Polynomials

Problem 1 Find all polynomials $P(x)$ such that

$$P(x) = \frac{P(x-1) + P(x+1)}{2}, \quad \text{for all } x \in \mathbb{R}.$$

Problem 2 Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$.

Problem 3 Find all polynomials P satisfying $xP(x-1) = (x-2)P(x)$.

Problem 4 Find all pairs of integers $m, n \in \mathbb{N}$ such that the polynomial

$$P(x) = 1 + x + x^2 + \ldots + x^m$$

divides the polynomial

$$Q(x) = 1 + x^n + x^{2n} \ldots + x^{mn}.$$

Problem 5 Show that if a polynomial $P(x)$ with real coefficients satisfies $P(x) \geq 0$ for all $x \in \mathbb{R}$, then it can be represented as

$$P(x) = Q_1^2(x) + \ldots + Q_n^2(x),$$

for some polynomials $Q_1, \ldots, Q_n$ with real coefficients.

Problem 6 Let polynomial $P(x)$ with real coefficients satisfy inequality $P(x) > 0$ for all $x > 0$. Show that there are polynomials $Q(x)$ and $R(x)$ with non-negative coefficients satisfying

$$P(x) = Q(x)/R(x).$$

Problem 7 Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z-r_1)(z-r_2)(z-r_3)(z-r_4), \end{aligned}$$

where a, b, c, d are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

Problem 8 Can polynomials

$$x^5 - x - 1 \quad \text{and} \quad x^2 + ax + b,$$

with $a, b \in \mathbb{Q}$, have a common complex root?