p-adic height pairings and integral points on hyperelliptic curves

Jennifer Balakrishnan
joint work with
Amnon Besser, J. Steffen Müller

University of Oxford

Warwick Number Theory Seminar
Monday, November 11, 2013

Motivation: Finding rational points

Theorem (Faltings, '83)

Let X be a curve of genus $g \geqslant 2$ over $\mathbf{Q}$. The set $X(\mathbf{Q})$ is finite.

Faltings' proof does not lead to an algorithm to compute $X(\mathbf{Q})$. However:

Chabauty's theorem

Theorem (Chabauty, '41)

Let X be a curve of genus $g \geqslant 2$ over $\mathbf{Q}$. Suppose the rank of the Mordell-Weil group of the Jacobian J of X is less than g. Then $X(\mathbf{Q}_p) \cap \overline{J(\mathbf{Q})}$ is finite. In particular, $X(\mathbf{Q})$ is finite.

To make Chabauty's theorem effective:

- Need to find a way to bound $X(\mathbf{Q}_p) \cap \overline{J(\mathbf{Q})}$
- Do this by constructing functions (p-adic integrals of 1-forms) on $J(\mathbf{Q}_p)$ that vanish on $J(\mathbf{Q})$ and restrict them to $X(\mathbf{Q}_p)$

The method of Chabauty-Coleman

Recall that the map $H^0(J_{\mathbf{Q}_p}, \Omega^1) \longrightarrow H^0(X_{\mathbf{Q}_p}, \Omega^1)$ induced by $X \hookrightarrow J$ is an isomorphism of $\mathbf{Q}_p$-vector spaces. Suppose ω_J restricts to ω. Then for $Q, Q' \in X(\mathbf{Q}_p)$, define

$$\int_Q^{Q'} \omega := \int_0^{[Q'-Q]} \omega_J.$$

If the Chabauty condition is satisfied, there exists $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ such that

$$\int_b^P \omega = 0$$

for all $P \in X(\mathbf{Q})$. Thus by studying the zeros of $\int \omega$, we can find the rational points of X.

Generalizing this approach

Our method to study integral points on hyperelliptic curves is in the spirit of the *nonabelian* Chabauty program:

- Kim's nonabelian Chabauty: aim is to generalize the Chabauty method, giving *iterated* p-adic integrals vanishing on rational or integral points on curves
- Explicit examples have been worked out in the case of
  - $\mathbf{P}^1 \setminus \{0, 1, \infty\}$
  - Elliptic curve $E \setminus \{\infty\}$, where rank $E = 0$ or 1
  - Odd degree genus g hyperelliptic curve $C \setminus \{\infty\}$, where we have rank $J(C) = g$

Digression: nonabelian Chabauty philosophy

Let $\mathcal{X} = \mathcal{E} \setminus O$ where $\mathcal{E}$ is an elliptic curve of rank 0 and squarefree discriminant. Fix a model of the form $y^2 = f(x)$, let p be a prime of good reduction, and let

$$\log(z) := \int_b^z \frac{dx}{2y}.$$

Let

$$\mathcal{X}(\mathbf{Z}_p)_1 = \{P \in \mathcal{X}(\mathbf{Z}_p) \mid \log(P) = 0\}.$$

So we have

$$\mathcal{X}(\mathbf{Z}_p)_1 = \mathcal{E}(\mathbf{Z}_p)_{\text{tors}} \setminus O.$$

For small p, it happens that $\mathcal{E}(\mathbf{Z})_{\text{tors}} = \mathcal{E}(\mathbf{Z}_p)_{\text{tors}}$, and hence that

$$\mathcal{X}(\mathbf{Z}) = \mathcal{X}(\mathbf{Z}_p)_1.$$

Extra points in classical Chabauty ("26a3")

```
E is:  26a3:: y^2 = x^3 + 621x + 9774

residue disks =  [(0 : 2 : 1), (0 : 3 : 1), (1 : 1 : 1), (1 : 4 : 1), (2 : 2 : 1), (2
     : 3 : 1), (3 : 2 : 1), (3 : 3 : 1)]
searching in disk:  (0 : 2 : 1)
zero of log:  (3*5 + 5^2 + 4*5^4 + 2*5^5 + 2*5^7 + 5^8 + 4*5^9 + O(5^10) : 2 + 3*5 +
     2*5^2 + 2*5^3 + 4*5^4 + 4*5^5 + 3*5^6 + 3*5^7 + 5^8 + 3*5^9 + O(5^10) : 1 + O
     (5^10))
searching in disk:  (0 : 3 : 1)
zero of log:  (3*5 + 5^2 + 4*5^4 + 2*5^5 + 2*5^7 + 5^8 + 4*5^9 + O(5^10) : 3 + 5 +
     2*5^2 + 2*5^3 + 5^6 + 5^7 + 3*5^8 + 5^9 + O(5^10) : 1 + O(5^10))
searching in disk:  (1 : 1 : 1)
zero of log:  (1 + 5 + 5^2 + 5^3 + 4*5^4 + 3*5^5 + 3*5^6 + 5^8 + 5^9 + O(5^10) : 1 +
     4*5 + 3*5^3 + 2*5^5 + 4*5^6 + 3*5^8 + 4*5^9 + O(5^10) : 1 + O(5^10))
searching in disk:  (1 : 4 : 1)
zero of log:  (1 + 5 + 5^2 + 5^3 + 4*5^4 + 3*5^5 + 3*5^6 + 5^8 + 5^9 + O(5^10) : 4 +
     4*5^2 + 5^3 + 4*5^4 + 2*5^5 + 4*5^7 + 5^8 + O(5^10) : 1 + O(5^10))
searching in disk:  (2 : 2 : 1)
zero of log:  (2 + 5 + 3*5^2 + 4*5^3 + 5^4 + 3*5^5 + 2*5^7 + 2*5^8 + 4*5^9 + O(5^10)
     : 2 + 3*5 + 4*5^2 + 5^3 + 2*5^4 + 2*5^5 + 3*5^6 + 4*5^7 + 4*5^8 + 2*5^9 + O
     (5^10) : 1 + O(5^10))
searching in disk:  (2 : 3 : 1)
zero of log:  (2 + 5 + 3*5^2 + 4*5^3 + 5^4 + 3*5^5 + 2*5^7 + 2*5^8 + 4*5^9 + O(5^10)
     : 3 + 5 + 3*5^3 + 2*5^4 + 2*5^5 + 5^6 + 2*5^9 + O(5^10) : 1 + O(5^10))
searching in disk:  (3 : 2 : 1)
zero of log:  (3 + O(5^10) : 2 + 3*5 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8
     + 4*5^9 + O(5^10) : 1 + O(5^10))
searching in disk:  (3 : 3 : 1)
zero of log:  (3 + O(5^10) : 3 + 5 + 4*5^2 + O(5^10) : 1 + O(5^10))
```

Clearly, we are finding more than the two integral points $(3, \pm 108)$.

Nonabelian Chabauty, continued

Then we do the following: consider the double (Coleman) integral

$$D_2(z) = \int_b^z \frac{dx}{2y}\ \frac{xdx}{2y},$$

and define the "level 2" set

$$\mathcal{X}(\mathbf{Z}_p)_2 = \{P \in \mathcal{X}(\mathbf{Z}_p) \mid \log(P) = 0 \text{ and } D_2(P) = 0\}.$$

Does

$$\mathcal{X}(\mathbf{Z}) = \mathcal{X}(\mathbf{Z}_p)_2?$$

Nonabelian Chabauty, $g = r = 1$ at "level 2"

The functions in nonabelian Chabauty are slightly different as we fix genus and go up in rank:

- For the elliptic curve $y^2 = x^3 + ax + b$, (with rank 1 and squarefree discriminant), consider

$$\log(z) := \int_b^z \frac{dx}{2y}, \qquad D_2(z) = \int_b^z \frac{dx}{2y}\frac{xdx}{2y}.$$

- By writing $\log(z)$ and $D_2(z)$ as p-adic power series and fixing one integral point P, one can consider

$$g(z) := D_2(z)\log^2(P) - D_2(P)\log^2(z).$$

- Kim showed: integral points on an elliptic curve are contained in the set of zeros of g.

Today: the analogue for hyperelliptic curves via p-adic heights

Notation

- $f \in \mathbf{Z}[x]$: monic and separable of degree $2g + 1 \geqslant 3$.
- $X/\mathbf{Q}$: hyperelliptic curve of genus g, given by

$$y^2 = f(x)$$

- $O \in X(\mathbf{Q})$: point at infinity
- $\mathrm{Div}^0(X)$: divisors on X of degree 0
- $J/\mathbf{Q}$: Jacobian of X
- p: prime of good ordinary reduction for X
- $\log_p$: branch of the p-adic logarithm

Special case: p-adic heights on elliptic curves

Let

- $p \geqslant 5$ prime
- $E/\mathbf{Q}$ elliptic curve with Weierstrass model $y^2 = f(x)$, good ordinary reduction at p

Take $P \in E(\mathbf{Q})$. If P reduces to O mod p and lies in $\mathcal{E}^0_{\mathbf{F}_l}$ at bad l, the cyclotomic p-adic height is given by

$$h_p(P) = \frac{1}{p} \log_p \left(\frac{\sigma(P)}{D(P)} \right) \in \mathbf{Q}_p.$$

$\sigma(P), d(P)$

Two ingredients:

- p-adic σ function σ: the unique odd function $\sigma(t) = t + \cdots \in t\mathbf{Z}_p[[t]]$ satisfying

  $$x(t) + c = -\frac{d}{\omega}\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right)$$

  (with ω the invariant differential $\frac{dx}{2y}$ and $c \in \mathbf{Z}_p$, which can be computed by Kedlaya's algorithm)
- denominator function $D(P)$: if $P = \left(\frac{a}{d^2}, \frac{b}{d^3}\right)$, then $D(P) = d$

σ as an iterated Coleman integral

Here's one way to think of σ:

$$x + c = -\frac{d}{\omega}\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right)$$

σ as an iterated Coleman integral

Here's one way to think of σ:

$$\omega(x+c) = -d\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right)$$

σ as an iterated Coleman integral

Here's one way to think of σ:

$$\int x\omega + cx = -\left(\frac{1}{\sigma}\frac{d\sigma}{\omega}\right)$$

σ as an iterated Coleman integral

Here's one way to think of σ:

$$\omega \int x\omega + c\omega = -\left(\frac{d\sigma}{\sigma}\right)$$

σ as an iterated Coleman integral

Here's one way to think of σ:

$$\int \omega \int x\omega + c\omega = -\log(\sigma),$$

which is a double *Coleman integral*.

p-adic heights on integral points

Suppose $P \in E(\mathbf{Z})$. Then

$$h(P) = \frac{1}{p}\log(\sigma(P))$$
$$= -\frac{1}{p}\int_b^P \omega(x\omega + c\omega)$$

Unfortunately, this definition of p-adic height is only valid for elliptic curves. To use p-adic heights to study integral points on higher genus curves, we must use the definition of Coleman and Gross.

Coleman-Gross p-adic height pairing

The Coleman-Gross p-adic height pairing is a symmetric bilinear pairing

$$h : \mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbf{Q}_p, \qquad \text{where}$$

- h can be decomposed into a sum of local height pairings $h = \sum_v h_v$ over all finite places v of $\mathbf{Q}$.
- $h_v(D, E)$ is defined for $D, E \in \mathrm{Div}^0(X \times \mathbf{Q}_v)$ with disjoint support.
- We have $h(D, \mathrm{div}(\beta)) = 0$ for $\beta \in k(X)^\times$, so h is well-defined on $J \times J$.
- The local pairings h_v can be extended (non-uniquely) such that $h(D) := h(D, D) = \sum_v h_v(D, D)$ for all $D \in \mathrm{Div}^0(X)$.
- We fix a certain extension and write $h_v(D) := h_v(D, D)$.

Local height pairings

Construction of h_v depends on whether $v = p$ or $v \neq p$.

- $v \neq p$: intersection theory, as in Ph.D. thesis of Müller ('10)
- $v = p$: logarithms, normalized differentials, Coleman integration (B. - Besser '11)

More on local heights at p

- $X_p := X \times \mathbf{Q}_p$
- Fix a decomposition

$$H^1_{\mathrm{dR}}(X_p) = \Omega^1(X_p) \oplus W, \tag{1}$$

  where W is unit root subspace
- ω_D: differential of the third kind on X_p such that
  - $\mathrm{Res}(\omega_D) = D$,
  - ω_D is normalized with respect to (1).
- If D and E have disjoint support, $h_p(D, E)$ is the Coleman integral

$$h_p(D, E) = \int_E \omega_D.$$

Theorem 1

- $\omega_i := \frac{x^i dx}{2y}$ for $i = 0, \ldots, g-1$
- $\{\bar{\omega}_0, \ldots, \bar{\omega}_{g-1}\}$: basis of W dual to $\{\omega_0, \ldots, \omega_{g-1}\}$ with respect to the cup product pairing.
- $\tau(P) := h_p(P - O)$ for $P \in X(\mathbf{Q}_p)$

Theorem 1 (B.–Besser–Müller)
We have

$$\tau(P) = -2 \int_O^P \sum_{i=0}^{g-1} \omega_i \bar{\omega}_i$$

- The integral is an iterated Coleman integral, normalized to have constant term 0 with respect to a certain choice of tangent vector at O.
- The proof uses Besser's p-adic Arakelov theory.

A result of Kim

Our second theorem is a generalization of the following:

Theorem (Kim, '10).

Let $X = E$ have genus 1 and rank 1 over $\mathbf{Q}$ such that the given model is minimal and all Tamagawa numbers are 1. Then

$$\frac{\int_O^P \omega_0 \, x\omega_0}{(\int_O^P \omega_0)^2},$$

normalized as above, is constant on non-torsion $P \in E(\mathbf{Z})$. With Besser, gave a simple proof of this result:

- By Theorem 1 we have $-2\int_O^P \omega_0 \, x\omega_0 = \tau(P)$.
- One can show that $h(P - O) = \tau(P)$ for non-torsion $P \in E(\mathbf{Z})$.
- Both $h(P - O)$ and $(\int_O^P \omega_0)^2$ are quadratic forms on $E(\mathbf{Q}) \otimes \mathbf{Q}$.

Theorem 2 (“Quadratic Chabauty”)

- For $i \in \{0, \ldots, g-1\}$, let $f_i(P) = \int_O^P \omega_i$ and $f_i(D) = \int_D \omega_i$
- Let $g_{ij}(D_k, D_l) = \frac{1}{2}(f_i(D_k)f_j(D_l) + f_j(D_k)f_i(D_l))$

Theorem 2 (B–Besser–Müller)
Suppose that the Mordell-Weil rank of $J/\mathbf{Q}$ is g and that the f_i induce linearly independent $\mathbf{Q}_p$-valued functionals on $J(\mathbf{Q}) \otimes \mathbf{Q}$. Then there exist constants $\alpha_{ij} \in \mathbf{Q}_p,\ i, j \in \{0, \ldots, g-1\}$ such that

$$\rho := \tau - \sum_{i \leqslant j} \alpha_{ij} g_{ij}$$

only takes values on $X(\mathbf{Z}[1/p])$ in an effectively computable finite set T.

Proof of Theorem 2

Sketch of proof. Set $\rho(P) := -\sum_{v\neq p} h_v(P-O)$, so we have

$$h(P-O) = h_p(P-O) + \sum_{v\neq p} h_v(P-O) = \tau(P) - \rho(P)$$

If the f_i induce linearly independent functionals on $J(\mathbf{Q}) \otimes \mathbf{Q}$, then the set g_{ij} is a basis of the space of $\mathbf{Q}_p$-valued quadratic forms on $J(\mathbf{Q}) \otimes \mathbf{Q}$. Since $h(P-O)$ is also quadratic in P, we can write

$$h(P-O) = \sum_{i\leqslant j} \alpha_{ij} f_i(P) f_j(P), \quad \alpha_{ij} \in \mathbf{Q}_p$$

and conclude

$$\rho(P) = \tau(P) - \sum_{i\leqslant j} \alpha_{ij} f_i(P) f_j(P).$$

Finite set of values T

Proposition

There is a proper regular model $\mathcal{X}$ of $X/\mathbf{Z}_q$ such that if z is p-integral, $h_q(z - O, z - O)$ depends solely on the component of the special fiber $\mathcal{X}_q$ that the section in $\mathcal{X}(\mathbf{Z}_q)$ corresponding to z intersects.

Algorithms

We have `Sage` code for the computation of the following objects:

- single and double Coleman integrals
- $h_p(D, E)$
- The main tool is Kedlaya's algorithm – computes the action of Frobenius and fix the global constant of integration.

We also have `Magma` code for the computation of:

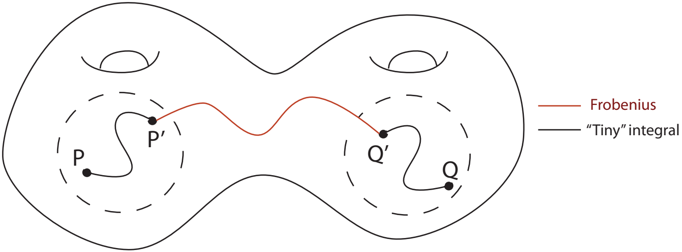
- $h_v(D, E)$ for $v \neq p$
- the set T
- The algorithms rely on Steve Donnelly's implementation of the computation of regular models in `Magma`.

Explicit Coleman integration

The Coleman integral is a p-adic line integral on the curve between points.

If points P, Q are in the same residue disk, use intuition from real-valued line integrals to compute Coleman integrals.

How do we integrate if P, Q aren't in the same residue disk? Coleman's key idea: use Frobenius to move between different residue disks (Dwork's "analytic continuation along Frobenius")



Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^* \omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij} \omega_j.$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^* \omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij} \omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = \int_{\phi(P')}^{\phi(Q')} \omega_i$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = \int_{P'}^{Q'} \phi^*\omega_i$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = \int_{P'}^{Q'} \left(df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j \right)$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting
$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$
- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system
$$\int_{P'}^{Q'} \omega_i = \int_{P'}^{Q'} df_i + \sum_{j=0}^{2g-1} M_{ij} \int_{P'}^{Q'} \omega_j$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = f_i(Q') - f_i(P') + \sum_{j=0}^{2g-1} M_{ij} \int_{P'}^{Q'} \omega_j.$$

Explicit Coleman integration, continued

So we need to

- calculate the action of Frobenius on differentials (Kedlaya's algorithm) and
- use this to set up a linear system to compute single and iterated Coleman integrals

Using a few more words (and equations):

- Calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = f_i(Q') - f_i(P') + \sum_{j=0}^{2g-1} M_{ij} \int_{P'}^{Q'} \omega_j.$$

- Eigenvalues of M are algebraic integers of norm $p^{1/2} \neq 1$.

Example 1

- $X : y^2 = x^3 - 3024x + 70416$: non-minimal model of "57a1"
- $X(\mathbf{Q})$ has rank 1 and trivial torsion.
- $p = 7$ is a good ordinary prime.
- $Q = (60, -324) \in X(\mathbf{Q})$
- Compute
$$\alpha_{00} = \frac{h(Q - O)}{\left(\int_O^Q \omega_0\right)^2}.$$
- Compute
$$T = \{i \cdot \log_7(2) + j \cdot \log_7(3) \ : \ i \in \{0, 2\}, j \in \{0, 2, 5/2\}\}.$$
- Compute
$$\{z \in X(\mathbf{Q}_7) : \rho(z) \in T\}.$$

Example 1, continued

- $X : y^2 = x^3 - 3024x + 70416$
- $T = \{i \cdot \log_7(2) + j \cdot \log_7(3) \ : \ i \in \{0, 2\}, j \in \{0, 2, 5/2\}\}$

There are 16 integral points on X; we have

P	$\rho(P)$
$(-48, \pm 324)$	$2\log_7(2) + \frac{5}{2}\log_7(3)$
$(-12, \pm 324)$	$2\log_7(2) + 2\log_7(3)$
$(24, \pm 108)$	$2\log_7(2) + 2\log_7(3)$
$(33, \pm 81)$	$\frac{5}{2}\log_7(3)$
$(40, \pm 116)$	$2\log_7(2)$
$(60, \pm 324)$	$2\log_7(2) + \frac{5}{2}\log_7(3)$
$(132, \pm 1404)$	$2\log_7(2) + 2\log_7(3)$
$(384, \pm 7452)$	$2\log_7(2) + \frac{5}{2}\log_7(3)$

Example 2

- $X : y^2 = x^3(x-1)^2 + 1$
- $J(\mathbf{Q})$ has rank 2 and trivial torsion.
- $Q_1 = (2, -3), Q_2 = (1, -1), Q_3 = (0, 1) \in X(\mathbf{Q})$ are the only integral points on X up to involution (computed by M. Stoll).
- Set $D_1 = Q_1 - O, D_2 = Q_2 - Q_3$, then
- $[D_1]$ and $[D_2]$ are independent.
- $p = 11$ is a good, ordinary prime.

Example 2, continued

- Compute

$$T = \left\{0, \frac{1}{2}\log_{11}(2), \frac{2}{3}\log_{11}(2)\right\}.$$

- Compute the height pairings $h(D_i, D_j)$ and the Coleman integrals $\int_{D_i} \omega_k \int_{D_j} \omega_l$ and deduce the α_{ij} from

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \end{pmatrix} = \begin{pmatrix} \int_{D_1} \omega_0 \int_{D_1} \omega_0 & \int_{D_1} \omega_0 \int_{D_1} \omega_1 & \int_{D_1} \omega_1 \int_{D_1} \omega_1 \\ \int_{D_1} \omega_0 \int_{D_2} \omega_0 & \frac{1}{2}\left(\int_{D_1} \omega_0 \int_{D_2} \omega_1 + \int_{D_1} \omega_1 \int_{D_2} \omega_0\right) & \int_{D_1} \omega_1 \int_{D_2} \omega_1 \\ \int_{D_2} \omega_0 \int_{D_2} \omega_0 & \int_{D_2} \omega_0 \int_{D_2} \omega_1 & \int_{D_2} \omega_1 \int_{D_2} \omega_1 \end{pmatrix}^{-1} \begin{pmatrix} h(D_1, D_1) \\ h(D_1, D_2) \\ h(D_2, D_2) \end{pmatrix}$$

- Use power series expansions of τ and of the double and single Coleman integrals to give a power series describing ρ in each residue disk.

Example 2, continued

How can we express τ as a power series on a residue disk $\mathcal{D}$?

- Construct the dual basis $\{\bar{\omega}_0, \bar{\omega}_1\}$ of W.
- Fix a point $P_0 \in \mathcal{D}$.
- Compute $\tau(P_0) = h_p(P_0 - O, P_0 - O)$ and use

$$\tau(P) = \tau(P_0) - 2\sum_{i=0}^{g-1}\left(\int_{P_0}^{P} \omega_i\bar{\omega}_i + \int_{P_0}^{P}\omega_i\int_{O}^{P_0}\bar{\omega}_i\right)$$

  to give a power series describing τ in the residue disk.
- The integral points $P \in \mathcal{D}$ are solutions to

$$\rho(P) = \tau(P) - \sum \alpha_{ij} f_i(P) f_j(P) \in T.$$

Example 2, continued

For example, on the residue disk containing $(0,1)$, the only solutions to $\rho(P) \in T$ modulo $O(11^{11})$ have x-coordinate $O(11^{11})$ or

$$4\cdot 11+7\cdot 11^2+9\cdot 11^3+7\cdot 11^4+9\cdot 11^6+8\cdot 11^7+11^8+4\cdot 11^9+10\cdot 11^{10}+O(11^{11})$$

Combine with the Mordell-Weil sieve to see that the "extra" points are not integral. These are the recovered integral points and their corresponding ρ values:

P	$\rho(P)$
$(2,\pm 3)$	$\frac{2}{3}\log_{11}(2)$
$(1,\pm 1)$	$\frac{1}{2}\log_{11}(2)$
$(0,\pm 1)$	$\frac{2}{3}\log_{11}(2)$

Example 3

Let X be the genus 3 hyperelliptic curve

$$y^2 = (x^3 + x + 1)(x^4 + 2x^3 - 3x^2 + 4x + 4).$$

- The prime $p = 7$ is good and ordinary
- $J(\mathbf{Q})$ has rank 3
- Let $P = (-1, 2), Q = (0, 2), R = (-2, 12), S = (3, 62)$, and let $\iota(P), \iota(Q), \iota(R), \iota(S)$ denote their respective images under the hyperelliptic involution ι.
- A set of generators of a finite-index subgroup of the Mordell-Weil group of the Jacobian of X is given by

$$\{D_1 = [P - O], D_2 = [S - \iota(Q)], D_3 = [\iota(S) - R]\}.$$

Example 3, continued

We first compute global 7-adic height pairings:

$$h(D_1, D_1) = 7 + 6 \cdot 7^2 + 4 \cdot 7^4 + 3 \cdot 7^6 + 5 \cdot 7^7 + 3 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})$$

$$h(D_1, D_2) = 4 \cdot 7 + 3 \cdot 7^2 + 2 \cdot 7^4 + 6 \cdot 7^5 + 7^6 + 7^7 + 7^8 + 2 \cdot 7^9 + O(7^{10})$$

$$h(D_1, D_3) = 2 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^6 + 5 \cdot 7^7 + 6 \cdot 7^8 + O(7^{10})$$

$$h(D_2, D_2) = 3 \cdot 7 + 2 \cdot 7^2 + 2 \cdot 7^3 + 7^4 + 5 \cdot 7^5 + 4 \cdot 7^6 + 7^7 + 2 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})$$

$$h(D_2, D_3) = 4 \cdot 7 + 6 \cdot 7^2 + 5 \cdot 7^3 + 3 \cdot 7^4 + 2 \cdot 7^5 + 7^6 + 6 \cdot 7^7 + 7^8 + 7^9 + O(7^{10})$$

$$h(D_3, D_3) = 7^2 + 3 \cdot 7^3 + 7^5 + 4 \cdot 7^6 + 3 \cdot 7^7 + 5 \cdot 7^8 + 7^9 + O(7^{10})$$

We use the height data and Coleman integration to find the α_{ij}:

$$\alpha_{00} = 3 \cdot 7^{-1} + 4 \cdot 7 + 5 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + 4 \cdot 7^5 + 6 \cdot 7^6 + 2 \cdot 7^7 + 4 \cdot 7^9 + O(7^{10})$$

$$\alpha_{01} = 2 \cdot 7^{-1} + 1 + 5 \cdot 7^2 + 3 \cdot 7^3 + 3 \cdot 7^4 + 6 \cdot 7^5 + 5 \cdot 7^7 + 4 \cdot 7^8 + 7^9 + O(7^{10})$$

$$\alpha_{02} = 5 \cdot 7^{-1} + 6 + 3 \cdot 7 + 6 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + 7^5 + 2 \cdot 7^6 + 6 \cdot 7^7 + 3 \cdot 7^8 + 2 \cdot 7^9 + O(7^{10})$$

$$\alpha_{11} = 4 + 3 \cdot 7 + 3 \cdot 7^2 + 4 \cdot 7^3 + 3 \cdot 7^4 + 3 \cdot 7^6 + 6 \cdot 7^7 + 6 \cdot 7^8 + 7^9 + O(7^{10})$$

$$\alpha_{12} = 2 \cdot 7^{-1} + 2 + 3 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 7^4 + 3 \cdot 7^6 + 3 \cdot 7^7 + 4 \cdot 7^8 + 7^9 + O(7^{10})$$

$$\alpha_{22} = 7^{-1} + 5 + 3 \cdot 7 + 7^2 + 5 \cdot 7^3 + 3 \cdot 7^4 + 4 \cdot 7^6 + 2 \cdot 7^7 + 7^8 + 4 \cdot 7^9 + O(7^{10})$$

Example 3, continued

With the α_{ij} data and dual basis, we find the following $\mathbf{Z}_7$-points having ρ-values in the set

$$T = \left\{ a\log(2) + b\log(31) : a \in \left\{0, 1, \frac{5}{4}, \frac{7}{4}\right\}, b \in \left\{0, \frac{1}{2}\right\}\right\}.$$

disk	$x(z)$
$\overline{(3, \pm 1)}$	$3 + 3\cdot 7 + 2\cdot 7^3 + 4\cdot 7^4 + 3\cdot 7^5 + 7^6 + O(7^7)$
	$3 + 3\cdot 7 + 3\cdot 7^2 + 2\cdot 7^3 + 4\cdot 7^5 + 2\cdot 7^6 + O(7^7)$
	$3 + 2\cdot 7 + 5\cdot 7^2 + 7^3 + 2\cdot 7^4 + 5\cdot 7^5 + 4\cdot 7^6 + O(7^7)$
	$3 + 2\cdot 7 + 7^2 + 2\cdot 7^3 + 6\cdot 7^4 + 5\cdot 7^5 + 7^6 + O(7^7)$
	$3 + 4\cdot 7^2 + 4\cdot 7^3 + 4\cdot 7^4 + 7^5 + 6\cdot 7^6 + O(7^7)$
	$\mathbf{3 + O(7^7)}$
	$3 + 3\cdot 7 + 5\cdot 7^2 + 6\cdot 7^4 + 4\cdot 7^5 + O(7^7)$
	$3 + 3\cdot 7 + 7^2 + 7^3 + 6\cdot 7^4 + 2\cdot 7^5 + 6\cdot 7^6 + O(7^7)$
$\overline{(4, \pm 1)}$	

Example 3, continued

disk	$x(z)$
$\overline{(0, \pm 2)}$	$4 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 6 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + O(7^7)$
	$5 \cdot 7 + 6 \cdot 7^2 + 3 \cdot 7^3 + 4 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + O(7^7)$
	$4 \cdot 7 + 2 \cdot 7^2 + 4 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^5 + O(7^7)$
	$5 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 2 \cdot 7^4 + 2 \cdot 7^5 + 3 \cdot 7^6 + O(7^7)$
	$\mathbf{O(7^7)}$
	$2 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 3 \cdot 7^4 + 3 \cdot 7^5 + 2 \cdot 7^6 + O(7^7)$
	$2 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + 6 \cdot 7^5 + 4 \cdot 7^6 + O(7^7)$
	$2 \cdot 7 + 6 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^6 + O(7^7)$
	$4 \cdot 7 + 2 \cdot 7^3 + 3 \cdot 7^4 + 4 \cdot 7^5 + O(7^7)$
	$5 \cdot 7 + 4 \cdot 7^2 + 2 \cdot 7^4 + 6 \cdot 7^6 + O(7^7)$
	$4 \cdot 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 4 \cdot 7^4 + 6 \cdot 7^5 + O(7^7)$
	$5 \cdot 7 + 7^3 + 5 \cdot 7^4 + 3 \cdot 7^5 + 2 \cdot 7^6 + O(7^7)$

Example 3, continued

disk	$x(z)$
$\overline{(5, \pm 2)}$	$5 + 6 \cdot 7 + 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + 5 \cdot 7^5 + 7^6 + O(7^7)$
	$5 + 4 \cdot 7 + 5 \cdot 7^2 + 6 \cdot 7^3 + 5 \cdot 7^4 + 5 \cdot 7^6 + O(7^7)$
	$5 + 6 \cdot 7 + 4 \cdot 7^2 + 2 \cdot 7^3 + 2 \cdot 7^4 + 2 \cdot 7^5 + O(7^7)$
	$5 + 4 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^5 + 7^6 + O(7^7)$
	$5 + 2 \cdot 7 + 2 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + 5 \cdot 7^6 + O(7^7)$
	$5 + 7 + 2 \cdot 7^2 + 7^3 + 3 \cdot 7^4 + 7^5 + 4 \cdot 7^6 + O(7^7)$
	$5 + 2 \cdot 7 + 7^2 + 5 \cdot 7^3 + 2 \cdot 7^4 + 3 \cdot 7^5 + 2 \cdot 7^6 + O(7^7)$
	$5 + 7 + 3 \cdot 7^2 + 4 \cdot 7^4 + 7^5 + 7^6 + O(7^7)$
	$\mathbf{5 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + O(7^7)}$
	$5 + 4 \cdot 7 + 7^3 + 4 \cdot 7^4 + 2 \cdot 7^5 + 3 \cdot 7^6 + O(7^7)$
	$5 + 6 \cdot 7 + 2 \cdot 7^2 + 5 \cdot 7^3 + 2 \cdot 7^5 + 4 \cdot 7^6 + O(7^7)$
	$5 + 4 \cdot 7 + 4 \cdot 7^2 + 5 \cdot 7^3 + 4 \cdot 7^4 + 5 \cdot 7^5 + 7^6 + O(7^7)$

Example 3, continued

disk	$x(z)$
$\overline{(6,\pm 2)}$	$6+3\cdot 7+4\cdot 7^3+4\cdot 7^4+3\cdot 7^5+O(7^7)$
	$6+7+3\cdot 7^2+7^3+3\cdot 7^4+7^5+4\cdot 7^6+O(7^7)$
	$6+3\cdot 7+3\cdot 7^2+7^3+4\cdot 7^4+3\cdot 7^5+2\cdot 7^6+O(7^7)$
	$6+7+2\cdot 7^3+6\cdot 7^4+2\cdot 7^5+2\cdot 7^6+O(7^7)$
	$\mathbf{6+6\cdot 7+6\cdot 7^2+6\cdot 7^3+6\cdot 7^4+6\cdot 7^5+6\cdot 7^6+O(7^7)}$
	$6+5\cdot 7+5\cdot 7^2+6\cdot 7^3+6\cdot 7^4+7^5+4\cdot 7^6+O(7^7)$
	$6+6\cdot 7+5\cdot 7^2+2\cdot 7^3+7^4+7^5+5\cdot 7^6+O(7^7)$
	$6+5\cdot 7+6\cdot 7^2+7^3+7^4+7^5+3\cdot 7^6+O(7^7)$
	$6+3\cdot 7+5\cdot 7^2+6\cdot 7^3+4\cdot 7^4+2\cdot 7^5+2\cdot 7^6+O(7^7)$
	$6+7+5\cdot 7^2+4\cdot 7^3+7^4+4\cdot 7^5+3\cdot 7^6+O(7^7)$
	$6+3\cdot 7+7^2+3\cdot 7^3+4\cdot 7^4+5\cdot 7^5+5\cdot 7^6+O(7^7)$
	$6+7+2\cdot 7^2+6\cdot 7^3+4\cdot 7^4+2\cdot 7^5+4\cdot 7^6+O(7^7)$

Example 3, continued

disk	$x(z)$
$\overline{(2,\pm 3)}$	$2+7^2+2\cdot 7^3+4\cdot 7^4+4\cdot 7^5+O(7^7)$
	$2+5\cdot 7+2\cdot 7^2+5\cdot 7^3+6\cdot 7^4+5\cdot 7^5+6\cdot 7^6+O(7^7)$
	$2+3\cdot 7^3+2\cdot 7^4+5\cdot 7^5+2\cdot 7^6+O(7^7)$
	$2+5\cdot 7+3\cdot 7^2+6\cdot 7^3+7^4+5\cdot 7^6+O(7^7)$
	$2+7+4\cdot 7^2+7^3+7^4+4\cdot 7^5+O(7^7)$
	$2+4\cdot 7+3\cdot 7^2+2\cdot 7^3+5\cdot 7^5+3\cdot 7^6+O(7^7)$
	$2+7+7^3+2\cdot 7^4+5\cdot 7^5+2\cdot 7^6+O(7^7)$
	$2+4\cdot 7+5\cdot 7^3+6\cdot 7^4+2\cdot 7^5+7^6+O(7^7)$
	$2+4\cdot 7^2+3\cdot 7^4+4\cdot 7^5+7^6+O(7^7)$
	$2+5\cdot 7+6\cdot 7^2+7^5+5\cdot 7^6+O(7^7)$
	$2+3\cdot 7^2+4\cdot 7^3+6\cdot 7^4+5\cdot 7^5+2\cdot 7^6+O(7^7)$
	$2+5\cdot 7+6\cdot 7^3+3\cdot 7^4+7^5+5\cdot 7^6+O(7^7)$

Future work

What next?

- Further explore the connection with Kim's nonabelian Chabauty.
- Higher rank?
- Theorem 2 also yields a bound on the number of integral points on X, but the bound needs computations of certain Coleman integrals. Improve on this to get a Coleman-like bound which only depends on simpler numerical data.
- Explicitly extend Theorems 1 and 2 to more general classes of curves.