Random walks and the component structure of the vacant set

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Warwick

- Introduction: random walks and cover time
- How to estimate 'un-visit probability' and some examples

- Vacant set definition and main results
- Vacant set in $G_{n,p}$
- Vacant set in random r-regular graphs

Discrete random walk on a finite graph

- $G = (V, E)$ is a finite graph with n vertices and m edges. $(|V| = n, |E| = m)$
- Assume that G is connected, so that all vertices in G have at least one neighbour vertex,
- Simple random walk: move to a randomly chosen neighbour at each step
- A simple random walk is a (unbiased) Markov process
- Suppose that at step t the walk is at vertex $X(t) = v$ Let $d(v) = |N(v)|$ be the degree of vertex v, then for $w \in N(v)$

$$\mathbf{Pr}(X(t+1) = w) = \frac{1}{d(v)}$$

Cover time: Introduction

- $G = (V, E)$ is a connected graph. ($|V| = n, |E| = m$)
- Random walk W_v on G starting at $v \in V$
  Let C_v be the expected time taken for W_v to visit every vertex of G
- The *cover time* of G is defined as $T_{cov} = \max_{v \in V} C_v$
- Starting vertex matters
- Graph with vertices u, v, w $\qquad C_v = C_u + 1$

Brief history

- AKLLR[1] For any connected graph $T_{cov} \leq 4mn$

- Application: Is there a path connecting vertex s to t?
  Test graph connectivity by random walk, in $O(n^3)$ steps with $O(\log n)$ storage.

- Good for exploring large networks.
  This led to increased interest in cover time

- Feige (1995). Bounds for any connected G:

$$(1 - o(1)) n \log n \leq T_{cov} \leq (1 + o(1)) \frac{4}{27} n^3$$

[1] Aleliunas, Karp, Lipton, Lovász and Rackoff. Random Walks, Universal Traversal Sequences, and the Complexity of Maze Problems. (1979)

Matthews bound for cover time

Hitting time from x to y, $\mathbf{E}_x T_y$, the expected time to reach destination vertex y, from start vertex x

$$T_{cov} \leq H_{\max}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = O(H_{\max} \log n)$$

where $H_{\max} = \max \mathbf{E}_x T_y$ is the maximum expected hitting time over all pairs (x, y)

e.g. Complete graph K_n, $H_{\max} = n - 1$, $T_{cov} \sim n \log n$

Geometric waiting time, probability visit y at next step $1/(n-1)$

Convergence of random walk to stationarity

- Transition matrix of walk $P = (p_{i,j})$,
  where $p_{i,j}$ is probability a walk at vertex i moves to vertex j
- Assume: graph connected, walk not periodic

$$P^t \to \Pi \quad \text{where} \quad \Pi_{i,j} = \pi_j$$

- For a random walk, stationary distribution

$$\pi_j = \frac{d(j)}{2|E|}$$

- The rate of convergence of the walk to π is given by

$$|P_u^t(x) - \pi_x| \le (\pi_x/\pi_u)^{1/2}\lambda_2^t,$$

  λ_2 is the second eigenvalue of the transition matrix P
- Many random graphs are expanders ($0 < \lambda_2 < 1$)
  Walk is Rapidly Mixing. After $t = O(\log n)$ steps

$$|P_u^{(t)}(v) - \pi_v| \le n^{-c}$$

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- (Informally) Waiting time of first visit to v tends to geometric distn, success probability $p_v \sim \pi_v/R_v$

Estimate of Unvisit Probability

T_{mix}, a suitable mixing time of the walk.
π_v, the stationary distribution of v.
R_v, the expect number of returns to v by the walk in time T_{mix}.

Unvisit Probability

$$\mathbf{Pr}(\mathcal{W}_u(\tau) \neq v : \tau = T_{mix}, \ldots, t) \sim e^{-t\pi_v/R_v}$$

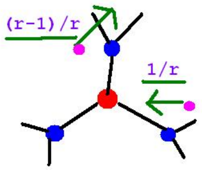
True under assumptions that hold for many e.g. random graph models. Asymptotics of generating function

If the walk is rapidly mixing $T_{mix} = O(\log n)$, we can 'ignore' effect of visits during the mixing time when calculating T_{cov}

For random graphs we can estimate R_v from the graph structure for most vertices, (and bound it for all vertices)

Example: r-regular random graphs

How to calculate R_v for random r-regular graphs ?
If v is tree-like (not near any short cycles) then $R_v \sim \frac{r-1}{r-2}$



Same as: biassed random walk on the half line $(0, 1, 2, ...)$

$\mathbf{Pr}($ go left $) = \frac{1}{r}$, $\mathbf{Pr}($ go right $) = \frac{r-1}{r}$

$f = \mathbf{Pr}($ walk returns to origin $) = \frac{1}{r-1}$ $R_v \sim \frac{1}{1-f} = \frac{r-1}{r-2}$

Example: r-regular random graphs

- $\pi_v = 1/n$
- T_{mix} the mixing time $O(\log n)$
- Most vertices are locally tree-like
  For such vertices $R_v \sim (r-1)/(r-2)$, expected number of returns to start in infinite r-regular tree

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- Size of set of unvisited vertices $\mathcal{R}(t) \sim ne^{-t(r-2)/(r-1)n}$

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- We know the size of $\mathcal{R}(t)$, the **vacant set**
- We can use this to calculate the cover time T_{cov}

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3. Web-graphs $G(m,n)$ where $m \geq 2$
$$T_{cov} \sim \frac{2m}{m-1}\, n \log n$$

Directed graphs: random digraphs $D_{n,p}$

The main challenge for $D_{n,p}$, was to obtain the stationary distribution

Theorem

Let $np = d \log n$ *where* $d = d(n)$, *and let* $m = n(n-1)p$

Let $\gamma = np - \log n$, *and assume* $\gamma = \omega(\log \log n)$

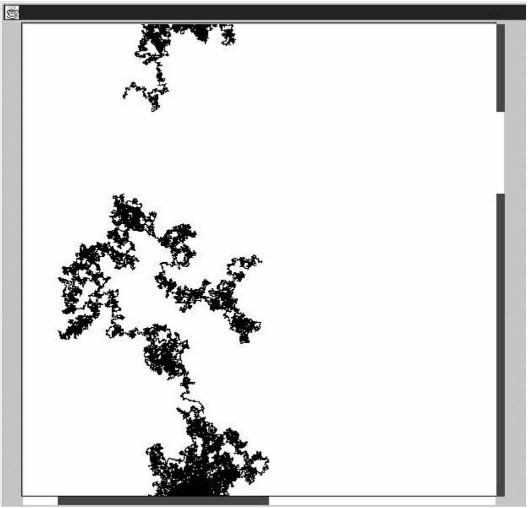
Then **whp**, *for all* $v \in V$,

$$\pi_v \sim \frac{\deg^-(v)}{m},$$

and

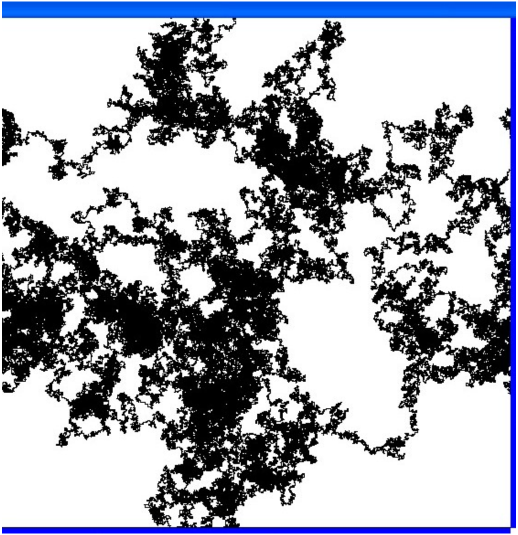
$$T_{cov} \sim d \log \left(\frac{d}{d-1} \right) n \log n$$

The vacant set of a random walk



Random walk on 600 $\times$ 600 toriodal grid. Black visited, white unvisited.

What is the component structure of vacant set?



Notation

Finite graph $G = (V, E)$.

$\mathcal{W}_u$ Simple random walk on G, starting at $u \in V$

The vacant set. Vertices not yet visited by the walk

Can think of vacant set $\mathcal{R}(t)$ as coloured red,
and visited vertices $\mathcal{B}(t)$ as colored blue

$\mathcal{R}(t)$ Set of vertices not visited by $\mathcal{W}_u$ up to time t
$\Gamma(t)$ Sub-graph of G induced by vacant set $\mathcal{R}(t)$

How large is $\mathcal{R}(t)$?
What is the likely component structure of $\Gamma(t)$?

Evolution of vacant set graph $\Gamma(t)$

It is a sort of random graph process in reverse
As the walk progresses the vacant set $\Gamma(t)$ is reduced from the whole graph G to a graph with no vertices

In the context of sparse random graphs, as the unvisited vertex set $\mathcal{R}(t)$ gets smaller, the edges inside $\Gamma(t)$ will get sparser and sparser.

Small sets of vertices don't induce many edges
One might expect that at some time $\Gamma(t)$ will break up into small components
This is basically what we prove

We say that $\Gamma(t)$ is sub-critical at step t, if all of its components are of size $O(\log n)$

We say that $\Gamma(t)$ is super-critical at step t, if it has a unique giant component, (of size $\Theta(\mathcal{R}(t))$) and all other components are of size $O(\log n)$

In the cases we consider there is a t^*, which is a (**whp**) threshold for transition from super-criticality to sub-criticality

Vacant set of $G_{n,p}$

We assume that

$$p = \frac{c \log n}{n}$$

where $(c-1)\log n \to \infty$ with n, and $c = n^{o(1)}$. Let

$$t(\epsilon) = n\left(\log\log n + (1+\epsilon)\log c\right)$$

Theorem

Let $\epsilon > 0$ be a small constant
Then **whp** *we have*
(i) $\Gamma(t)$ is super-critical for $t \le t(-\epsilon)$
(ii) $\Gamma(t)$ is sub-critical for $t \ge t(\epsilon)$

Giant component of $\mathcal{R}(t)$ until $t > n \log\log n$
For $c > 1$ constant, Cover time T_{cov} of $G_{n,p}$ is $T_{cov} \sim n \log n$

Random graphs $G_{n,r}$

For $r \geq 3$, constant, let

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\; n$$

Theorem

Let $\epsilon > 0$ be a small constant. Then **whp** *we have*

(i) $\Gamma(t)$ *is super-critical for* $t \leq (1-\epsilon)t^*$

(ii) *For* $t \leq (1-\epsilon)t^*$*, size of giant component is* $\Omega(n)$

(iii) $\Gamma(t)$ *is sub-critical for* $t \geq (1+\epsilon)t^*$

e.g. for 3-regular random graphs $r = 3$, and $t^* = (6\log 2)\; n$
Giant component for about $t^* = (6\log 2)n$ steps
Cover time $T_{cov} \sim 2n\log n$

Related Work

Benjamini and Sznitman; Windisch:
Considered the d-dimensional toroidal grids $d \geq 5$.
Super-critical below $C_1 n$, sub-critical above $C_2 n$

Černy, Teixeira and Windisch:
Considered random r-regular graphs $G_{n,r}$
They showed sub-criticality for $t \geq (1+\epsilon)t^*$
and existence of a unique giant component for $t \leq (1-\epsilon)t^*$
These proofs use the concept of random interlacements of continuous time random walks

Our proof: Discrete time

- Simple. Based on established random graph results
- Gives results for $G_{n,p}$
- Completely characterizes the component structure
- Proves that in the super-critical phase $t \leq t^*$, the second largest component of $G_{n,r}$ has size $O(\log n)$ **whp**
  Gives the small tree structure of $\Gamma(t)$

Subsequent Work: Černy, Teixeira and Windisch:
Consider random r-regular graphs $G_{n,r}$
Investigate scaling window around t^* using annealed model

Component structure of vacant set of $G_{n,p}$

Distribution of edges in $\Gamma(t)$

Lemma

Consider a random walk on $G_{n,p}$
Conditional on $N = |\mathcal{R}(t)|$, $\Gamma(t)$ is distributed as $G_{N,p}$.

Proof This follows easily from the principle of deferred decisions. We do not have to expose the existence or absence of edges between the unvisited vertices of $\mathcal{R}(t)$ □

Thus to find the super-critical/ sub-critical phases, we only need high probability estimates of $|\mathcal{R}(t)|$ as t varies

This, we know how to do, from our work on cover time of random graphs

Size of vacant set $\mathcal{R}(t)$ in $G_{n,p}$

Analysis of $G_{n,p}$ for $p = c \log n/n$

1. $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$

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1. $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$
2. Almost all vertices have $\sim$ average degree $np = c \log n$
   Thus $\pi_v \sim 1/n$

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2. Almost all vertices have $\sim$ average degree $np = c \log n$
   Thus $\pi_v \sim 1/n$
3. Probability of retracing an edge at next step $1/d(v) = o(1)$
   Thus $R_v = 1 + o(1)$ for all $v \in V$
4. Size of vacant set $\mathbf{E}(|\mathcal{R}(t)|) \sim ne^{-(1+o(1))t/n}$.
5. We can use Chebyshev to show that $|\mathcal{R}(t)|$ is concentrated

If $t_\theta = n(\log \log n + (1+\theta) \log c)$ then

$$\mathbf{E}(|\mathcal{R}(t)|) \sim \frac{n}{c^{1+\theta} \log n} = \frac{1}{c^\theta p}$$

Size of 'giant' component

- Recall that $np = c \log n$,
  If $t_\theta = n(\log\log n + (1+\theta)\log c)$ then $\mathbf{E}(|\mathcal{R}(t)|) \sim 1/(c^\theta p)$
  So, at t_θ,
  $$\mathbf{E}(|\mathcal{R}(t_\theta)|p) \sim \frac{1}{c^\theta}$$
- Threshold criteria for random graph $G_{N,p}$ is $Np \sim 1$
- When $\theta = 0$, then $\mathbf{E}(|\mathcal{R}(t_\theta)|p) \sim 1$
- The threshold t^* occurs at around $\theta = 0$ i.e.
  $$t^* \sim n(\log\log n + \log c)$$
- Size of giant is order $|\mathcal{R}(t_\theta)|$. As $t \to t^*$ from below, size of 'giant' is order $|\mathcal{R}(t^*)| \sim 1/p = n/(c \log n)$
- Above t^* max component size collapses to $O(\log n)$

Component structure of vacant set of random r-regular graphs for $r \geq 3$, constant.

Reminder: Vacant set of r-regular random graphs

- $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$

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- $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$
- $\pi_v = 1/n$
- Most vertices are locally tree-like
  For such vertices $R_v \sim (r-1)/(r-2)$,
  expected number of returns to start in infinite r-regular tree
- $$\mathbf{Pr}(v \text{ unvisited in } T_{mix}, \ldots, t) \sim e^{-t(r-2)/(r-1)n}$$
- A similar upper bound can be obtained for the $o(n)$ non-tree-like vertices
- Size of vacant set $\mathcal{R}(t) \sim ne^{-t(r-2)/(r-1)n}$

Threshold

Let

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\, n.$$

Theorem

Let $\epsilon > 0$ be a small constant. Then **whp** *we have*

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(ii) *For* $t \leq (1-\epsilon)t^*$*, size of giant component is* $\Omega(n)$

(iii) $\Gamma(t)$ *is sub-critical for* $t \geq (1+\epsilon)t^*$ *and*

Proof outline for r-regular random graph

- Generate the graph in the configuration model using the random walk
- Graph $\Gamma(t)$ induced by vacant set $\mathcal{R}(t)$ is random
- Estimate un-visit probability of vertices to find size of $\mathcal{R}(t)$
- Estimate degree sequence $\boldsymbol{d}$ of $\Gamma(t)$in the configuration model, using size of vacant set $\mathcal{R}(t)$, and number of unvisited edges $\mathcal{U}(t)$
- Given the degree sequence $\boldsymbol{d}$ of $\Gamma(t)$, we can use Molloy-Reed condition for existence of giant component in a random graph with fixed degree sequence
- Estimate number of small trees in configuration model

Degree sequence of $\Gamma(t)$

Vacant set size

$|\mathcal{R}(t)| = (1 + o(1))N_t$ where $N_t = ne^{-\frac{(r-2)t}{(r-1)n}}$

Vertex degree

Let $D_s(t)$ the number of unvisited vertices of $\Gamma(t)$ with $\boldsymbol{r - s}$ visited neighbours and of degree s in $\Gamma(t)$

For $0 \leq s \leq r$, and for ranges of t given below, **whp**

$$D_s(t) \sim N_t \binom{r}{s} p_t^s \,(1 - p_t)^{r-s}$$

where

$$p_t = e^{-\frac{(r-2)^2}{(r-1)r}\frac{t}{n}}$$

Range of validity is $o(n) \leq t \leq \Theta(n \log n)$

Includes t^*

Uniformity

Lemma

Consider a random walk on G_r. Conditional on $N = |\mathcal{R}(t)|$ and degree sequence $\mathbf{d} = d_{\Gamma(t)}(v), v \in \mathcal{R}(t)$, then $\Gamma(t)$ is distributed as $G_{N,\mathbf{d}}$, the random graph with vertex set $[N]$ and degree sequence $\mathbf{d}$.

Proof Basic idea: Reveal G_r using the random walk.
Suppose that we condition on $\mathcal{R}(t)$ and the *history of the walk*, $\mathcal{H} = (W_u(0), W_u(1), \ldots, W_u(t))$. If G_1, G_2 are graphs with vertex set $\mathcal{R}(t)$ and if they have the same degree sequence then substituting G_2 for G_1 will not conflict with $\mathcal{H}$.
Every extension of G_1 is an extension of G_2 and vice-versa. □

Thus we only need:
Good model of component structure of $G_{N,\mathbf{d}}$
High probability estimates of the degree sequence $D_s(t)$ of $\Gamma(t)$.

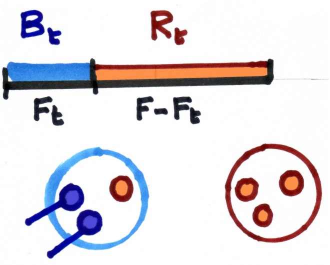
Main variables

By calculating un-visit probabilities in various ways, we can estimate the size at step t of

- $\mathcal{R}(t)$ the set of unvisited vertices
- $\mathcal{U}(t)$ the set of unvisited edges
- $D_s(t)$ the number of unvisited vertices of degree s in $\Gamma(t)$ ie number of unvisited vertices with $r - s$ edges incident with visited vertices $\mathcal{B}(t)$

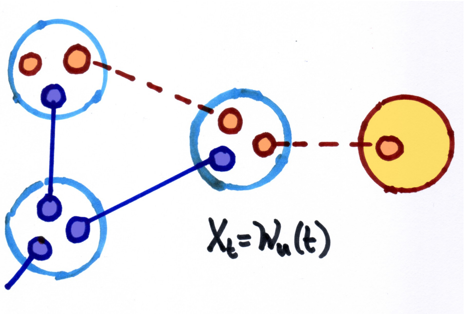
Annealed process

We use the random walk to generate the graph in the configuration model as a random pairing F



- B_t blue conifg. points at step t
  which form discovered pairing F_t
- R_t red conifg. points at step t
  This will form un-generated pairing $F - F_t$
- Visited vertices may have config. points in R_t, corresponding to unexplored edges

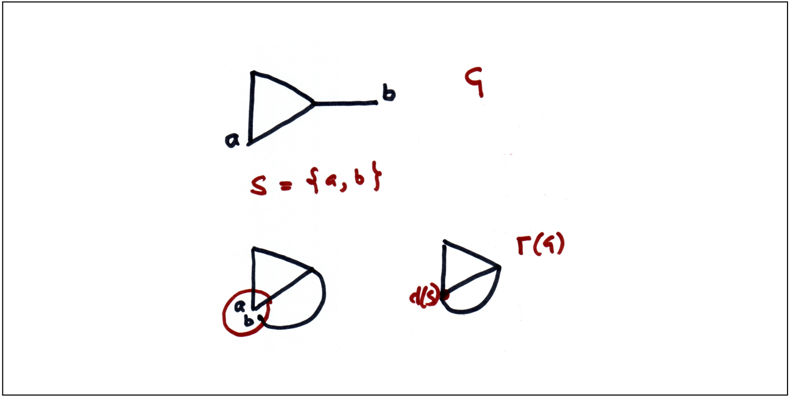
Next configuration pairing



At step t walk located at vertex $X_t \in \mathcal{B}(t)$
Probability walk moves to an unvisited vertex?
Given the walk selects a red config. point of X_t (if any), the probability this is paired with an config. point in $\mathcal{R}(t)$ is

$$\frac{r|\mathcal{R}(t)|}{|R_t|-1}$$

Shrinking Vertices: First visit to a set of vertices S



S subset of vertices of G

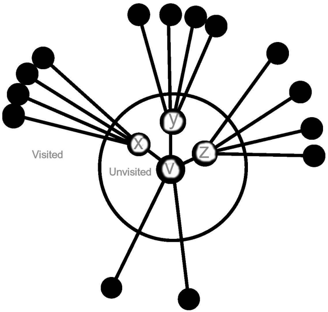
$\gamma(S)$ is S shrunk to a vertex

$H(G)$ is G with S shrunk to $\gamma(S)$

$$\mathbf{Pr}_G(S \text{ unvisited at step } t) \sim \mathbf{Pr}_{H(G)}(\gamma(S) \text{ unvisited at step } t)$$

Degree of unvisited vertex

Vertex v has 3 unvisited neighbours x, y, z and 2 visited neighbours a, b, so $s = 3$, $r - s = 2$



Calculate probability that exactly $\{v, x, y, z\}$ are unvisited, and a, b visited from probability that $\{v, x, y, z\}$ are unvisited, $\{v, x, y, z, a\}$ are unvisited etc. Contract e.g. $\{v, x, y, z\}$ to a single vertex γ of degree 20 with 3 loops

The degree sequence of $\mathcal{R}(t)$

To analyse the degree sequence of $\Gamma(t)$ we prove

Lemma

If the neighbours of v in G are $w_1, w_2, \ldots, w_r$ then

$$\mathbf{Pr}(v, w_1, \ldots, w_s \in \mathcal{R}_t,\ w_{s+1}, \ldots, w_r \in \mathcal{B}(t)) \sim e^{-\frac{(r-2)t}{(r-1)n}}\ p_t^s\ (1 - p_t)^{r-s}$$

where $p_t = e^{-\frac{t(r-2)^2}{n(r-1)r}}$

We write

$$\mathbf{Pr}_{\mathcal{W}}(\{v, w_1, \ldots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \ldots, w_r\} \subseteq \mathcal{B}(t))$$
$$= \sum_{X \subseteq [s+1,r]} (-1)^{|X|} \mathbf{Pr}_{\mathcal{W}}((\{v, w_1, \ldots, w_s\} \cup X) \subseteq \mathcal{R}(t))$$
$$\sim \sum_{X \subseteq [s+1,r]} (-1)^{|X|} \, e^{-tp_{\gamma_X}},$$

where

$$p_{\gamma_X} \sim \frac{((r-2)(s+|X|)+r)(r-2)}{r(r-1)n}.$$

To prove this we contract $\{v, w_1, \ldots, w_s\} \cup X$ to a single vertex γ_X creating $\Gamma_X(t)$.
We then estimate the probability that γ_X hasn't been visited by a random walk on $\Gamma_X(t)$. (Unvisit probability)

For this we argue that $|\{v, w_1, \ldots, w_s\} \cup X| = s + |X| + 1$

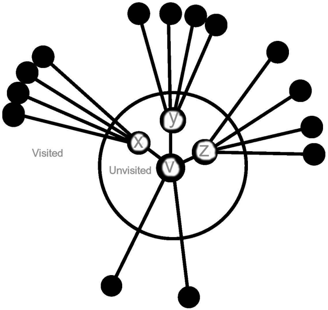
$$\pi_{\gamma_X} = \frac{r(s + |X| + 1)}{rn}$$

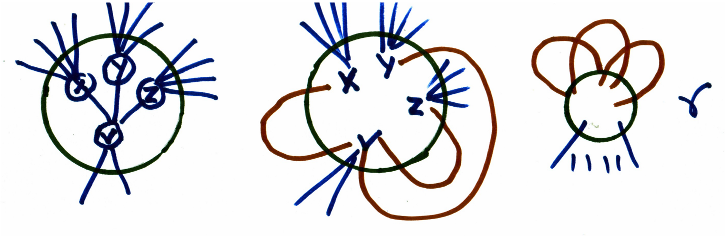
and

$$R_{\gamma_X} \sim \frac{(s + |X| + 1)r(r-1)}{((r-2)(s + |X|) + r)(r-2)}$$

Expression for R_{γ_X} is obtained by considering the expected number of returns to the origin in an infinite tree with branching factor $r - 1$ at each non-root vertex. At the root there are $s + |X|$ loops and $(r-2)(s + |X|) + r$ branching edges..

Example

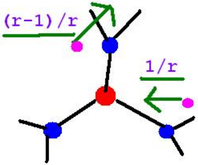




Reminder, R_v for random r-regular graphs

A transition on the loops returns to γ_X immediately, and a transition on any other edge is (usually) like a walk in a tree

If v is tree-like (not near any short cycles) then $R_v \sim \frac{r-1}{r-2}$



Same as: random walk on the line $(0, 1, 2, ...)$
Pr(go left) $= \frac{1}{r}$, **Pr**(go right) $= \frac{r-1}{r}$

Molloy-Reed Condition

Theorem

Let $\lambda_0, \lambda_1, \ldots, \lambda_r \in [0,1]$ *be such that* $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$*.*
Suppose that $\boldsymbol{d} = (d_1, d_2, \ldots, d_N)$ *satisfies*
$|\{j : d_j = s\}| = (1 + o(1))\lambda_s N$ *for* $s = 0, 1, \ldots, r$*.*
Let $G_{n,\boldsymbol{d}}$ *be chosen randomly from graphs with vertex set* $[N]$ *and degree sequence* $\boldsymbol{d}$*. Let*

$$L = \sum_{s=0}^{r} s(s-2)\lambda_s.$$

(a) *If* $L < 0$ *then* **whp** $G_{n,\boldsymbol{d}}$ *is sub-critical.*
(b) *If* $L > 0$ *then* **whp** $G_{n,\boldsymbol{d}}$ *is super-critical.*

Furthermore the unique giant component has size βn *where* β *is the solution to an equation derived from the degree sequence*

Threshold for collapse of giant component

Degree sequence of $\Gamma(t)$ is (approximately) binomial $Bin(r, p_t)$ where $p_t = e^{-\frac{t(r-2)^2}{n(r-1)r}}$

Once we know the degree sequence we can use the Molloy-Reed criterion to see whether or not there is a giant component. G has a giant component iff $L > 0$, where

$$L = \sum_v d_v(d_v - 2).$$

Direct calculation gives $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\, n$ as the critical value

Heuristically, $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$ can be obtained from the degree sequence of unvisited vertices

Branching outward from an unvisited vertex
The probability an edge goes to another unvisited vertex:

$$p_t = e^{-\frac{(r-2)^2 t}{(r-1)rn}}$$

We need branching factor $(r-1)p_t > 1$, to have a chance to get a large component

At $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$

$$\begin{aligned} (r-1)p_t &= (r-1)e^{-\frac{(r-2)^2 t}{(r-1)rn}} \\ &= (r-1)e^{-\log(r-1)} \\ &= 1 \end{aligned}$$

Enumerating tree components

These are small subgraphs of the underlying graph

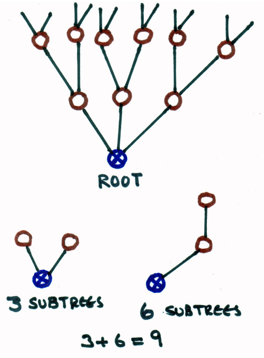
How to count subgraphs of a given graph?

Rooted subtrees of the infinite r-regular tree

How to count subgraphs of a given graph?
Number of rooted k-subtrees of the infinite r-regular tree

$$\frac{r}{((r-2)k+2)}\binom{(r-1)k}{k-1}$$

Example $r = 3, ; k = 3$



Number of small components in $\Gamma(t)$

$N_t = \mathbf{E}|\mathcal{R}(t)|$. Expected size of vacant set
p_t probability of a red edge
$N(k,t)$: Number of unvisited tree components of $\Gamma(t)$ with k vertices

Theorem

Let ϵ be a small positive constant. Let $1 \le k \le \epsilon \log n$ and $\epsilon n \le t \le (1-\epsilon) t_{k-1}$. Then **whp***:*

$$N(k,t) \sim \frac{r}{k((r-2)k+2)} \binom{(r-1)k}{k-1} \quad N_t \, p_t^{k-1} (1-p_t)^{k(r-2)+2}$$

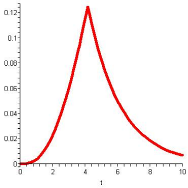
Vertices on small components of vacant set

Let

$$t^* = n \frac{r(r-1)}{(r-2)^2} \log(r-1).$$

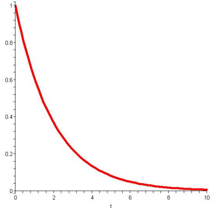
Theorem

Let $\mu(t)$ be the expected proportion of vertices on small trees
The function $\mu(t)$ increases from 0 *at $t = 0$,*
to a maximum value $\mu^* = 1/(r-1)^{r/(r-2)}$ as $t \to t^*$,
and decreases to 0 as $t \to (r-1)/(r-2)\, n \log n$

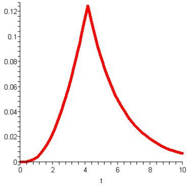


Example: $r = 3$. Vacant set as a function of $\tau = t/n$

Proportion of vertices in vacant set $N(t)/n \sim e^{-t/n((r-2)/(r-1))}$

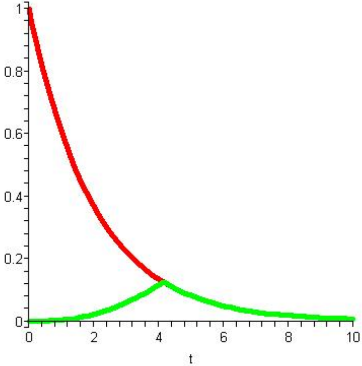


Proportion of vertices in unvisited tree components



Threshold: $r = 3$, $t^* = 6 \log 2$

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$$



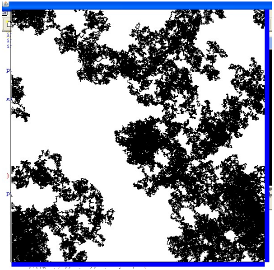
Propn. of vertices in vacant set, and on small tree components

Closing observations

- Random graphs $G(n, p)$ and random r-regular graphs exhibit threshold behavior
- The size of the giant component can be estimated in the super-critical range
- The number of small tree components of a given size can be estimated

- The technique can be applied to other problems e.g.
- Vacant net: sub-graph induced by the unvisited edges
- Upper bounds on sub-critical threshold for hypercube, high degree grids,...

THANK YOU



QUESTIONS