IMC Preparation
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Analysis II

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.
   a) If f is continuous and $\text{range}(f) = \mathbb{R}$, then f is monotonic.
   b) If f is monotonic and $\text{range}(f) = \mathbb{R}$, then f is continuous.
   c) If f is monotonic and f is continuous, then $\text{range}(f) = \mathbb{R}$.

2. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

3. (a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ there exists a function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{Q}$. (b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

4. (a) A sequence $x_1, x_2, \ldots$ of real numbers satisfies $x_{n+1} = x_n \cos x_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values x_1? (b) A sequence $y_1, y_2, \ldots$ of real numbers satisfies $y_{n+1} = y_n \sin y_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values y_1?

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$,
$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$
   Prove that for all $x \in [0, \infty)$,
$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

6. (HW, due 31 Jan) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a; b])$ is a closed interval of length $b - a$.

7. (HW, due 31 Jan) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a shadow point if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that all the points of the open interval $I = (a, b)$ are shadow points, but a and b are not shadow points. Prove that
   a) $f(x) \leq f(b)$ for all $a < x < b$;
   b) $f(a) = f(b)$.

8. (HW, due 31 Jan) Is it true that there can be at most countably many pairwise disjoint letter T's in the plane?