A Numerical Method for Partial Differential Equations on Surfaces

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Motivation - PDEs on Surfaces

PDEs on surfaces arise in various areas, for instance

- materials science: enhanced species diffusion along grain boundaries,
- fluid dynamics: surface active agents,
- cell biology: phase separation on biomembranes, diffusion processes on plasma membranes, chemotaxis.

Neutrophil

Outline

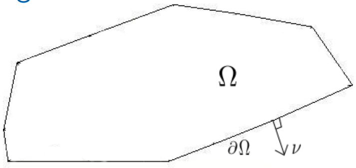
1. Finite Element Method

2. Surface Finite Element Method

4. Numerical Tests

Outline

Strong Problem Formulation



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$$-\Delta u + u = f \text{ in } \Omega$$
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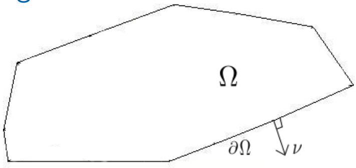

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Here $u = u(x, y)$, $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$, $\nabla \cdot \underline{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}$, $\Delta u := \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

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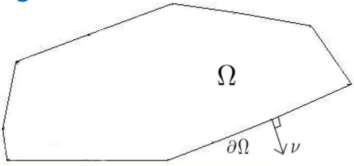

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Integration by parts formula:

$$\int_\Omega v \nabla \cdot \underline{w} \; dx = -\int_\Omega \underline{w} \cdot \nabla v \; dx + \int_{\partial\Omega} v \underline{w} \cdot \nu \; ds$$

where ν: *outward unit normal* of Ω.

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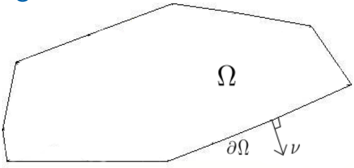

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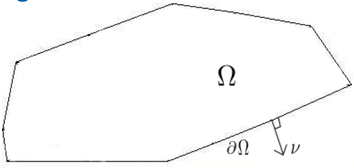

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$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v + uv \; dx = \int_\Omega fv \; dx.$$

Weak Problem Formulation

For weak formulation, need appropriate function space.
Sobolev spaces:

$$H^m(\Omega) := \{u \in L^2(\Omega) \ : D^\alpha u \in L^2(\Omega) \ \forall |\alpha| \leq m\}$$

with corresponding norm

$$\|u\|_{H^m(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2} .$$

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Weak problem (**P**): Find $u \in V := H^1(\Omega)$ such that

$$a(u, v) = \int_\Omega fv\ dx,\ \forall v \in V.$$

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where $a(u, v) := \int_\Omega \nabla u \cdot \nabla v + uv \ dx$.

By the Lax-Milgram theorem, if $f \in L^2(\Omega)$ then there is a unique solution $u \in V$ to the weak problem (which we call the weak solution). In addition, under certain regularity conditions on Ω, it satisfies the stability estimate

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Finite Element Approximation - Idea

We now consider the weak problem in a finite dimensional (linear) subspace $V_h \subset V$.
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FEM problem $(\mathbf{P}_h)$: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \int_\Omega f v_h \, dx \ \forall v_h \in V_h.$$

By Lax-Milgram, there is a unique $u_h \in V_h$ satisfying the discrete problem.

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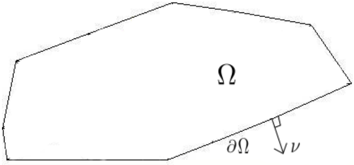

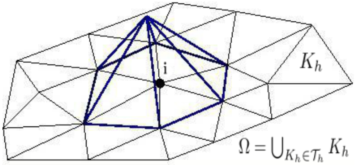

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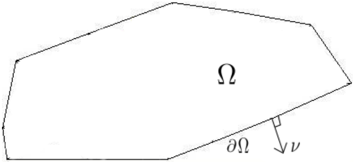

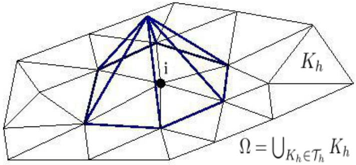

FEM space:

$$V_h := \left\{ v_h \in C^0(\Omega) \ : \ v_h\big|_{K_h} \in P^1(K_h) \ \forall K_h \in \mathcal{T}_h \right\} \subset V.$$

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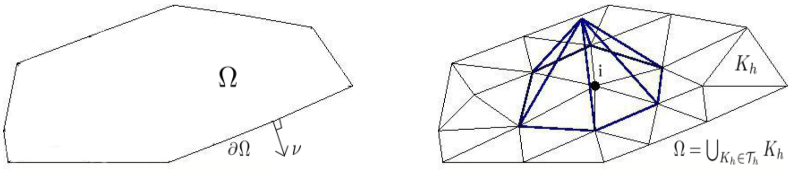

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Let $\{\phi_i^h\}_{i=1}^{N_h}$ denote a basis of V_h (one such basis function shown above). For all $u_h \in V_h$ there uniquely exists a vector $(\alpha_1, ..., \alpha_{N_h}) \in \mathbb{R}^{N_h}$ such that

$$u_h = \sum_{i=1}^{N_h} \alpha_i \phi_i^h$$

Finite Element Approximation - Convergence

We wish to show that, as $h \to 0$,

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Numerical analysts wish to also know the convergence order $k > 0$ at which the above converges to zero:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^k$$

where $C > 0$ does not depend on h.

Finite Element Approximation - Convergence

Recall that $a(u, v) := \int_\Omega \nabla u \cdot \nabla v + uv \; dx$.

Tools for convergence result.

Boundedness:

$$a(w_1, w_2) \leq C_b \|w_1\|_{H^1(\Omega)} \|w_2\|_{H^1(\Omega)} \; \forall w_1, w_2 \in V.$$

Coercivity:

$$a(w, w) \geq C_s \|w\|^2_{H^1(\Omega)} \; \forall w \in V.$$

Interpolation estimate:

$$\|u - I_h u\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \quad \big(\leq Ch\|f\|_{L^2(\Omega)} \big).$$

where $I_h u \in V_h$ is the linear interpolant of u.

Galerkin orthogonality:

$$a(u - u_h, v_h) = 0 \; \forall v_h \in V_h.$$

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$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = \int_\Omega f v_h - \int_\Omega f v_h = 0$$

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a(e_h, e_h) &= a(u - u_h, u - u_h) \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h).
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&= a(u - u_h, u - v_h) + \cancelto{0}{a(u - u_h, v_h - u_h)} \\
&\leq C_b \|e_h\|_{H^1(\Omega)} \|u - v_h\|_{H^1(\Omega)}
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$\Rightarrow$

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch \|f\|_{L^2(\Omega)}.$$

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- Apply coercivity and boundedness of $a(\cdot, \cdot)$.
- Choose $v_h = I_h u \in V_h$ and apply interpolation estimate.
- Divide by $\|e_h\|_{H^1(\Omega)}$ and use stability estimate $\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$.

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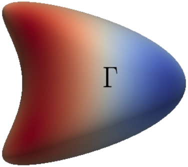
Strong Problem Formulation



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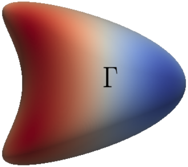

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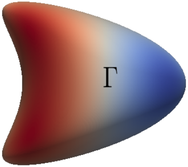

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Integration by parts formula on surfaces:

$$\int_\Gamma v \nabla_\Gamma \cdot \underline{w} \; dA = - \int_\Gamma \left(\underline{w} \cdot \nabla_\Gamma v + v \underline{w} \cdot \kappa \right) \; dA + \int_{\partial\Gamma} v \underline{w} \cdot \mu \; ds$$

where μ: *outer co-normal* of Γ on $\partial\Gamma$, $\kappa = \alpha\nu$: *mean curvature vector*, ν: *outward unit normal*.

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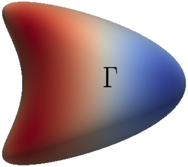

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$$- \cancelto{0 \text{ (no boundary)}}{\int_{\partial\Gamma} v \nabla_\Gamma u \cdot \mu \; ds}.$$

Weak Problem Formulation

Sobolev spaces:

$$H^m(\Gamma) := \{u \in L^2(\Gamma) \ : \nabla_\Gamma^\alpha u \in L^2(\Gamma) \ \forall |\alpha| \leq m\}$$

with corresponding norm

$$\|u\|_{H^m(\Gamma)} := \left(\sum_{|\alpha| \leq m} \|\nabla_\Gamma^\alpha u\|^2_{L^2(\Gamma)} \right)^{1/2} .$$

Problem ($\mathbf{P}_\Gamma$): Find $u \in V := H^1(\Gamma)$ such that

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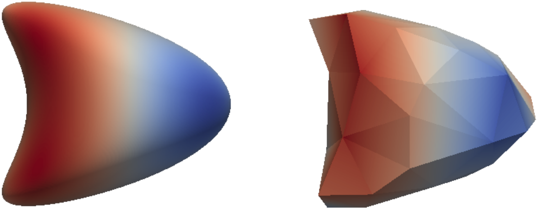
Theorem (Aubin 1982)

If $f \in L^2(\Gamma)$ then there is a unique weak solution $u \in V$ to $(\mathbf{P}_\Gamma)$ which satisfies

$$\|u\|_{H^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}.$$

Triangulated Surfaces

- Γ is approximated by a polyhedral surface Γ_h composed of planar triangles K_h.
- The vertices sit on $\Gamma \quad \Rightarrow \quad \Gamma_h$ is its linear interpolation.
- Triangulate Γ_h as we have done for Ω in the flat case.



Γ $\qquad$ $\Gamma_h = \bigcup_{K_h \in \mathcal{T}_h} K_h$

FEM Problem

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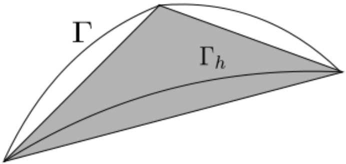
where $a_{\Gamma_h}(u_h, v_h) := \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h + u_h v_h \, dA_h$.

Note that Γ_h is not a subset of Γ. So problems $(\mathbf{P}_\Gamma)$ and $(\mathbf{P}_{\Gamma_h})$ live on different domains!



$\Gamma_h \not\subset \Gamma$

Surface FEM - Convergence

Want to show $\|u - u_h\|_{H^1(\Gamma)} \leq Ch\|f\|_{L^2(\Gamma)}$ as $h \to 0$.

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<u>But</u>

$$a_\Gamma(u - u_h, v_h) \leq Ch^2\|f\|_{L^2(\Gamma)}.$$

$\Rightarrow$

$$\|u - u_h\|_{H^1(\Gamma)} \leq Ch\|f\|_{L^2(\Gamma)} + Ch^2\|f\|_{L^2(\Gamma)} \text{``} \leq \text{''} Ch\|f\|_{L^2(\Gamma)}.$$

Convergence Statement

Theorem

Let $u \in H^2(\Gamma)$ and $u_h \in V_h$ denote the solutions to $(\mathbf{P}_\Gamma)$ and $(\mathbf{P}_{\Gamma_h})$, respectively. Then

$$\|u - u_h\|_{H^1(\Gamma)} \leq Ch\|f\|_{L^2(\Gamma)},$$
$$\|u - u_h\|_{L^2(\Gamma)} \leq Ch^2\|f\|_{L^2(\Gamma)}.$$

Outline

Test Problem on Unit Sphere

Solve:

$$-\Delta_\Gamma u + u = f$$

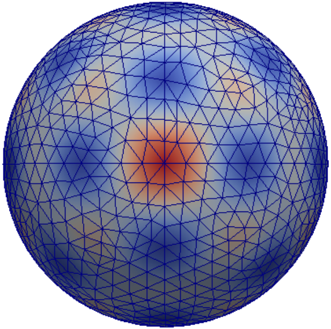
on the unit sphere

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}.$$

The right-hand side f is chosen such that

$$u(x_1, x_2, x_3) = \cos(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3)$$

is the exact solution.



EOCs for Sphere Test Problem

Elements	h	L_2-error	L_2-eoc	DG-error	$H1$-eoc
623	0.223929	0.171459		5.07662	
2528	0.112141	0.0528817	1.70	2.64273	0.94
10112	0.0560925	0.0146074	1.86	1.3151	1.01
40448	0.028049	0.00378277	1.95	0.653612	1.01
161792	0.0140249	0.000957472	1.98	0.325961	1.00
647168	0.00701247	0.000240483	1.99	0.162822	1.00

Test Problem on Dziuk Surface

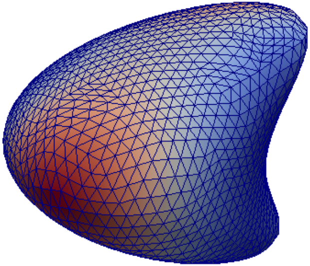
Dziuk surface

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ (x_1 - x_3^2)^2 + x_2^2 + x_3^2 = 1\}.$$

The right-hand side f is chosen such that

$$u(x) = x_1 x_2$$

is the exact solution.



EOCs for Dziuk Surface

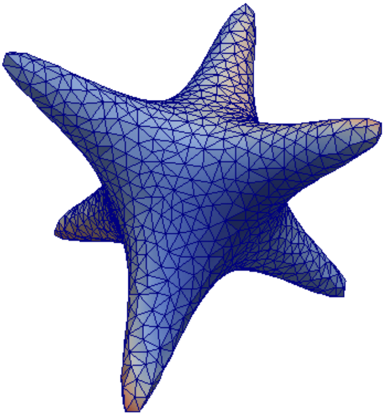
Elements	h	L_2-error	L_2-eoc	$H1$-error	$H1$-eoc
92	0.704521	0.243493		0.894504	
368	0.353599	0.0842372	1.53	0.490805	0.87
1472	0.176993	0.0268596	1.65	0.263808	0.90
5888	0.0885231	0.00637826	2.07	0.135162	0.97
23552	0.0442651	0.00171047	1.90	0.0685366	0.98
94208	0.022133	0.00041636	2.04	0.0343677	1.00
376832	0.0110666	0.00010427	2.00	0.0171891	1.00
1507328	0.0055333	2.60734e-05	2.00	0.0085934	1.00

Test Problem on Enzensberger-Stern Surface

Enzensberger-Stern surface

$$\Gamma = \{x \in \mathbb{R}^3 \; : \; 400(x^2y^2 + y^2z^2 + x^2z^2) - (1 - x^2 - y^2 - z^2)^3 - 40 = 0.\}$$

The right-hand side f is again chosen such that $u(x) = x_1 x_2$ is the exact solution.



EOCs for Enzensberger-Stern Surface

Elements	h	L_2-error	L_2-eoc	DG-error	DG-eoc
2358	0.163789	0.476777		0.998066	
9432	0.0817973	0.175293	1.44	0.472241	1.08
37728	0.040885	0.0160606	3.45	0.150144	1.65
150912	0.0204411	0.00139698	3.52	0.0703901	1.09
603648	0.0102204	0.000338462	2.04	0.0347345	1.02
2414592	0.00511	7.86713e-05	2.10	0.0172348	1.01

Thanks for your attention!

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