IMC SEMINAR 2016
Linear Algebra

Things to remember:

- Linear independence, spanning, properties of determinant and trace, column and row operations.
- Eigenvalues, Jordan canonical form, Cayley-Hamilton theorem.
- Vandermonde determinant, density of invertible matrices.

Warm up.

1. (IMC 2005 1.1) Let A be the $n \times n$ matrix, whose (i, j)th entry is $i + j$ for all $i, j = 1, 2, \ldots, n$. What is the rank of A?
Solution. The matrix A is given by

$$\begin{pmatrix} 2 & 3 & \ldots & n+1 \\ 3 & 4 & \ldots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \ldots & 2n \end{pmatrix}$$

And by row operations (which do not change the rank), that is, by replacing the ith row by its difference with the $(i-1)$th, $i = 2, \ldots, n$, the matrix can be put into the form,

$$\begin{pmatrix} 2 & 3 & \ldots & n+1 \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & \ldots & n+1 \\ 1 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

The rank of this last matrix is 2 as the two top rows are linearly independent. All this applies for $n \geq 2$, therefore the rank of A is 2. For $n = 1$ we have $\operatorname{rank}(A) = 1$. ∎

2. Let A and B be $n \times n$ matrices and I be the identity matrix. Show that $\det(I + AB) = \det(I + BA)$.
Solution. First let us assume that A is invertible. Then,

$$\begin{aligned} \det(I + AB) &= \det(A^{-1}) \det(I + AB) \det(A), \\ &= \det(A^{-1}(I + AB)A), \\ &= \det(I + BA). \end{aligned}$$

Hence the result holds. Now let us fix B, for any matrix A with entries (a_{ij}) we note that the expression

$$\det(I + AB) - \det(I + BA)$$

is a polynomial on (a_{ij}), therefore it is a continuous function from $\mathbb{R}^{n^2}$ to $\mathbb{R}$. Moreover, the set of invertible matrices form a dense subset in $\mathbb{R}^{n^2}$ and we have just proved that $\det(I + AB) - \det(I + BA)$ vanishes on that subset. Thus, by continuity we conclude that $\det(I + AB) - \det(I + BA) = 0$ for all $A \in \mathbb{R}^{n^2}$. Finally since B was arbitrary, then the previous arguments applies also to all $n \times n$ matrices B.

Homework.

Bring your solutions next session on February 3rd.

1. Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$$

2. Let A, $B \in M_{2\times 2}(\mathbb{R})$ such that $A^2 + B^2 = AB$. Show that $(AB - BA)^2 = 0$.

3. Does any rotation of a d-dimension sphere have a fixed point?

4. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \dots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \le i, j \le n + 1$. Prove that $v_1, \dots, v_{n+1}$ are linearly dependent over the rationals.
Hint: Use the identity $-2\langle v_i, v_j\rangle = |v_i - v_j|^2 - |v_i|^2 - |v_j|^2$.