Lecture notes

Fundamental inequalities: techniques and applications

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Abstract

Inequalities are ubiquitous in Mathematics (and in real life). For example, in optimization theory (particularly in linear programming) inequalities are used to described constraints. In analysis inequalities are frequently used to derive a priori estimates, to control the errors and to obtain the order of convergence, just to name a few. Of particular importance inequalities are Cauchy-Schwarz inequality, Jensen's inequality for convex functions and Fenchel's inequalities for duality. These inequalities are simple and flexible to be applicable in various settings such as in linear algebra, convex analysis and probability theory.

The aim of this mini-course is to introduce to undergraduate students these inequalities together with useful techniques and some applications. In the first section, through a variety of selected problems, students will be familiar with many techniques frequently used. The second section discusses their applications in matrix inequalities/analysis, estimating integrals and relative entropy. No advanced mathematics is required.

This course is taught for Warwick's team for International Mathematics Competition for University Students, 24th IMC 2017. The competition is planned for students completing their first, second, third or fourth year of university education.

Contents

1 Fundamental inequalities: Cauchy-Schwarz inequality, Jensen's inequality for convex functions and Fenchel's dual inequality

In this section, we review three basic inequalities that are Cauchy-Schwarz inequality, Jensen's inequality for convex functions and Fenchel's inequality for duality. For simplicity of presentation, we only consider simplest underlying spaces such as $\mathbb{R}^n$ or a finite set. These inequalities, however, can be stated in much more complex situations. Many techniques for proving inequalities are presented via selected examples and exercises in Section 1.5.

1.1 Cauchy-Schwarz inequality

Let $\mathbf{u}, \mathbf{v}$ be two vectors of an inner product space (over $\mathbb{R}$, for simplicity). The Cauchy-Schwarz inequality states that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Proof. If $\mathbf{v} = 0$, the inequality is obvious true. Let $\mathbf{v} \neq 0$. For any $\lambda \in \mathbb{R}$, we have

$$0 \leq \|\mathbf{u} + \lambda \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\lambda \langle \mathbf{u}, \mathbf{v} \rangle + \lambda^2 \|\mathbf{v}\|^2.$$

Consider this as a quadratic function of λ. Therefore we have

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 \leq \|u\|^2 \|\mathbf{v}\|^2,$$

which implies the Cauchy-Schwarz inequality. □

1.2 Convex functions

Definition 1.1 (Convex function). Let $X \subset \mathbb{R}^n$ be a convex set.

- A function $f : X \to \mathbb{R}$ is call *convex* if for all $x_1, x_2 \in X$ and $t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

- A function $f : X \to \mathbb{R}$ is call *strictly convex* if for all $x_1, x_2 \in X$ and $t \in (0, 1)$

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

Example 1.1. Examples of convex functions:

- affine functions: $f(x) = a \cdot x + b$, for $a, b \in \mathbb{R}^n$,
- exponential functions: $f(x) = e^{ax}$ for any $a \in \mathbb{R}$,

- Euclidean-norm: $f(x) = \|x\| = \Big(\sum\limits_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}$.

Verifying convexity of a differentiable function Note that in Definition 1.1, the function f does not require to be differentiable. The following criteria can be used to verify the convexity of f when it is differentiable.

(1) If $f : X \to \mathbb{R}$ is differentiable, then it is convex if and only if

$$f(x) \geq f(y) + \nabla f(y) \cdot (x - y) \quad \text{for all} \quad x, y \in X.$$

(2) If $f : X \to \mathbb{R}$ is twice differentiable, then it is convex if and only if its Hessian $\nabla^2 f(x)$ is semi-positive definite for all $x \in X$.

Some important properties of convex functions

Lemma 1.2. *Below are some important properties of convex functions.*

(1) If f and g are convex functions, then so are $m(x) = \max\{f(x), g(x)\}$ and $s(x) = f(x) + g(x)$.

(2) If f and g are convex functions and g is non-decreasing, then $h(x) = g(f(x))$ is convex.

Proof. These properties can be directly proved by verifying the definition. □

1.3 Jensen's inequality

Theorem 1.3 (Jensen's inequality). *Let f be a convex function, $0 \leq \alpha_i \leq 1; i = 1, \ldots, n$ such that $\sum\limits_{i=1}^{n} \alpha_i = 1$. Then for all $x_1, \ldots, x_n$, we have*

$$f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) \leq \sum_{i=1}^{n} \alpha_i f(x_i). \tag{1}$$

Proof. We prove by induction. The cases $n = 1, 2$ are obvious. Suppose that the statement is true for $n = 1, \ldots, K - 1$. Suppose that $\alpha_1, \ldots, \alpha_K$ are non-negative numbers such that $\sum_{i=1}^{K} \alpha_i = 1$. We need to prove that

$$f\Big(\sum_{i=1}^{K} \alpha_i x_i\Big) \leq \sum_{i=1}^{K} \alpha_i f(x_i).$$

Since $\sum_{i=1}^{K} \alpha_i = 1$, at least one of the coefficients α_i must be strictly positive. Assume that $\alpha_1 > 0$. Then by the conducting assumptions, we obtain

$$\begin{aligned} f\Big(\sum_{i=1}^{K} \alpha_i x_i\Big) &= f\Big(\alpha_1 x_1 + (1-\alpha_1)\sum_{i=2}^{K} \frac{\alpha_i}{1-\alpha_1} x_i\Big) \\ &\leq \alpha_1 f(x_1) + (1-\alpha_1) f\Big(\sum_{i=2}^{K} \frac{\alpha_i}{1-\alpha_1} x_i\Big) \\ &\leq \alpha_1 f(x_1) + (1-\alpha_1)\sum_{i=2}^{K} \frac{\alpha_i}{1-\alpha_1} f(x_i) \\ &= \sum_{i=1}^{K} \alpha_i f(x_i), \end{aligned}$$

where we have used the fact that $\sum\limits_{i=2}^{K} \frac{\alpha_i}{1-\alpha_1} = 1$. □

Remark 1.4 (**Jensen's inequality- probabilistic form**). Jensen's inequality can also be stated using probabilistic form. Let (Ω, A, μ) be a probability space. If g is a real-valued function that is $\mu-$ integrable and if f is a convex function on the real line, then

$$f\Big(\int_\Omega g\, d\mu\Big) \leq \int_\Omega f \circ g\, d\mu.$$

□

Example 1.2 (**Examples of Jensen's inequality**).

1) For all real numbers $x_1, \ldots, x_n$, it holds

$$\Big(\sum_{i=1}^{n} x_i\Big)^2 \leq n \sum_{i=1}^{n} x_i^2.$$

   Proof. Since $f(x) = x^2$ is convex, we have

$$\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big)^2 = f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \leq \sum_{i=1}^{n} \frac{1}{n} f(x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i^2,$$

   which is the desired statement. □

2) **Arithmetic-Geometric (AM-GM) Inequality**. Let $(x_i)_{1\leq i\leq n}$ and $(\lambda_i)_{1\leq i\leq n}$ be real number satisfying

$$x_i \geq 0, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1.$$

Then, with the convention $0^0 = 1$,

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}. \tag{2}$$

In particular, taking $\lambda_1 = \ldots = \lambda_n = \frac{1}{n}$ yields

$$\sum_{i=1}^{n} x_i \geq n \sqrt[1/n]{x_1 \ldots x_n}.$$

Proof. By taking the logarithmic both sides, (2) is equivalent to

$$\sum_{i=1}^{n} \lambda_i \ln(x_i) \leq \ln\Big(\sum_{i=1}^{n} \lambda_i x_i\Big).$$

This is exactly Jensen's inequality applying to the convex function $f(x) = -\ln(x)$. □

1.4 Convex conjugate and Fenchel's inequality

The convex conjugate of a function $f : \mathbb{R}^d \to \mathbb{R}$ is, $f^* : \mathbb{R}^d \to \mathbb{R}$, defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - f(y)\}.$$

Fenchel's inequality: for $x, y \in \mathbb{R}^d$, we have

$$f(x) + f^*(y) \geq x \cdot y.$$

Example 1.3. Examples of Fenchel's inequality

1) $f(x) = \frac{|x|^2}{2}$, then $f^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - |x|^2\} = \frac{|y|^2}{2}$. Fenchel's inequality reads

$$\frac{1}{2}(|x|^2 + |y|^2) \geq x \cdot y.$$

2) $f(x) = \frac{1}{p}|x|^p$ where $p > 1$. Then $f^*(y) = \sup_{x \in \mathbb{R}^d} \{x \cdot y - \frac{1}{p}|x|^p\} = \frac{1}{q}|y|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.
Fenchel's inequality becomes: for $x, y \in \mathbb{R}^d$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\frac{1}{p}|x|^p + \frac{1}{q}|y|^q \geq x \cdot y.$$

1.5 Some techniques to prove inequalities

In practises, three inequalities introduced in previous sections often do not appear in standard forms. It is crucial to recognize them. In this section, through exercises we will learn some techniques to prove inequalities.

Exercise 1. Let $a_i, b_i \in \mathbb{R}, b_i > 0$ for $i = 1, \dots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}.$$

Proof. By the Cauchy-Scharz inequality we have

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = \Big(\sum_{i=1}^{n} \frac{a_i}{\sqrt{b_i}}\sqrt{b_i}\Big)^2 \leq \Big(\sum_{i=1}^{n} \frac{a_i^2}{b_i}\Big)\Big(\sum_{i=1}^{n} b_i\Big).$$

□

Exercise 2 (Problem 6, IMC 2015)**.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

Proof. By the AM-GM Inequality we have

$$2(n+1) - 1 = n + (n+1) > 2\sqrt{n(n+1)},$$

which implies that

$$2(n+1) - 2\sqrt{n(n+1)} > 1.$$

Dividing both sides by $\sqrt{n}(n+1)$ yields

$$\frac{1}{\sqrt{n}(n+1)} < 2\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big).$$

Hence by summing up over n we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2\sum_{n=1}^{\infty} \Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big) = 2.$$

□

Exercise 3. Let A, B, C be three angles of a triangle. Prove that

$$\sin A + \sin B + \sin C \leq 3\frac{\sqrt{3}}{2}.$$

Proof. Consider the function $f(x) = \sin x$. Since $f''(x) = -\sin^2(x) \leq 0$, f is concave. Therefore,

$$\frac{\sin A + \sin B + \sin C \leq}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

□

Exercise 4 (Problem 3, IMC 2016)**.** Let n be a positive integer and $a_1, \ldots, a_n; b_1, \ldots, b_n$ be real number such that $a_i + b_i > 0$ for $i = 1, \ldots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n}(a_i + b_i)}. \tag{3}$$

Proof. We notice the similar form of both sides of (3). For $A, B \in \mathbb{R}$ we have

$$\frac{AB - B^2}{A+B} = B - \frac{2B^2}{A+B} \tag{4}$$

Applying (4) for $A = a_i, B = b_i$, we get

$$LHS = \sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} = \sum_{i=1}^{n} \left(b_i - \frac{2b_i^2}{a_i + b_i}\right) = \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} \frac{2b_i^2}{a_i + b_i}.$$

Similarly now applying (4) for $A = \sum_{i=1}^{b} a_i, B = \sum_{i=1}^{n} b_i$, we obtain

$$RHS = \sum_{i=1}^{n} b_i - \frac{2\left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} a_i + b_i}.$$

Therefore (3) is equivalent to

$$\frac{\left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} a_i + b_i} \leq \sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i} \tag{5}$$

By Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} b_i\right)^2 = \left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{a_i + b_i}} \sqrt{a_i + b_i}\right)^2 \leq \left(\sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i}\right) \cdot \left(\sum_{i=1}^{n} a_i + b_i\right),$$

which implies (5) as desired. □

Exercise 5 (Problem 1, IMC 2010)**.** Let $0 < a < b$. Prove that

$$\int_a^b (x^2 + 1)e^{-x^2}\, dx \geq e^{-a^2} - e^{-b^2}$$

Proof. By the AM-GM Inequality $x^2+1 \geq 2x > 0$ for any $0 < a \leq x \leq b$, we have

$$\int_a^b (x^2+1)e^{-x^2}\,dx \geq \int_a^b 2xe^{-x^2}\,dx = e^{-x^2}\Big|_a^b = \int_a^b (x^2+1)e^{-x^2}\,dx.$$

□

Exercise 6 (Problem 6, IMC 2001)**.** Let n be an integer and let $f_n(x) = \sin x \cdot \sin(2x) \cdot \ldots \sin(2^n x)$. Prove that

$$|f_n(x)| \leq \frac{2}{\sqrt{3}}|f_n(\pi/3)|.$$

Proof. Let $g(x) = |\sin x| \cdot |\sin(2x)|^{1/2}$. We have

$$\begin{aligned} |g(x)| = |\sin x| \cdot |\sin(2x)|^{1/2} &= \frac{\sqrt{2}}{\sqrt[4]{3}}\Big(\sqrt[4]{|\sin x| \cdot |\sin x| \cdot |\sin x| \cdot |\sqrt{3}\cos x|}\Big)^2 \\ &\leq \frac{\sqrt{2}}{\sqrt[4]{3}}\;\frac{3\sin^2 x + 3\cos^2 x}{4} = \Big(\frac{\sqrt{3}}{2}\Big)^2 = g(\pi/3). \end{aligned}$$

Note we have use the AM-GM inequality that

$$3\sin^2 x + 3\cos^2 x = \sin^2 x + \sin^2 x + \sin^2 x + (\sqrt{3}\cos x)^2 \geq 4\sqrt[4]{|\sin x||\sin x||\sin x||\sqrt{3}\cos x|}.$$

Therefore, let $K = \frac{2}{3}\Big[1 - (-1/2)^n\Big]$, we have

$$\begin{aligned} |f_n(x)| &= |\sin x| \cdot |\sin(2x)| \ldots |\sin(2^n x)| \\ &= \Big(|\sin x||\sin(2x)|^{1/2}\Big) \cdot \Big(|\sin(2x)||\sin(4x)|^{1/2}\Big)^{1/2}\Big(|\sin(4x)||\sin(8x)|^{1/2}\Big)^{3/4} \times \\ &\qquad \times \ldots \times \Big(|\sin(2^{n-1}x||\sin(2^n x)|^{1/2}\Big)^K \Big(|\sin(2^n x)|\Big)^{1-K/2} \\ &= g(x) \cdot g(2x)^{1/2} \ldots g(2^{n-1}x)^K \Big(|\sin(2^n x)|\Big)^{1-K/2} \\ &\leq g(\pi/3)g(x) \cdot g(2\pi/3)^{1/2} \ldots g(2^{n-1}\pi/3)^K \\ &= |f_n(\pi/3)|/|\sin(2^n\pi/3)^{1-K/2}| \\ &= |f_n(\pi/3)|\Big(\frac{2}{\sqrt{3}}\Big)^{1-K/2} \leq |f_n(\pi/3)|\frac{2}{\sqrt{3}}. \end{aligned}$$

This is the desired inequality. □

Exercise 7 (IMO 1995)**.** Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Proof. Let $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. Then x, y, z are positive real numbers and $xyz = 1$. We have

$$\frac{1}{a^3(b+c)} = \frac{1}{\frac{1}{x^3}\left(\frac{1}{y} + \frac{1}{z}\right)} = \frac{x^2}{y+z}.$$

Similarly

$$\frac{1}{b^3(c+a)} = \frac{y^2}{z+x}, \quad \frac{1}{c^3(a+b)} = \frac{z^2}{x+y}.$$

By Cauchy-Scharz inequality (see also Exercise 1) and the Arithmetic-Geometric Inequality we have

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2} \geq \frac{3\sqrt[3]{xyz}}{2} = \frac{3}{2}.$$

□

Exercise 8 (Problem 1, The 26th Annual Vojtech Jarnik International Competition 2016)**.** Let a, b, c be positive real number such that $a + b + c = 1$. Prove that

$$\left(\frac{1}{a} + \frac{1}{bc}\right)\left(\frac{1}{b} + \frac{1}{ca}\right)\left(\frac{1}{c} + \frac{1}{ab}\right) \geq 1728$$

Proof. By the AM-GM inequality we have

$$\frac{1}{a} + \frac{1}{bc} = \frac{1}{a} + \frac{1}{3bc} + \frac{1}{3bc} + \frac{1}{3bc} \geq 4\frac{1}{\sqrt[4]{ab^3c^3}},$$

and

$$\frac{1}{27} = \left(\frac{a+b+c}{3}\right)^3 \geq abc, \quad \text{i.e.,} \quad \frac{1}{abc} \geq 27.$$

Therefore,

$$\begin{aligned}\left(\frac{1}{a} + \frac{1}{bc}\right)\left(\frac{1}{b} + \frac{1}{ca}\right)\left(\frac{1}{c} + \frac{1}{ab}\right) &\geq \left(4\frac{1}{\sqrt[4]{ab^3c^3}}\right)\left(4\frac{1}{\sqrt[4]{ba^3c^3}}\right)\left(4\frac{1}{\sqrt[4]{ca^3b^3}}\right) \\ &= \frac{64}{abc} \geq 64 \times 27 = 1728.\end{aligned}$$

□

Exercise 9. Let $x, y \in \mathbb{R}, y > 0$. Prove that

$$e^x + y(\ln y - 1) \geq x \cdot y.$$

Proof. Let $f(x) = e^x$. Then for $y > 0$, we have $f^*(y) = \sup_{x \in \mathbb{R}}\{x \cdot y - e^x\} = y(\ln y - 1)$. By Fenchel's inequality we have $f(x) + f^*(y) = e^x + y(\ln y - 1) \geq x \cdot y$ as desired. □

Exercise 10 (IMO 2001)**.** Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Proof. Since the expression on the LHS does not change when we replace (a, b, c) by (ka, kb, kc) for arbitrary $k \in \mathbb{R}$, we can assume that $a + b + c = 1$. Since $x \mapsto \frac{1}{\sqrt{x}}$ is convex for $x > 0$, applying Jensen's inequality we obtain

$$\begin{aligned}\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} &\geq \frac{1}{\sqrt{a(a^2+8bc)+b(b^2+8ca)+c(c^2+8ab)}}\\ &= \frac{1}{\sqrt{a^3+b^3+c^3+24abc}}. \qquad (6)\end{aligned}$$

Next we show that

$$a^3+b^3+c^3+24abc \leq 1 = (a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+6abc, \qquad (7)$$

which is equivalent to show that

$$(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b) \geq 6abc.$$

This is indeed true because of the AM-GM inequality

$$(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b) \geq 6\sqrt[6]{a^2b\cdot a^2c\cdot b^2a\cdot b^2c\cdot c^2a\cdot c^2b} = 6abc.$$

The desired inequality follows from (6) and (7). □