Homework problems

Problem 6 Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

1. $p(r_i) = 0, \qquad i = 1, 2, \dots, n$, and

2. $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \qquad i = 1, 2, \dots, n-1$,

where $p'(x)$ denotes the derivative of $p(x)$.

Problem 7 Find all real solutions to the equation

$$2^x + 5^x = 3^x + 4^x.$$

Problem 8 Let f be an infinitely differentiable real-valued function defined on the real numbers such that

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \qquad \text{for every } n = 1, 2, 3, \dots.$$

Determine the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \dots$.