X}$.

Now a chain supported in a preimage of a cell

is isomorphic to $\mathbb{Z}G$.

$$\partial_1 = (1-a_1, 1-a_2, \ldots, 1-a_n, 1-t)$$

∂_2 is a $n \times n+1$ matrix over $\mathbb{Z}G$.

$\varphi : G \to \mathbb{Z} \Rightarrow \varphi$ cannot kill all generators of G.

Set $t = a_{n+1}$.

$\exists i$: $\varphi(a_i) \neq 0$.

$\therefore$ $1 - a_i$ is invertible on $\widehat{\mathbb{Z}G}^{\varphi}$!

So for $C \otimes \widehat{\mathbb{Z}G}^{\varphi}$:

compute H_1

$$\begin{tikzcd} & \widehat{\mathbb{Z}G}^{\varphi\, n} \arrow[ddr, "(1-a_i, \dots)"] & \\ \widehat{\mathbb{Z}G}^{\varphi\, n} \arrow[ur, "A"] \arrow[dr, "\partial_2"'] & \oplus \arrow[r, phantom, "\partial_1"] & {} \\ & \mathbb{Z}G^{\varphi} \arrow[r, "1-a_i", "\cong"'] & \widehat{\mathbb{Z}G}^{\varphi} \end{tikzcd}$$

So we can get rid of the last part.

<u>So</u> $H_1(G; \widehat{\mathbb{Z}G}^{\varphi}) = 0 \Leftrightarrow A$ is onto (right-invertible).

Key point: the same A works for all φ with $\varphi(a_i) \neq 0$ (almost all φ!).

<u>So</u> (almost)

fixed matrix

$\varphi \in \Sigma(G) \Leftrightarrow H_1(G; \widehat{\mathbb{Z}G}^{\varphi}) = 0 \Leftrightarrow A \otimes \widehat{\mathbb{Z}G}^{\varphi}$ is right invertible.

Thm [K] Let G be a f.g. agrarian group.

($\mathbb{Z}G \subset$ skew-field.)

Let A be a square matrix over $\mathbb{Z}G$.

Then $\exists$ polytope $P(A) \subseteq H_1(G;\mathbb{R})$ with vertices in $H_1(G;\mathbb{Z})$ with some vertices marked s.t.

$\forall\, \varphi : G \to \mathbb{R}$ non zero: $A \otimes \widehat{\mathbb{Z}G}^{\varphi}$ is invertible $\iff$ φ attains its minimum on $P(A)$ precisely at a marked vertex.

Fact $F_n \rtimes_f$, $\pi_1(\Sigma_g) \rtimes \mathbb{Z}$ are agrarian (Atiyah conjecture).

This finishes the proof for $F_n \rtimes f$.

For 3-mflds:

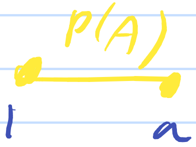
$$C : \quad C_3 \xrightarrow{\partial_3 = \partial_1^T} C_2 \longrightarrow C_1 \xrightarrow{\partial_1} C_0$$

$$\begin{tikzcd} \widehat{\mathbb{Z}G}^{\varphi} \arrow[dr] & \widehat{\mathbb{Z}G}^{\varphi 4} \arrow[r, "A"] \arrow[r, shift right=2] & \widehat{\mathbb{Z}G}^{\varphi 4} & \\ & \oplus\ \widehat{\mathbb{Z}G}^{4} & \oplus\ \widehat{\mathbb{Z}G}^{4} \arrow[r] & \widehat{\mathbb{Z}G}^{\varphi} \end{tikzcd} \quad \square$$

Compute H_1.

Recall $G=\mathbb{Z}^2$, $1-a\in\mathbb{Z}G$.

$A=(1-a)$ 1×1 matrix.



controls invertibility of $A\otimes\mathbb{C}^q$.

Def **Newton polygon** of $x\in\mathbb{Z}G$ is

the convex hull in $H_1(G;\mathbb{R})$ of

supp(x).

We want $P(A)$ to be "the Newton polygon

of the matrix A".

Idea: take det! $G=\mathbb{Z}^2$.

$\mathbb{Z}G$ is an abelian ring.

$A=(a_{ij})$, $a_{ij}\in\mathbb{Z}G$

$$\det A = \sum_{\sigma\in S_n} a_{i\sigma(i)} \quad \text{Standard formula.}$$

Define $P(A)=$ Newton$(\det A)$.

Idea Take $\det A\in\mathbb{Z}G$, take P to

be the Newton polytope.

How to define $\det A$?

Fact G is agrarian iff $\mathbb{Z}G \hookrightarrow$ skew-field $\mathbb{K}$.

$A \otimes \mathbb{K}$ is a square matrix over a (skew) field!

So $\det A \otimes \mathbb{K}$ exists! (Dieudonné det).

But $\det A \otimes \mathbb{K} \notin \mathbb{Z}G$.

Almost true: $\det A \otimes \mathbb{K} = p/q$, $p, q \in \mathbb{Z}G$ (rational function).

We define $P(A)$ to be "the difference" $\mathrm{Newton}(p) - \mathrm{Newton}(q)$.

Thm [K] In fact, $P(A)$ is a polytope.

[Note for $G = \mathbb{Z}^2$ this follows from $\det A \in \mathbb{Z}G$, using st. formula.

But for non-abelian rings (groups) the st. formula fails!].