Principles of tensor numerical methods for multi-dimensional PDEs

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Traditional numerical approximations of PDEs in $\mathbb{R}^d$ are only computationally tractable for a moderate number of spacial variables due to the "curse of dimensionality", i.e. storage demands and complexity costs grow exponentially with d.

The breaking through approach to a data-sparse representation of d-variate functions and operators on large $n^{\otimes d}$-grids is based on the principle of separation of variables. The modern grid-based tensor methods [4] employ the commonly used low-parametric canonical, Tucker and matrix product states (tensor train) data formats, thus requiring merely linear in d storage costs for tensors of size n^d, $O(dn)$.

The novel method of quantized tensor approximation (QTT) is proven to provide a logarithmic data-compression for a wide class of discrete functions and operators [1]. It enables to discretize and solve multi-dimensional steady-state and dynamical problems with a logarithmic complexity in the volume size of the computational $n^{\otimes d}$-grid, $O(d \log n)$.

Several examples of successful applications of the *grid-based tensor numerical methods* will be discussed. First, we demonstrate how the grid-based QTT tensor approximation applies to the integration of multidimensional or/and strongly oscillating functions (say, many-electron integrals, integrals with the lattice sum of interaction potentials [3] or with high-frequency oscillators [5]).

The second part of the talk addresses the low-rank QTT tensor representation of discrete FEM elliptic operators and their inverse, as well as of some important classes of matrices, such as discrete FFT, convolution and wavelet vector transforms, all with $O(\log n)$ cost in the vector size.

We also outline the tensor approach to the solution of multi-parametric elliptic equations arising in the Galerkin-collocation discretization of stochastic PDEs, and discuss the simultaneous space-time tensor approximation of parabolic problems using the example of the multi-parametric chemical master equation [2].

References

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