Yulan Qing.

Quasi-geodesics in $Out(F_n)$ Joint work with K. Rafi

Space: $Out(F_n) = Aut(F_n) / Inn(F_n)$

Theorem (Nielsen, McCool) $Out(F_n)$ is finitely generated and finitely presented.

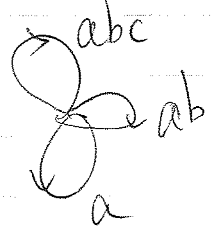
Typical generators: $\begin{cases} s_i \to s_i s_k \\ s_j \to s_j & j \neq i \end{cases}$

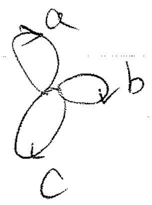
Metric: Word metric

Q: Given $\phi \in Out(F_n)$, what is the word length $\|\phi\|$?

First, consider Stallings Folding Algorithm:

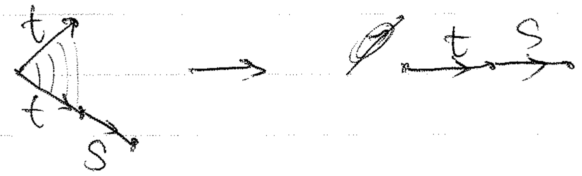
Let elts of $Out(F_n)$ be represented by labelled roses: (up to change of basepoint)

R_0 



R_0

Stallings Folding: A quotient map from graph to graph that identifies two edges with the same label and incident to ~~a~~ same vertex.



Rmk: First perform subdivision

Stallings Folding Theorem: $\exists$ a finite folding path from R to R_0.

After making some choices, such Stallings Folding Paths (SFP) can be shadowed into $\mathrm{Out}(F_n)$, giving a finite path in $\mathrm{Out}(F_n)$. This provides a coarse upper bd.

Q: Do Stallings folds also give lower bds?

A: No. Consider the example

$$\langle a, b, c\underbrace{(ab^m)^n}_{w_3}\rangle \;\text{- - - - - -}\; \langle a, b, c\rangle$$

SFP: $(m+1)n$ steps

Alternatively

$$\begin{tikzcd}
\langle a, b, w_3\rangle \arrow[d, "m \text{ steps}"] & & \langle a, b, c\rangle \\
\langle ab^m, b, w_3\rangle \arrow[rr, "n"'] & & \langle ab^m, b, c\rangle \arrow[u, "m"']
\end{tikzcd}$$

$2m+n$ steps

$\Rightarrow$ Choosing large enough m, n, one can show that SFP are not quasi-geodesics.

Rmk: high power in the marking provides a lower bd.

Therefore, given the corresponding definition that captures high power,

$\mathrm{twist}_\phi(d)$ [Guirardel, Clay-Pettet]

One can show a lower bd:

Let $l_R(\alpha)$ denote the combinatorial length of α, a conjugacy class in the graph R.

Theorem 1 (Q – Rafi) Let m be the length of a path in $Out(F_n)$ connecting R to R_0, $\phi(R)=R_0$.
Let

$$L := \min_{R_i} \{ \cancel{R}\, l_{R_i}(\alpha) \}$$

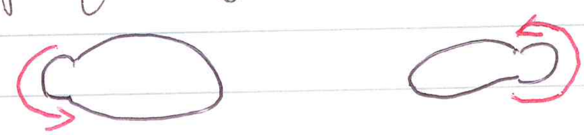
(Denote the path as $R=R_m, R_{m-1} \cdots R_2, R_1, R_0$)

Then, $$m \geq \max\left(\left(twist_\alpha(\phi) \log_{10} L - 2 \right) - 1,\ twist_\alpha(\phi) \right).$$

Rmk:
- $twist_\alpha$ is a coarse lower bd.
- Making a loop short saves time in $Out(F_n)$
- log ~~in~~ the equation is sharp.

i.e. $\exists$ example in which it takes $\log L$ steps to twist around a loop of length L.

Strategy



Example: $\langle (bc)^m a,\ db,\ \varphi^s(d)c,\ d,\ e \rangle$

$\varphi : \langle d, e\rangle \to \langle d, e\rangle$ iwip.

$\alpha : [bc]$

Explanation: omitted.

What does Theorem 1 imply about distance estimate?

Background: Mapping Class Group Distance Formula

Let S be orientable connected surface.

$$\mathrm{MAP}(S) := \mathrm{Homeo}^+(S)/\text{Isotopy}.$$

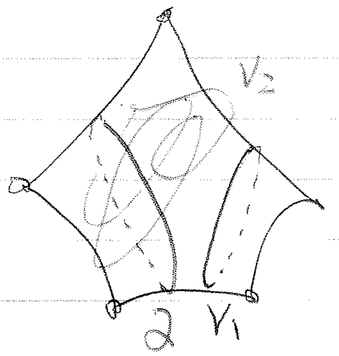
Theorem.
(Masur, Minsky) Given $\phi \in \mathrm{MAP}(S)$, $\mathcal{M}$: generating set

$$\|\phi\|_{\mathcal{M}} \overset{K,C}{\asymp} \sum_{Y \subseteq S} [d_{C(Y)}(\mu_0, \phi(\mu_0))]_K + \sum_{\alpha \text{ s.c.c. in } S} [\mathrm{twist}_\alpha(\phi)]_K$$

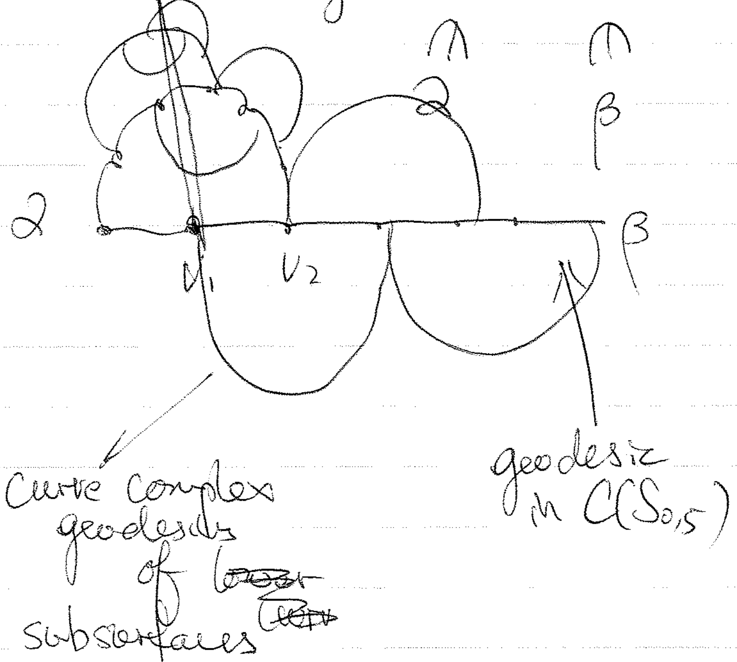
Y: nonannular subsurface.

$C(Y)$: curve graph of Y.

Example: $S_{0,5}$



two markings. P_1, P_2



1. α. v_2 lies on the same side of v_1. $S_{0,4}$

Slice: marking

Hierarchy Paths

- Quasi-geodesics
- do not backtrack in any subsurfaces (in the curve complexes of subsurfaces).
- Bounded geodesic Image Theorem: at any time the MAP(S) elt on the Hierarchy path is only acting on boundedly many subsurfaces.

Rmk: • Theorem 1 is a weak analogue of Bounded geodesic Image Theorem.

• Given Theorem 1, we show that

Thm 2. $\exists\ \phi \in Out(F_n)$, and free factor A such that

Any quasi-geodesic connecting id. to ϕ backtracks in $\mathcal{F}(A)$.

Rmk:

Dictionary	Out(F_n)	MAP(S)
	words	loops
	$twist_\phi$(CP)	twist (MM)
	Free Factor Complex	Curve Complex

Proof of Theorem 2:

$\langle a, b, \; c(\psi^s(a))^t \rangle$ - - - - - - - - - - > $\langle a, b, c \rangle$

$\alpha = [\psi^s(a)]$ $\qquad\qquad$ $\psi : \langle a, b \rangle \to \langle a, b \rangle$

$\mathrm{twist}_\alpha(\phi) = t$ $\qquad\qquad$ $a \to aba$

$A : \langle a, b \rangle$ $\qquad\qquad$ $b \to ba$ $\qquad \|\psi\| = 2.$

Strategy: 1 Use the ⊔ path to produce an upper bd for any quasi-geodesic

2. Use Theorem 1 to produce lower bd.

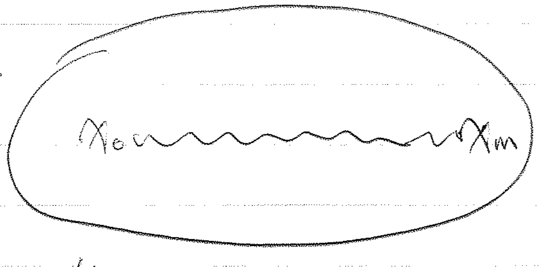
3. Choose constants to get contradiction.

Step 1: The ⊔ path: $4s + t$ steps

$\Rightarrow$ Any K, C Quasi-geodesic from R_m to R_0:

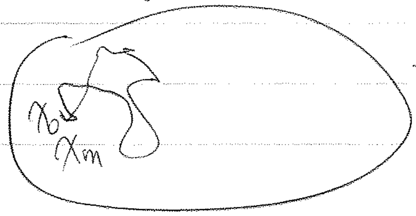
$$m \leq K(4s+t) + C. \qquad (1)$$

Step 2.



$\mathrm{Out}(F_n)$ Let $x_0, x_1 \ldots x_m$ dene the K, C Quasi-geodesic

$\Downarrow$ project to $\mathcal{F}(A)$



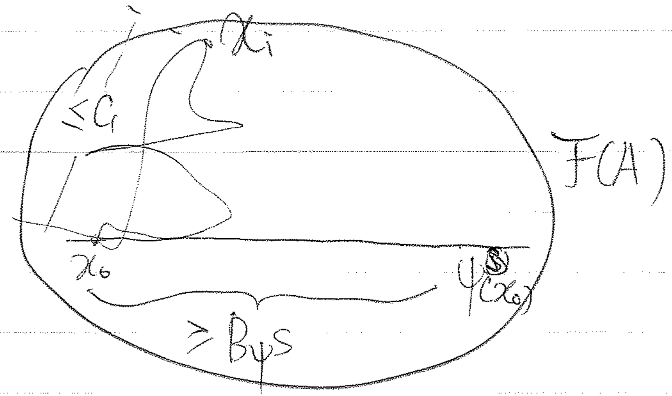
$\mathcal{F}(A)$ Suppose the projection is a K_1, C_1 quasi-geodesic in $\mathcal{F}(A)$, for any x_i on the path:

$$d_A(x_0, x_i) + d_A(x_i, x_m) < K_1 \underbrace{d_A(x_0, x_m)}_{\overset{\Downarrow}{0}} + C_1$$

$$2\, d_A(x_0, x_i) \leq C_1$$

$$d_A(x_0, x_i) \leq C_1 \qquad \forall i.$$

Meanwhile,



ψ acts loxodromically,
$\Rightarrow \exists$ constant B_ψ s.t.
$d_A(x_0, \psi^s(x_0)) \geq B_\psi s$

By reverse Δ-inequality, $d_A(x_i, \psi^{(s)}(x_0)) \geq B_\psi s - C_1$

$\Rightarrow$ By Bestvina–Feighn $\quad l_{x_i}(\phi) \geq \dfrac{B_\psi s - C_1 - 13}{6} \quad \forall i$.

Let $\dfrac{B_\psi s - C_1 - 13}{6} := L$.

By Theorem 1, $\quad m \geq \operatorname{twist}_2(\phi) \log_{10} L - 2 - 1$

$$\geq \operatorname{twist}_2(\phi) \log_{10} \frac{B_\psi s - C_1 - 13}{6} - 2 - 1.$$

Step 3.

$$\mathrm{twist}_\alpha(\phi) \le \frac{m}{\log_{10} L - 2} \le \frac{K(4s+t)+C}{\log_{10} L - 2}$$

① Make s large enough s.t. $\log_{10} L - 2 \ge 2K$.

② Make $\underset{\mathrm{twist}_\alpha(\phi)}{\underset{\uparrow}{t}}$ large enough s.t. $2s + \frac{C}{2K} \le \frac{t}{4}$

~~The contradiction~~

$$t \le \frac{K(4s+t)+C}{\log_{10} L - 2} \le 2s + \frac{t}{2} + \frac{C}{2K}$$

$$\le \frac{3}{4}t \qquad \lightning$$

$\blacksquare$

Lastly, we state the corollary on greedy folding path in CV_n.

Corollary 3: (Q-R) $\exists$ ~~ϕ~~ x, y in CV_n s.t. $[x,y]_{gf}$ does not give a path as its shadow that is a Quasi-geodesic in $\mathrm{Out}(F_n)$.

Proof:

$$\begin{tikzcd} CV_n \arrow[r, "\text{Bestvina Feighn}", "f"'] \arrow[d, shift left, "S"] & \mathcal{FF}(A) \\ \mathrm{Out}(F_n) \arrow[u, shift left, "\bar{S}"] \arrow[ur, "\pi"', "\text{Theorem 2.}"] & \end{tikzcd}$$

If $\bar{S}$ ~~takes~~ takes Qg to Qg then $f \circ \bar{S} = \pi$ takes Qg to Qg, contradicting Theorem 2. $\lightning$

$\blacksquare$