The abstract commensurator of $Out(F_3)$ (j.w. Bridson & Horbez)

I Background

Let G be a group. Let $\Omega(G)$ be the set of isomorphisms between finite index subgroups of G. $f_1, f_2 \in \Omega$ are equivalent if they agree on some ~~common~~ finite index subgroup contained in their domains

$$\mathrm{Comm}(G) = \Omega(G)/\sim$$

is the abstract commensurator of G, and is a group with multiplication given by composition (after possibly restricting maps so that it's well-defined).

Ex $\mathbb{Z}$ $\quad Aut(\mathbb{Z}) = \mathbb{Z}/2 \quad \mathrm{Comm}(\mathbb{Z}) = \mathbb{Q}^\times$

$F_2 \quad Aut(F_2) = $ v. free $\quad Aut(F_n) \leq \mathrm{Comm}(F_2) \quad \forall n$

[Bartholdi: $\mathrm{Comm}(F_2)$ is ∞-generated]

Surprisingly, mapping class groups and $Out(F_n)$ (for large enough complexity of the surface and $n \geq 3$) are algebraically rigid.

["Rigid" in this context means G is f.i. in $\mathrm{Comm}(G)$]

- (Ivanov, McCarthy) $\quad \mathrm{Comm}(MCG(S_g)) = MCG^{\pm}(S_g) \quad g \geq 3$
- Generalizations by Brendle-Margalit, Bridson-Pettet-Souto, McLeay
- ~~for varying $N \trianglelefteq$~~ for many $N \trianglelefteq MCG(S_g)$, $\mathrm{Comm}(N) = MCG^{\pm}(S_g)$.
- Bavard-Dowdall-Rafi - versions for big mapping class groups.

In $Out(F_n)$:

- Khramtsov, Bridson-Vogtmann - $Out(Out(F_n)) = Out(F_n) \quad n \geq 3$
- Farb-Handel - $\mathrm{Comm}(Out(F_n)) = Out(F_n) \quad n \geq 4$
- Clearly does not hold for $n = 2$ as $Out(F_2)$ is v. free.

Thm (Bridson-Horbez-W)

$$\mathrm{Comm}(Out(F_3)) = Out(F_3)$$

[Related results later...]

II Blueprints for proofs.

You have a favorite group G and suspect that $\mathrm{Comm}(G) \cong G$ ...

Step 1 Find a 'rigid' graph with a G action.

i.e. $G \curvearrowright X$ and the map $G \to \mathrm{Aut}(X)$ is an isomorphism.

Step 2 Show that the vertex and edge stabilizers are distinguished algebraically.

[I'm being deliberately vague here but note that for commensurator rigidity, any notions you use better be invariant up to passing to finite index subgroups.]

Step 3 We have:

$$\begin{tikzcd} G \arrow[r, "ad"] \arrow[dr, "\Phi"'] & \mathrm{Comm}(G) \arrow[d, dashed, "\Psi"] \\ & \mathrm{Aut}(X) \end{tikzcd}$$

Ψ ← given by step 2

Ψ is surjective as Φ is. If $[f] \in \ker \Psi$, $g \in H = \mathrm{dom}(f)$

$$ad_{f(g)} = f \circ ad_g \circ f^{-1} \Rightarrow \Phi(f(g)) = \Phi(g)$$

$\Rightarrow g + f(g)$ have the same action on $X \Rightarrow f|_H = id$.

[Rmks • This strategy implies something stronger than $\mathrm{Comm}(G) \cong G$ - any isomorphism between f.i. subgroups is induced by conjugation.

• Can run a similar arguments with $N \trianglelefteq G$ to show $\mathrm{Comm}(N) \cong G$.]

III Rigid $Out(F_n)$-graphs.

There are a few to pick from:

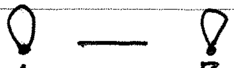
- Spine of Outer space (Bridson - Vogtmann)
- Free splitting graph (Aramayona - Souto)
- Non-sep. splitting graph (Pandit) ⊛
- Cyclic splitting graph (+ friends) (Horbez - W)
- Free factor graph (Bestvina - Bridson)

NS_n Non-separating splitting graph.

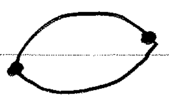
Each vertex is a one-edge HNN extension of F_n with trivial edge group.

$v =$  A $\qquad$ A = corank 1 free factor.

* The vertex splitting is determined by the conjugacy class of A, so $stab(v) = stab([A])$.



$A \qquad B$ have an edge between them if the splittings have a common refinement:

 A_1 $\qquad$ A_1  A_2

$rank(A_1) = n-2 \qquad rank(A_1) + rank(A_2) = n-2$

When $n=3$ $\quad A_i \cong \mathbb{Z}$

stab () ~~commensurable~~ virtually $\mathbb{Z}^3$

stab () virtually $\mathbb{Z}^2$

IV Characterizing stab([A]) algebraically.

$\exists$ an index 2 subgroup s.t.:

$$1 \to A \times A \to \text{stab}^\circ([A]) \to \text{Out}([A]) \to 1$$

The kernel is given by 'twists'. If t is ~~the~~ some stable letter of the HNN extension then there is a subgroup of twists

$$t \longmapsto w_1 t w_2 \qquad w_1, w_2 \in A.$$

Proposition/ If $H \leqslant \text{Out}(F_3)$ is f.g. + has a normal subgroup which is the product of two free groups.
+ $\text{vcd}(H) = 3$ (maximal)

Then $H \leqslant \text{stab}([A])$ for some corank 2 free factor. (possibly after passing to a finite index subgroup).

IVa Some hyperbolic geometry.

Prop Let H act on a Gromov hyperbolic space X. If $K_1 \times K_2 \trianglelefteq H$ then either

(1) Either K_1 or K_2 has a finite orbit in ∂X
(2) $H \curvearrowright X$ has bounded orbits.

Sketch If (1) fails then neither K_1 nor K_2 has an accumulation point on $\partial X \Rightarrow \exists R > 0$ s.t.

$$S = \{ x : d(x, gx) \leqslant R \ \forall g \in K_1 \times K_2 \}$$

is non-empty and bounded. As $K_1 \times K_2 \trianglelefteq H$, S is preserved by H so H has bounded orbits.

IV 6 The black box (Bestvina-Feighn, Bestvina-Reynolds, Hamenstädt, Handel-Mosher, Guirardel-Horbez, Guirardel-Levitt)

Handel-Mosher (Guirardel-Horbez) — general case (c.f. the lecture of Guirardel)

I If $H \leq Out(F_n)$, after passing to a f.i. subgroup either:

(1) H fixes a one-edge free splitting A or $A_1 — A_2$

(2) There exists a hyperbolic graph X such that $H \curvearrowright X$ with unbounded orbits (this is a relative free factor graph)

II (Guirardel-Levitt) If $n = 3$ then ~~the set~~ in case (2), the stabilizers of all points are ~~virtual~~ in ∂X are virtually abelian.

Proof of proposition (1)

If H has a normal subgroup that is the product of two free groups and $H \curvearrowright X$ with X δ-hyp, then if point stabilizers in ∂X are virtually abelian, then H has bounded orbits.

$\Rightarrow$ H is contained in A or $stab(\mathbb{Z} — \mathbb{Z} * F_2)$

in the second case, the $vcd(stab(A_1 — A_2)) = 2$, so $H \leq stab(A)$.

V A sketch for characterizing $\mathrm{Stab}([A]) \cap \mathrm{Stab}([B])$ $(n=3)$

$A = \langle a_1, a_2 \rangle$ $\quad$ $\mathrm{Stab}([A]) = \mathrm{Stab}([a_1, a_2])$

$\mathrm{Stab}([A]) \cap \mathrm{Stab}([B])$ is a McCool group

$$\mathrm{Stab}(\{[a_1,a_2], [b_1, b_2]\})$$

$H \leq \mathrm{Out}(F_n)$ $\quad$ $\mathcal{C}$ = conjugacy classes fixed by H

$$\underset{\text{Guirardel-Levitt}}{H \leq M_c(\mathcal{C}_H)}$$

$H \rightsquigarrow$ JSJ deformation space over $\mathcal{C}$

If $\alpha = [a_1, a_2]$, $\beta = [b_1, b_2]$ then F_3 is ~~one-ended~~ freely indecomposable relative to $\{\alpha, \beta\}$. ~~If H is virtually abelian~~
$\rightsquigarrow$ tree of cylinders $=: T_H$

- T_H has no QH vertices if H is abelian
$\Rightarrow$ H is commensurable with the group of twists on T_H

- Invariant free factors can be 'read off' from T_H

- ~~If~~ $N(H)$ ~~is comm~~ can be read off from T_H

- If $N(H)$ is commensurable with $H \rightsquigarrow$ all invariant free factors are compatible.

VI ~~Further~~ Generalizations

Theorem If $N \trianglelefteq Out(F_n)$ for $n \geqslant 4$ and N contains a power of every twist associated to a cyclic splitting then $Comm(N) \cong Out(F_n)$

~~Extra~~ **E.g.** Can take N to be the kernel of the map to the outer automorphism group of the free Burnside group $B_{n,p}$. N is infinite in some cases (see Coulon, Coulon - Hilion)

Extra tools

- stabilizers of boundary points of rel. free factor graphs can contain free subgroups. Argument for characterizing $Stab([A])$ still applies but needs more care.
- $Stab([A]) \cap Stab([B])$ is no longer a McCool group; instead use Feighn-Handel's work on abelian subgroups.

Hopes Should work for IA_n + possibly other terms in the Johnson filtration ($n=3,4$ seem fine for IA_n; $n \geqslant 5$ - harder!)