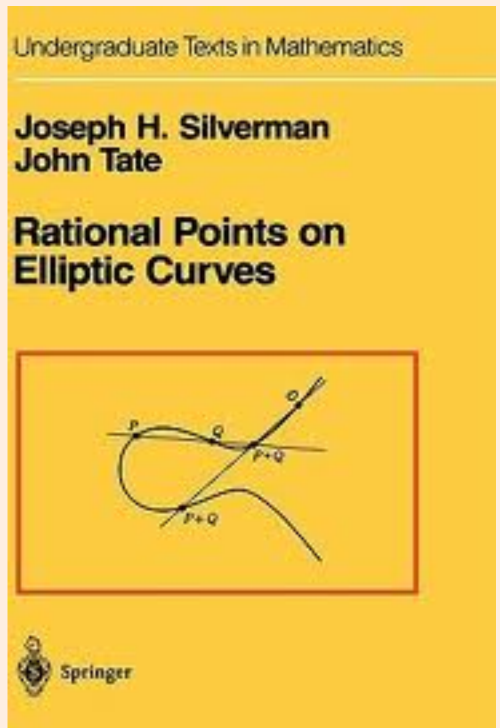
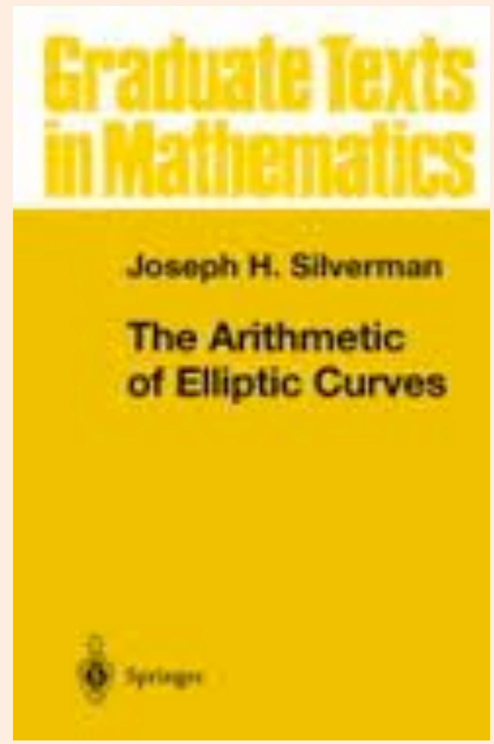
Elliptic curves and the Birch and Swinnerton-Dyer Conjecture (B.S.D.)

Elliptic curves and the Birch and Swinnerton-Dyer Conjecture

MAIN REFERENCES





Elliptic curves over $\mathbb{Q}$

A curve given by an equation of the form :

$$E \; : y^2 = x^3 + ax + b \qquad a,b \in \mathbb{Q}$$

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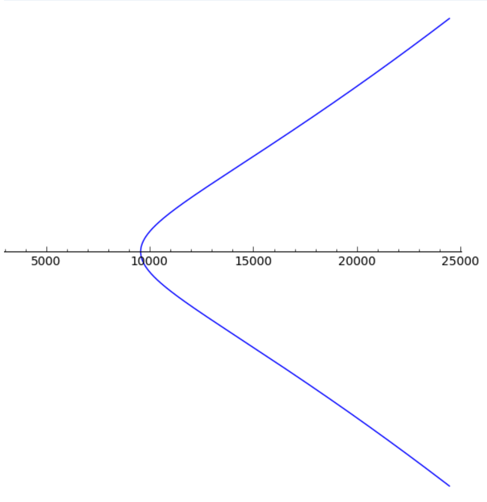

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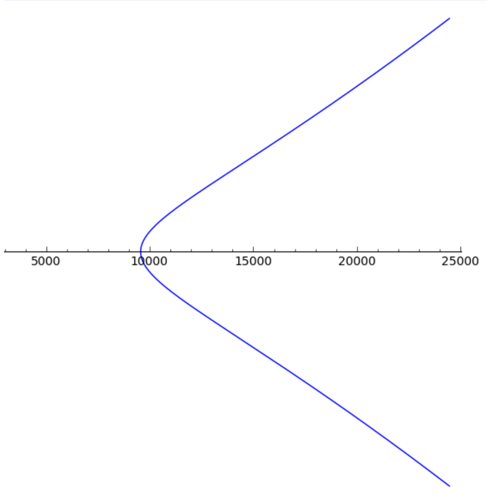

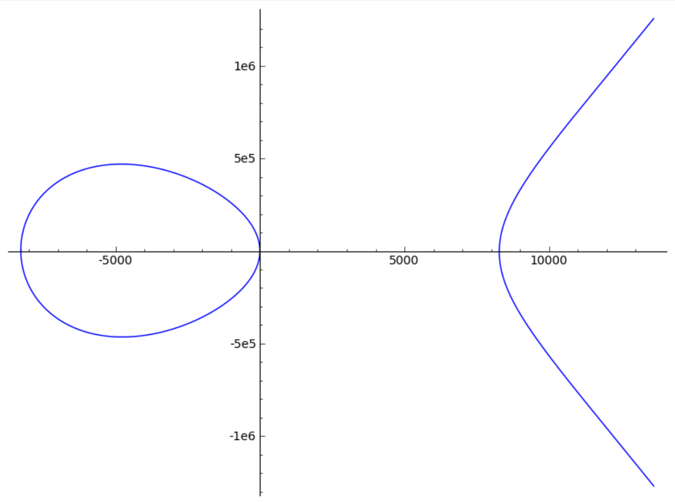

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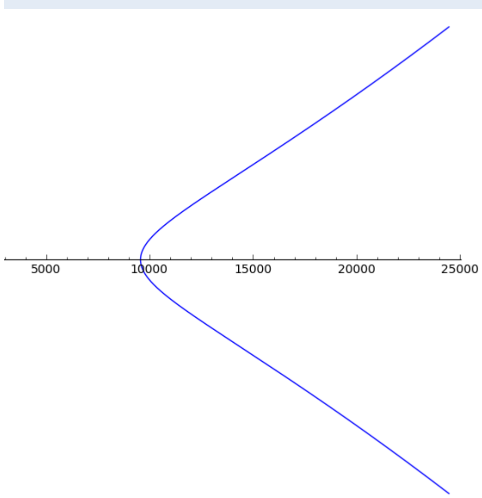
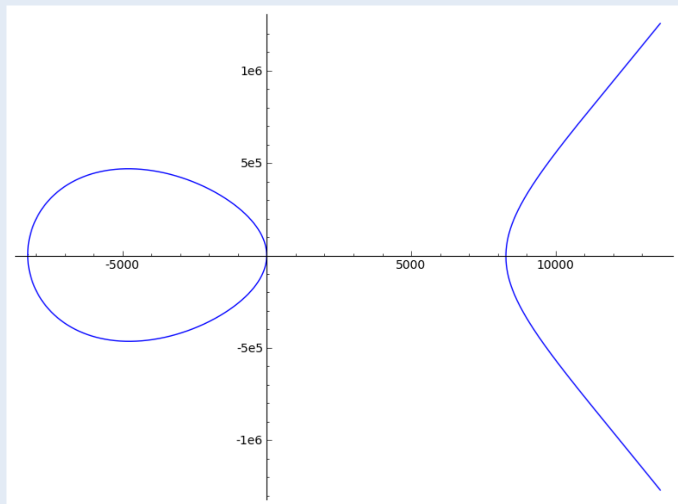
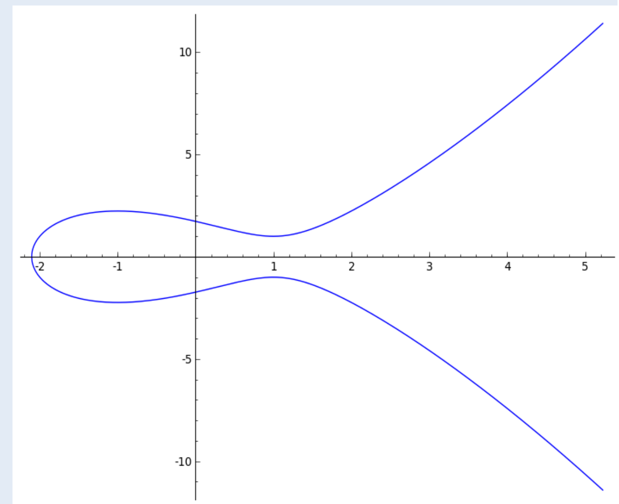






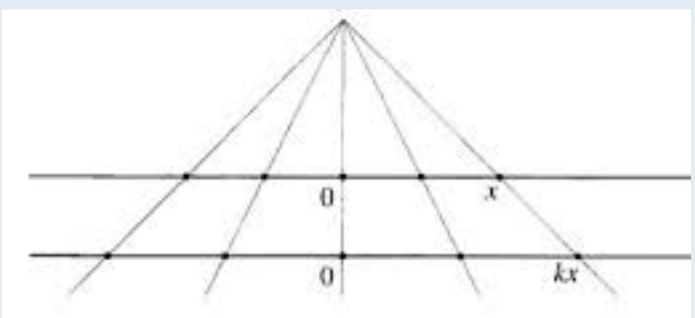
Elliptic curves over $\mathbb{Q}$

$E(\mathbb{Q})$ is the set :

$$\{(x,y)\in\mathbb{Q}^2 : y^2 = x^3 + ax + b\}\cup\{\mathcal{O}\}$$

where $\mathcal{O}$ denotes the point at infinity.

$\mathcal{O}$

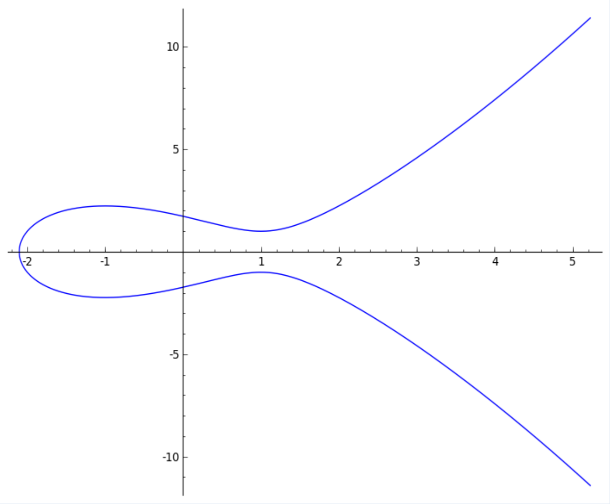


Elliptic curves over $\mathbb{Q}$

A specific example : E1080k1

$$y^2 = x^3 - 3x + 3$$

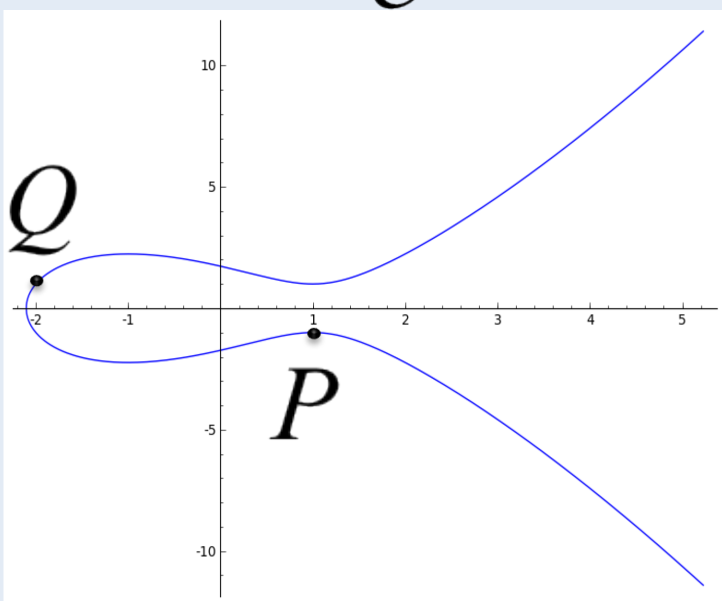
$$\mathcal{O}$$



Points on a curve

$$y^2 = x^3 - 3x + 3$$

$\mathcal{O}$



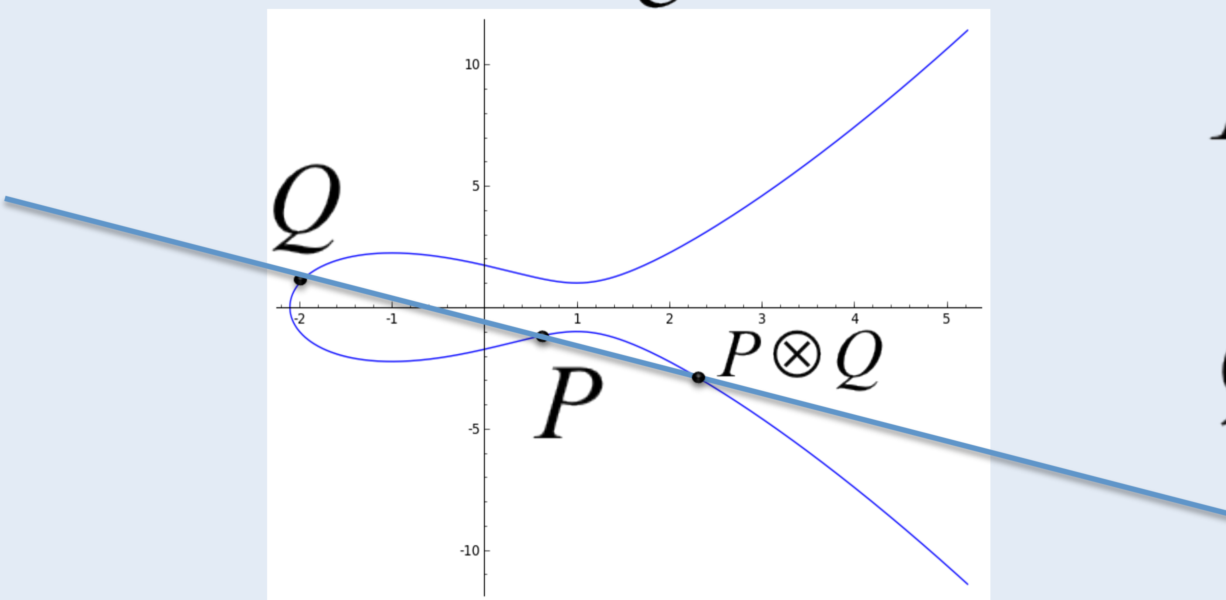
$P(1,-1)$

$Q(-2,1)$

Points on a curve

$$y^2 = x^3 - 3x + 3$$

$\mathcal{O}$



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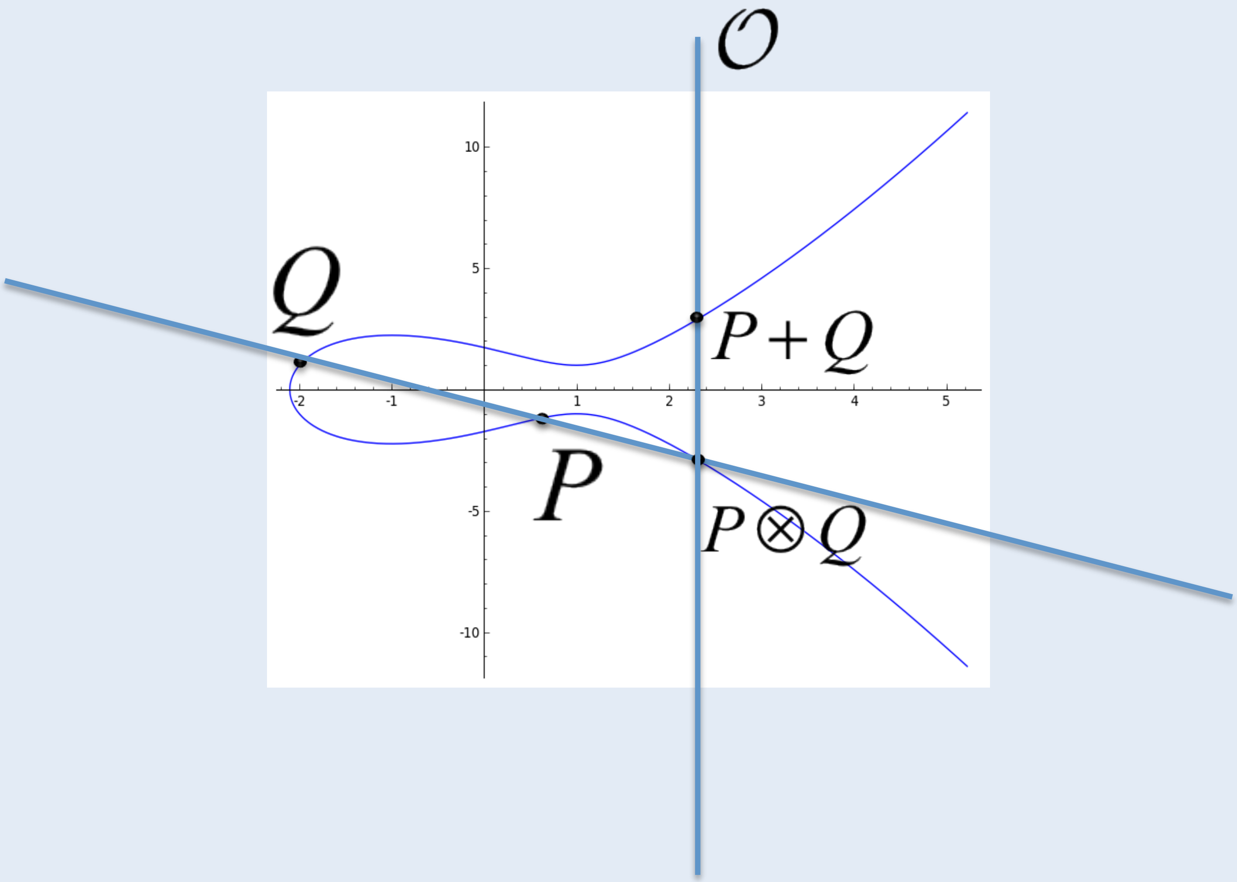

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Points on a curve

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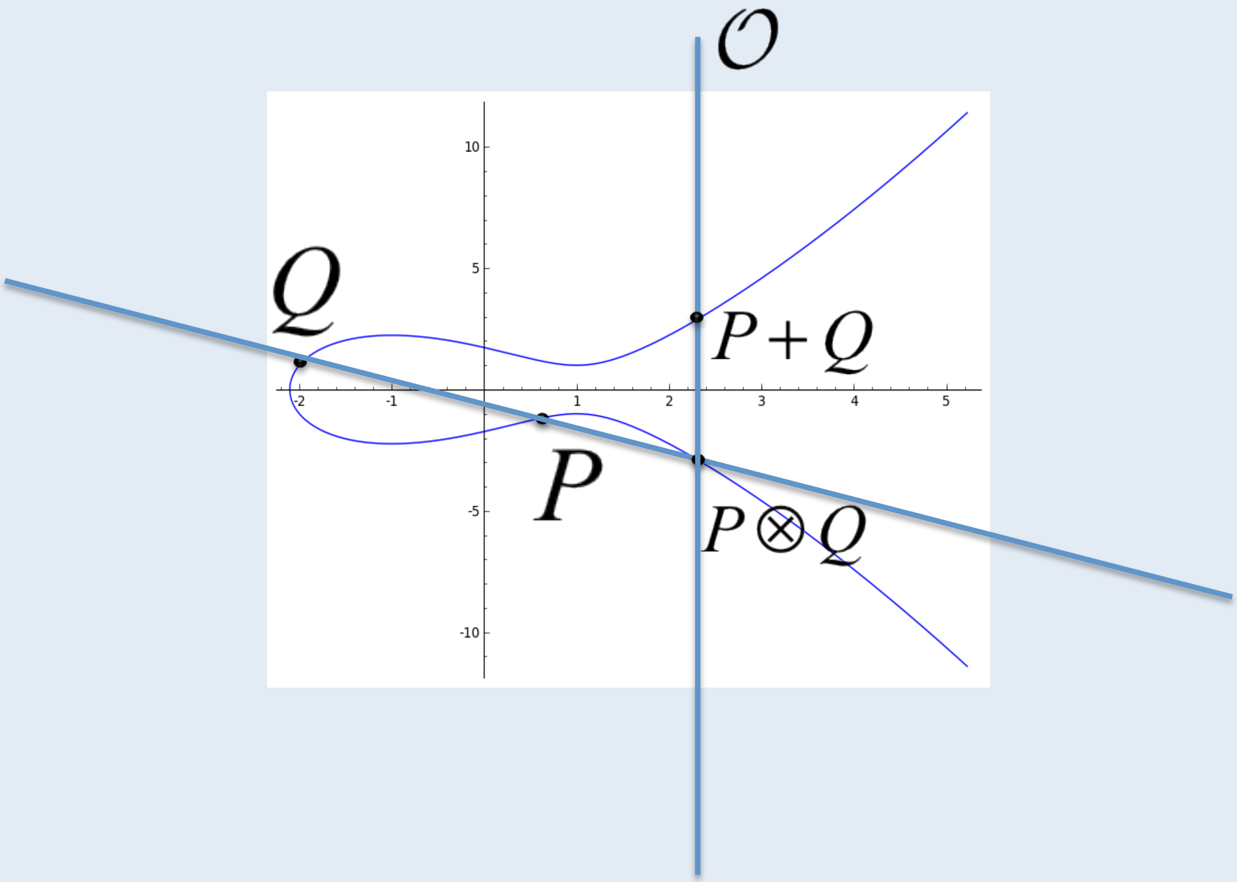


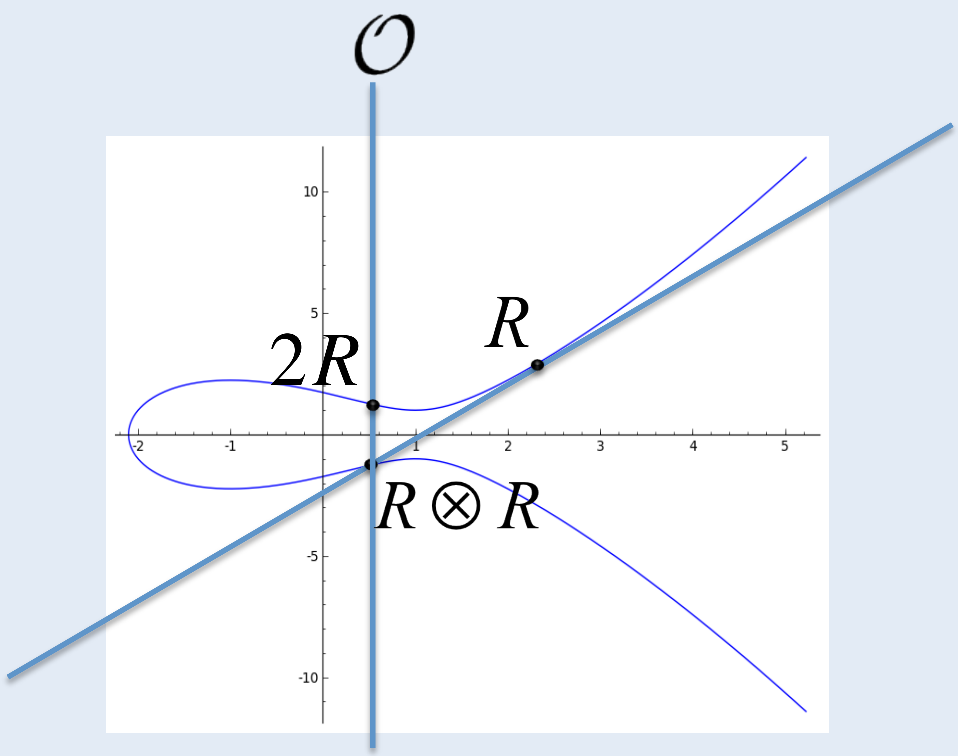
$$P(1,-1)$$

$$Q(-2,1)$$

$$P+Q(\frac{13}{9},\frac{35}{27})$$

Points on a curve

$$y^2 = x^3 - 3x + 3$$

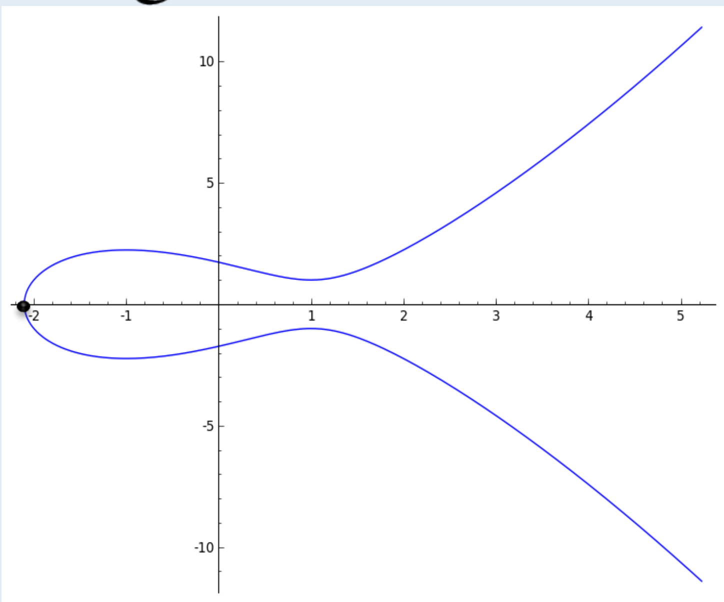


Points on a curve

$$y^2 = x^3 - 3x + 3$$

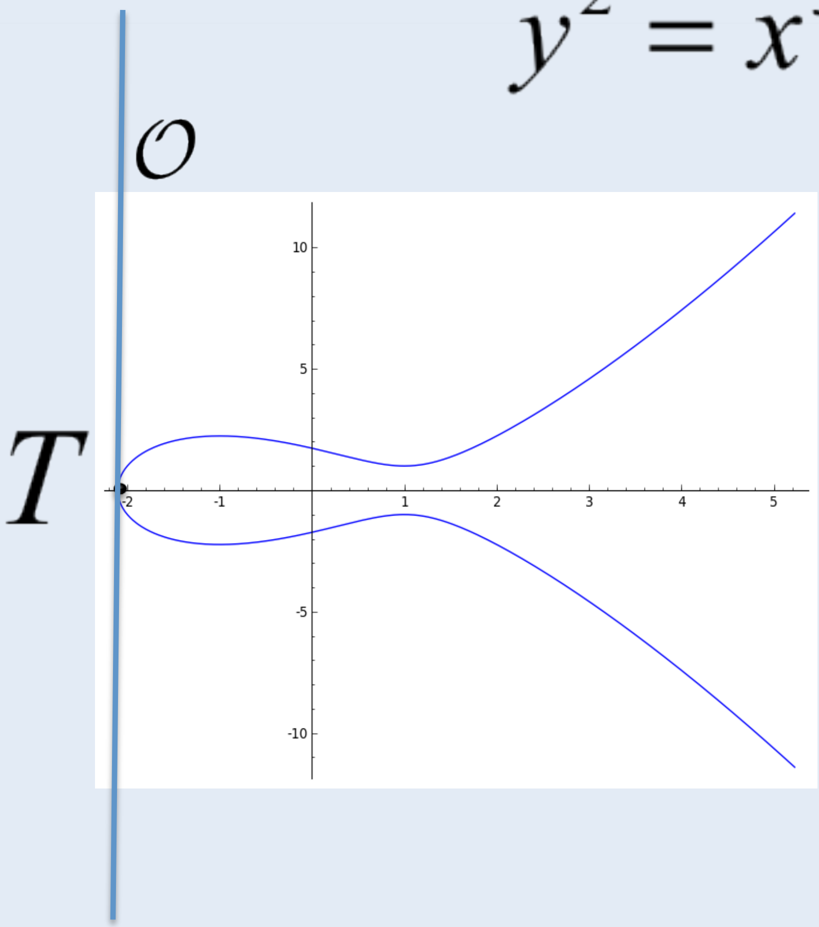
$\mathcal{O}$

T



Points on a curve : Torsion Points

$$y^2 = x^3 - 3x + 3$$

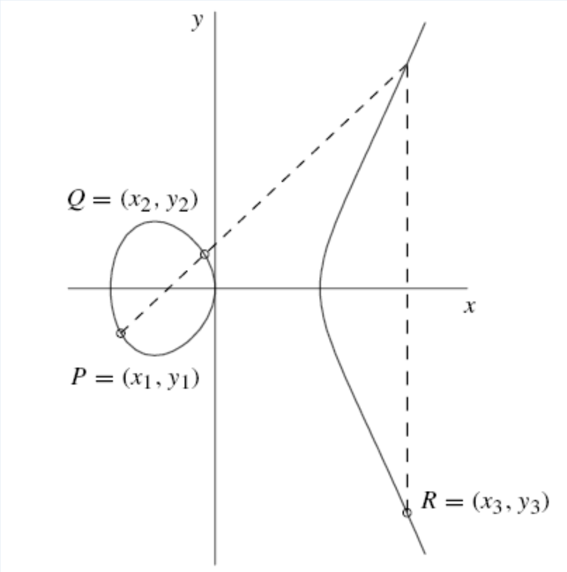


$$T + T = \mathcal{O}$$

T has order 2

Points on a curve : Group Structure

$$\{(x,y)\in\mathbb{Q}^2 : y^2 = x^3 + ax + b\}\cup\{\mathcal{O}\}$$



finitely generated abelian group with identity $\mathcal{O}$

Mordell-Weil group of E over $\mathbb{Q}$

denoted $E(\mathbb{Q})$

Points on a curve : Group Structure

$$E(\mathbb{Q}) \simeq E_{tors} \times \mathbb{Z}^r$$

r = rank of E

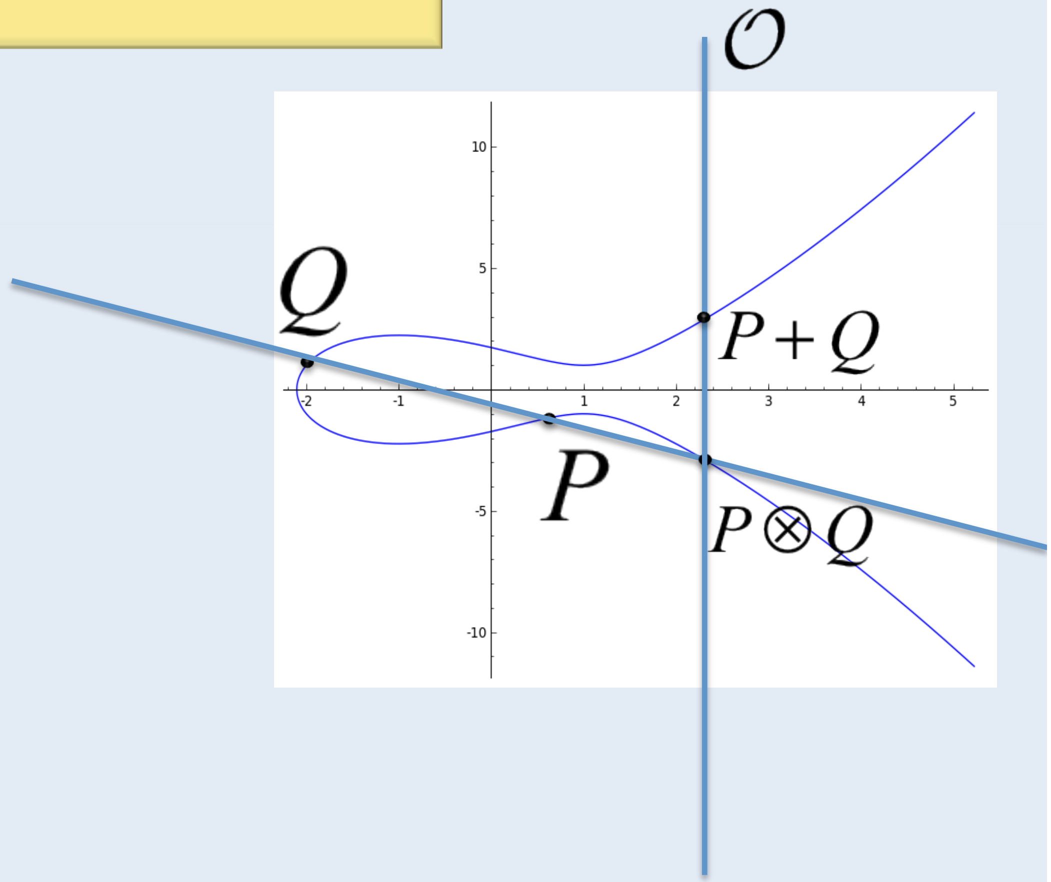
r is finite

Points on a curve : Group Structure

$P(1,-1)$

$Q(-2,1)$

$P+Q(\frac{13}{9},\frac{35}{27})$



sage: P+R
(97/4 , -953/8)
sage: P+P+R
(541/961 , -36359/29791)
sage: P+P+P+R
(-14426/11025 , 2505707/1157625)
sage: P+P+P+P+R
(1474561/674041 , 1454330989/553387661)
sage: P+P+P+P+P+R
(89285857/14531344 , -815124641617/55393483328)

Points on a curve : Height function

Growth of x-coordinate of nP as $n \longrightarrow \infty$

1
-2
13/9
97/4
541/961
-14426/11025
1474561/674041
89285857/14531344
-302582687/8290648809
-534970997642/1270419782641
1579333165209181/393146054623921
792028838279095201/276884829830340900
-581568593115008433779/6749993960755967410 81
41876637094683583095383 8/1425649711854942207996961
1634226998889903175810182691 3/150194810260945416161461140 9
674784339850162676294570167791 37/386201378671996768490296270080 64
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11674095627067036379387449997833409020174/1469049849251009612535759098767836566422 5
161784096240284236105955889403189998997269198 1/164093827491820628246935001614220902101841 21
5262324005913287342303052206496834328469051694441 7/436435684197435889825666728423994908515050755940 84
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16115983483474693875238621239964054464900601516851544973689 18/13324880414299162614293353464101485860583843553491774387768 81
1510359085566510509804795486590234511030332005949918103373335657761/158534242391200742507265622713246191282938909105811495036262488 41

Points on a curve : Height function

Growth of x-coordinate of nP as $n \to \infty$

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1579333165209181/393146054623921
792028838279095201/276884829830340900
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52623240059132873423030522064968343284690516944417/43643568419743588982566672842399490851505075594084
-10337897268963445482495959907423248137859970977589121187/4913926622576435575040721307165339723211630353326217809
1611598348347469387523862123996405446490060151685154497368918/13324880414299162614293353464101485860583843555349177438776881
1510359085566510509804795486590234511030332005949918103373335657761/15853424239120074250726562271324619128293890910581149503626248841

measure the growth

$$h \quad : E(\mathbb{Q}) \,/\, E_{tors} \to [0, \infty)$$

Points on a curve : Height function

height of points given by 2 generators : nP + mQ

Table 1: Height(aP+bQ)

*	0	1	2	3	4	5	6	7
-10	32.7	32.0	32.2	33.5	35.6	38.8	42.8	47.9
-9	26.5	25.9	26.3	27.6	29.9	33.1	37.3	42.5
-8	20.9	20.5	21.0	22.4	24.8	28.2	32.5	37.8
-7	16.0	15.7	16.3	17.8	20.4	23.8	28.3	33.6
-6	11.8	11.5	12.3	14.0	16.6	20.2	24.7	30.2
-5	8.17	8.06	8.91	10.7	13.5	17.2	21.8	27.4
-4	5.23	5.24	6.20	8.12	11.0	14.8	19.6	25.3
-3	2.94	3.07	4.15	6.18	9.17	13.1	18.0	23.8
-2	1.31	1.55	2.75	4.90	8.00	12.1	17.1	23.0
-1	0.327	0.687	2.00	4.27	7.48	11.7	16.8	22.9
0	0.000	0.477	1.91	4.29	7.62	11.9	17.2	23.3
1	0.327	0.921	2.47	4.97	8.42	12.8	18.2	24.5
2	1.31	2.02	3.68	6.30	9.88	14.4	19.9	26.3
3	2.94	3.77	5.55	8.28	12.0	16.6	22.2	28.8
4	5.23	6.18	8.08	10.9	14.7	19.5	25.2	31.9
5	8.17	9.23	11.2	14.2	18.2	23.0	28.8	35.6
6	11.8	13.0	15.1	18.2	22.2	27.2	33.1	40.1
7	16.0	17.3	19.6	22.8	26.9	32.1	38.1	45.1
8	20.9	22.3	24.7	28.0	32.3	37.5	43.7	50.8
9	26.5	28.0	30.5	33.9	38.3	43.7	50.0	57.2
10	32.7	34.4	36.9	40.5	45.0	50.5	56.9	64.2

Points on a curve : Height function

$$h \quad : E(\mathbb{Q}) / E_{tors} \rightarrow [0, \infty)$$

positive definite quadratic form

real vector space $\quad E(\mathbb{Q}) \otimes \mathbb{R}$

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Points on a curve : Regulator

$$h \quad : E(\mathbb{Q}) / E_{tors} \to [0,\infty)$$

$$< P,Q > \ := h(P+Q) - h(P) - h(Q)$$

structure of Euclidean Space to $E(\mathbb{Q}) \otimes \mathbb{R}$

$$Reg(E(\mathbb{Q}))$$

number of points up to a certain height

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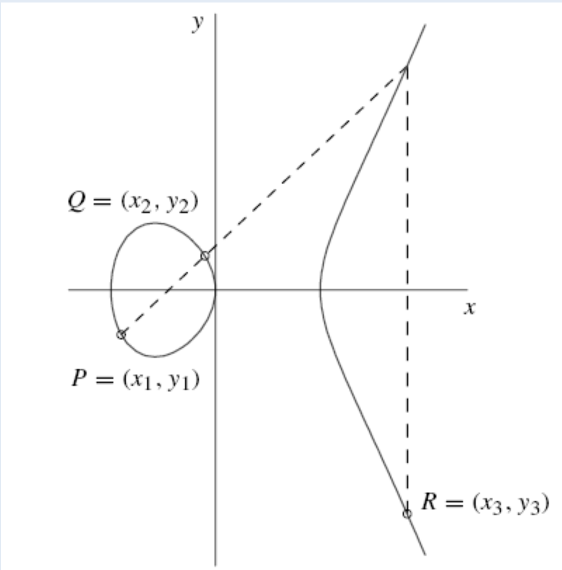


Mordell-Weil group
finitely generated abelian group

$$E(\mathbb{Q}) \simeq E_{tors} \times \mathbb{Z}^r$$

Arithmetic Invariants

$$|E_{tors}|^2$$

$$r$$

$$Reg(E(\mathbb{Q}))$$

Example : E2478g3

$$y^2 = x^3 - 88791220251x - 10183642628382666$$

$r = 1$

$P(31511, 5361297)$

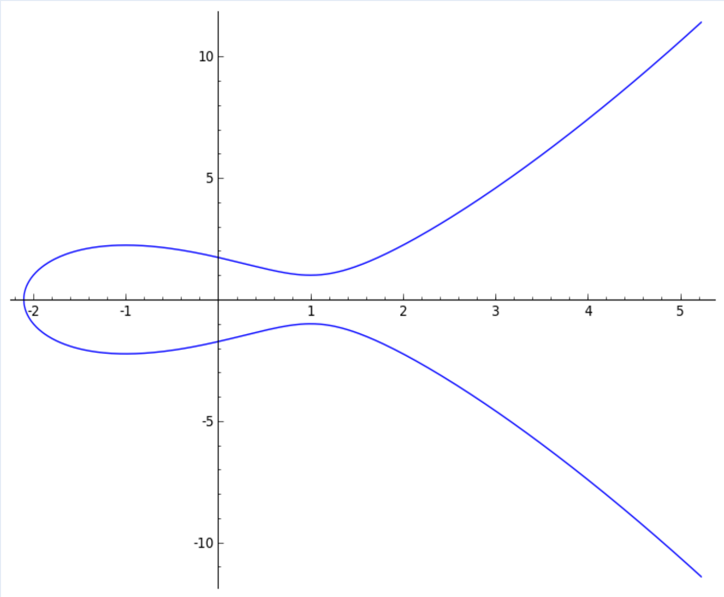
$|E_{tors}| = 2$

$Reg(E(\mathbb{Q}))$

9.905178234307795324557391664963729405700

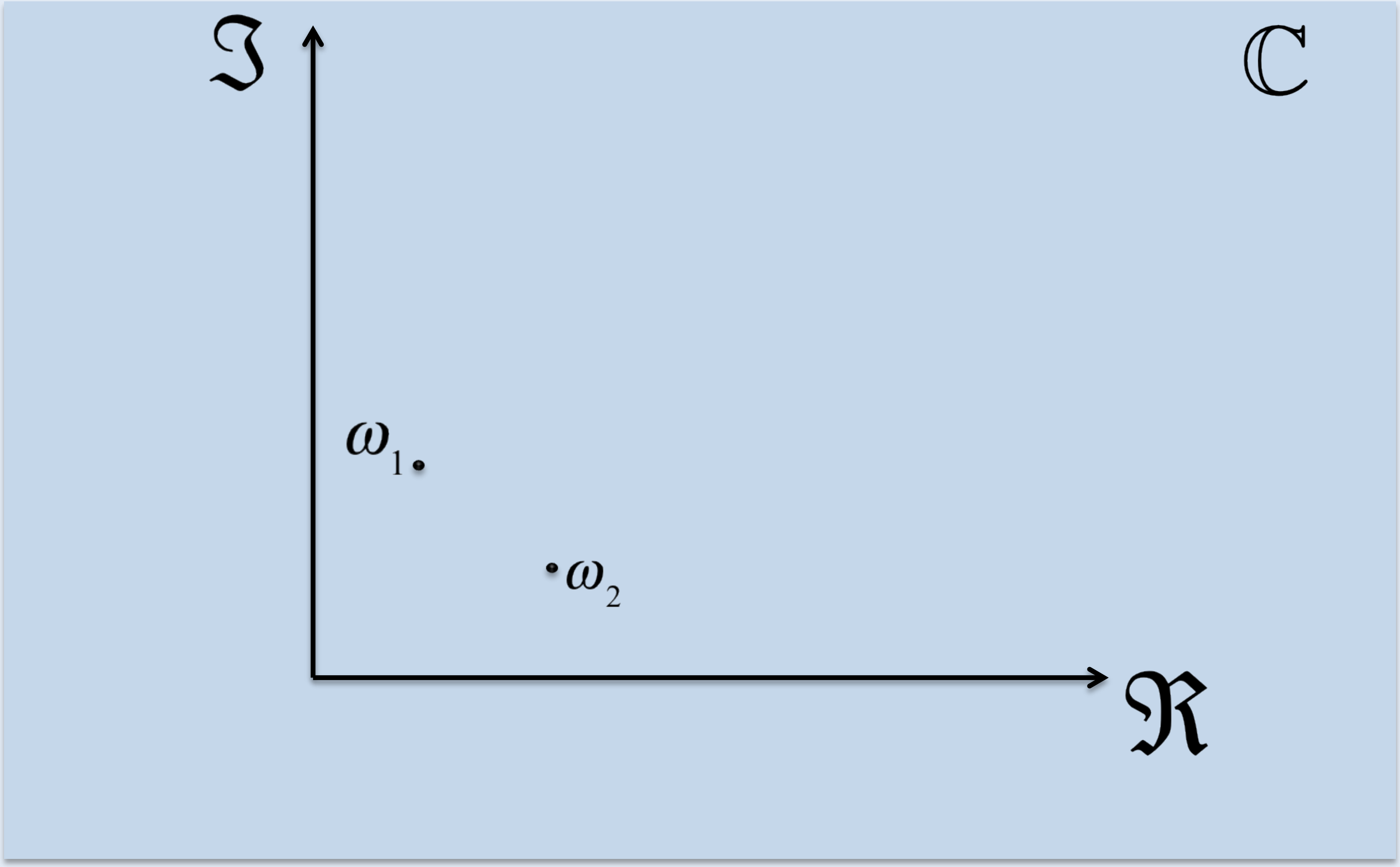
Elliptic curves over $\mathbb{C}$

Over $\mathbb{R}$

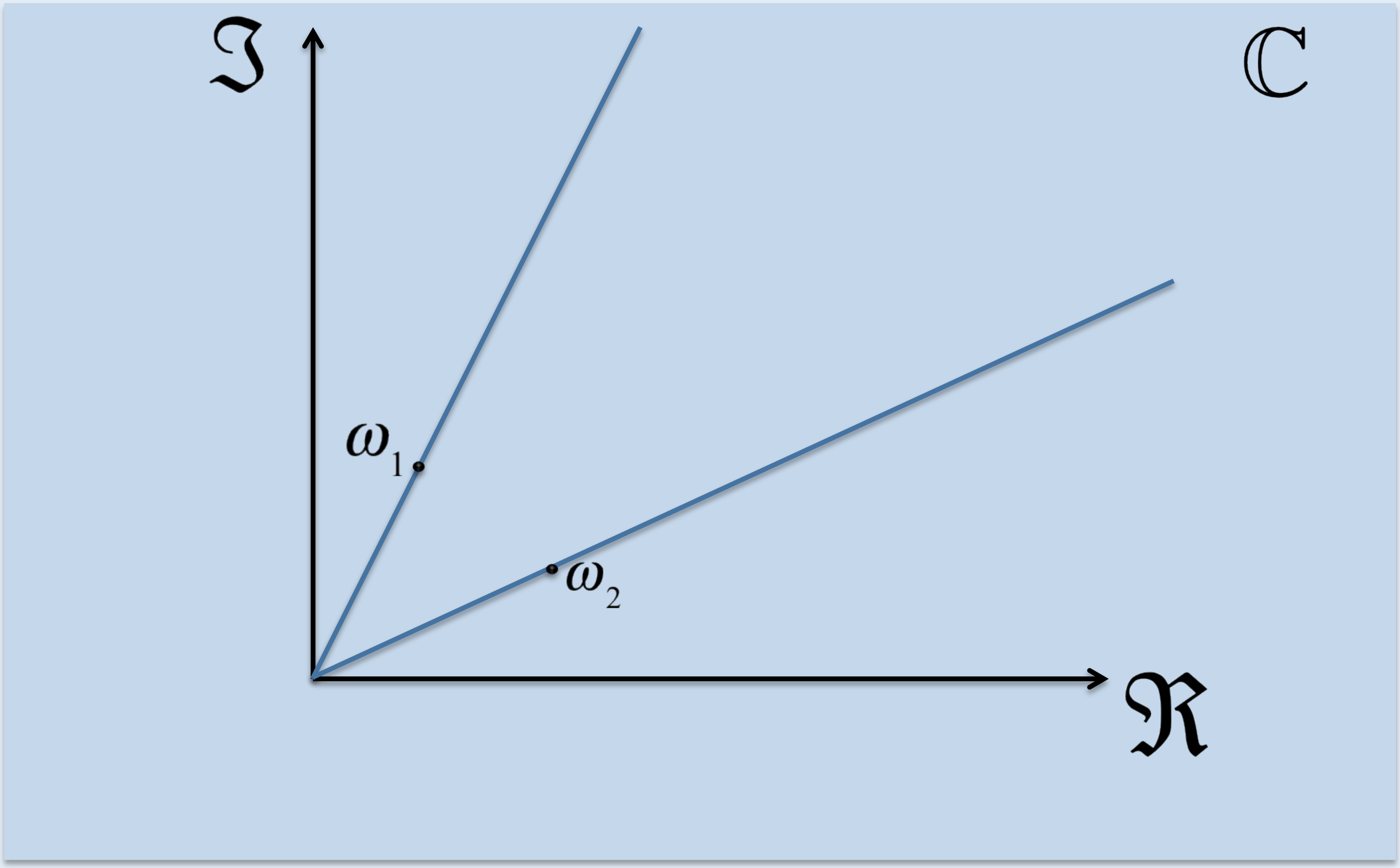


What happens over $\mathbb{C}$

Elliptic curves over $\mathbb{C}$



Elliptic curves over $\mathbb{C}$



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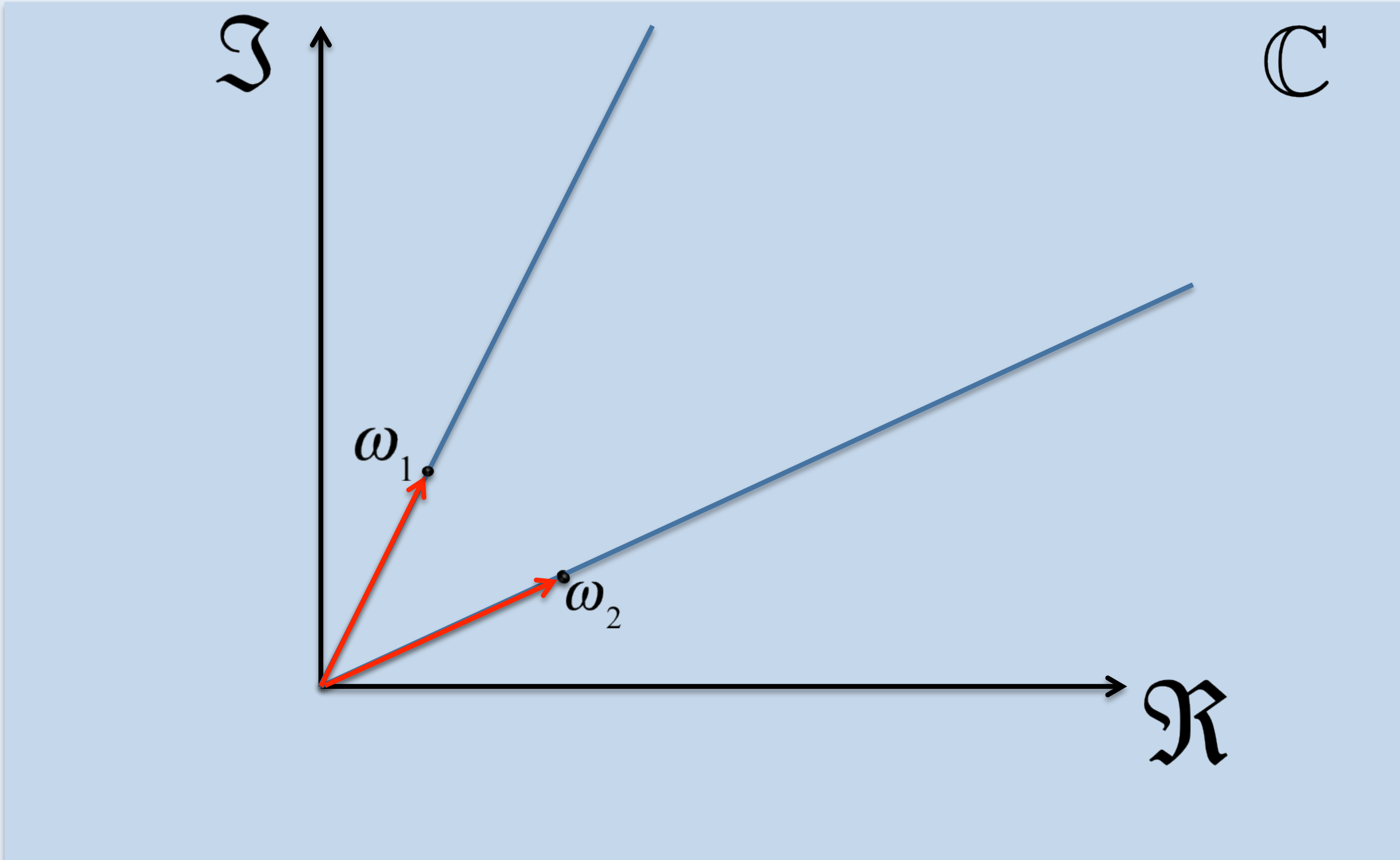

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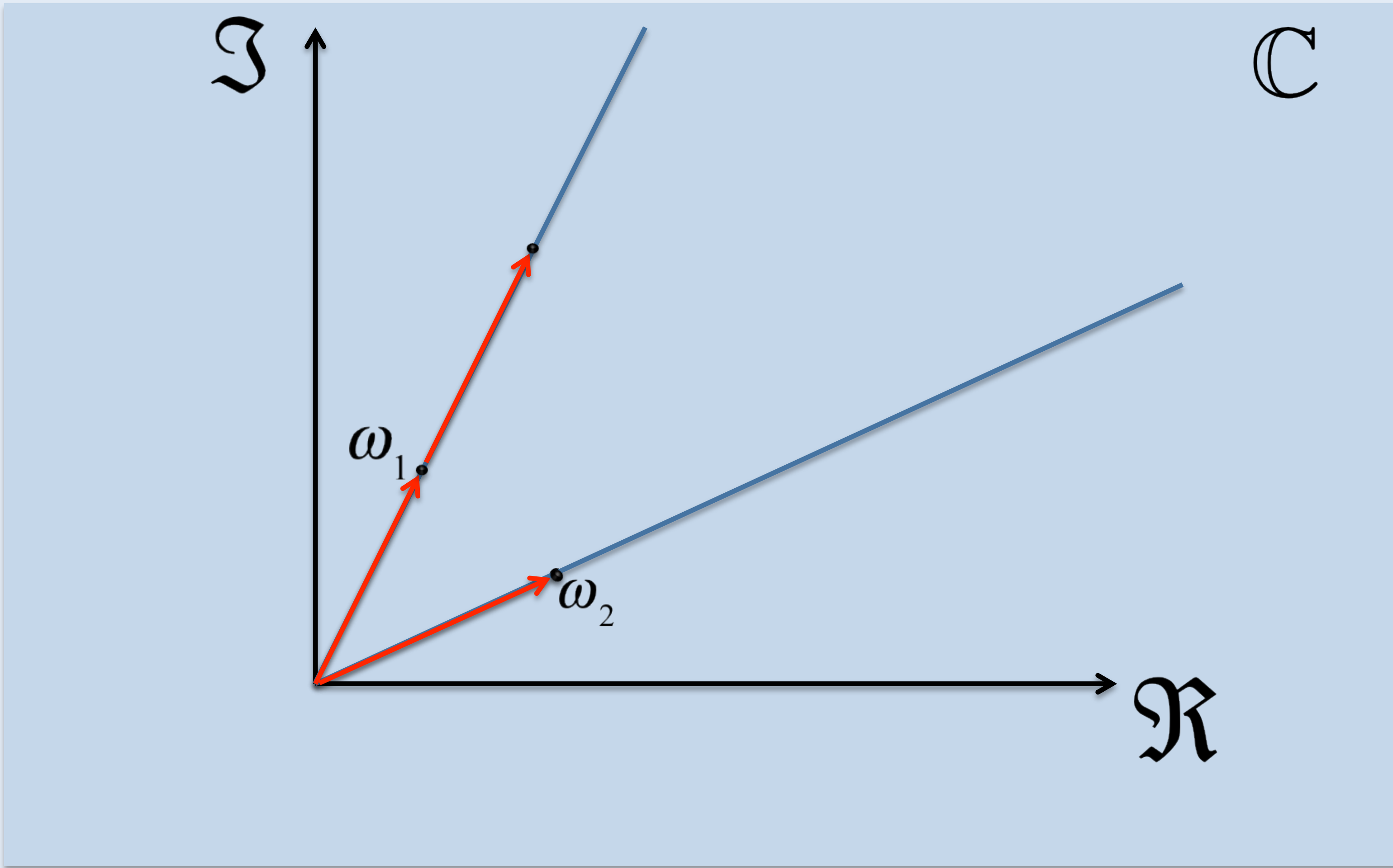

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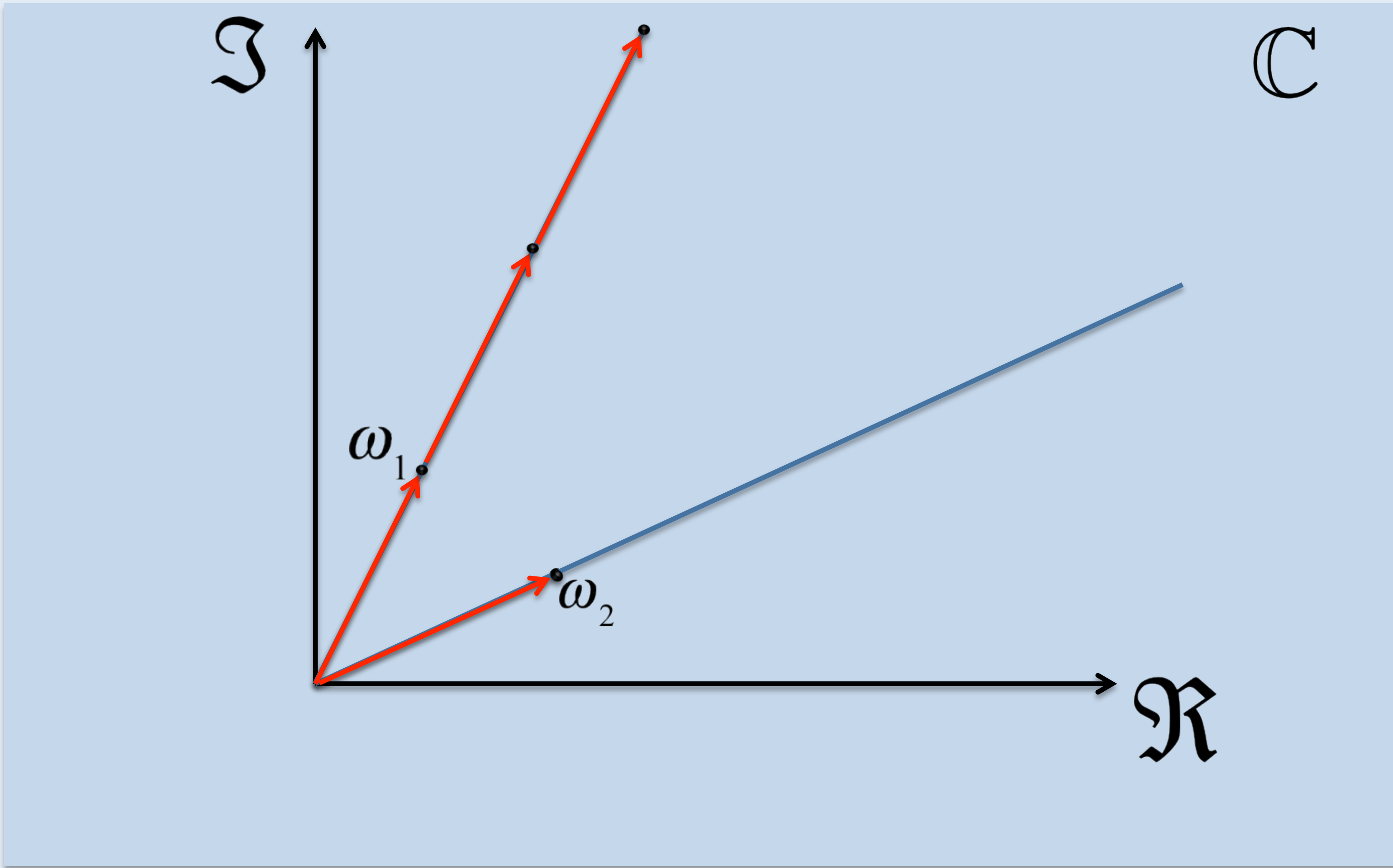

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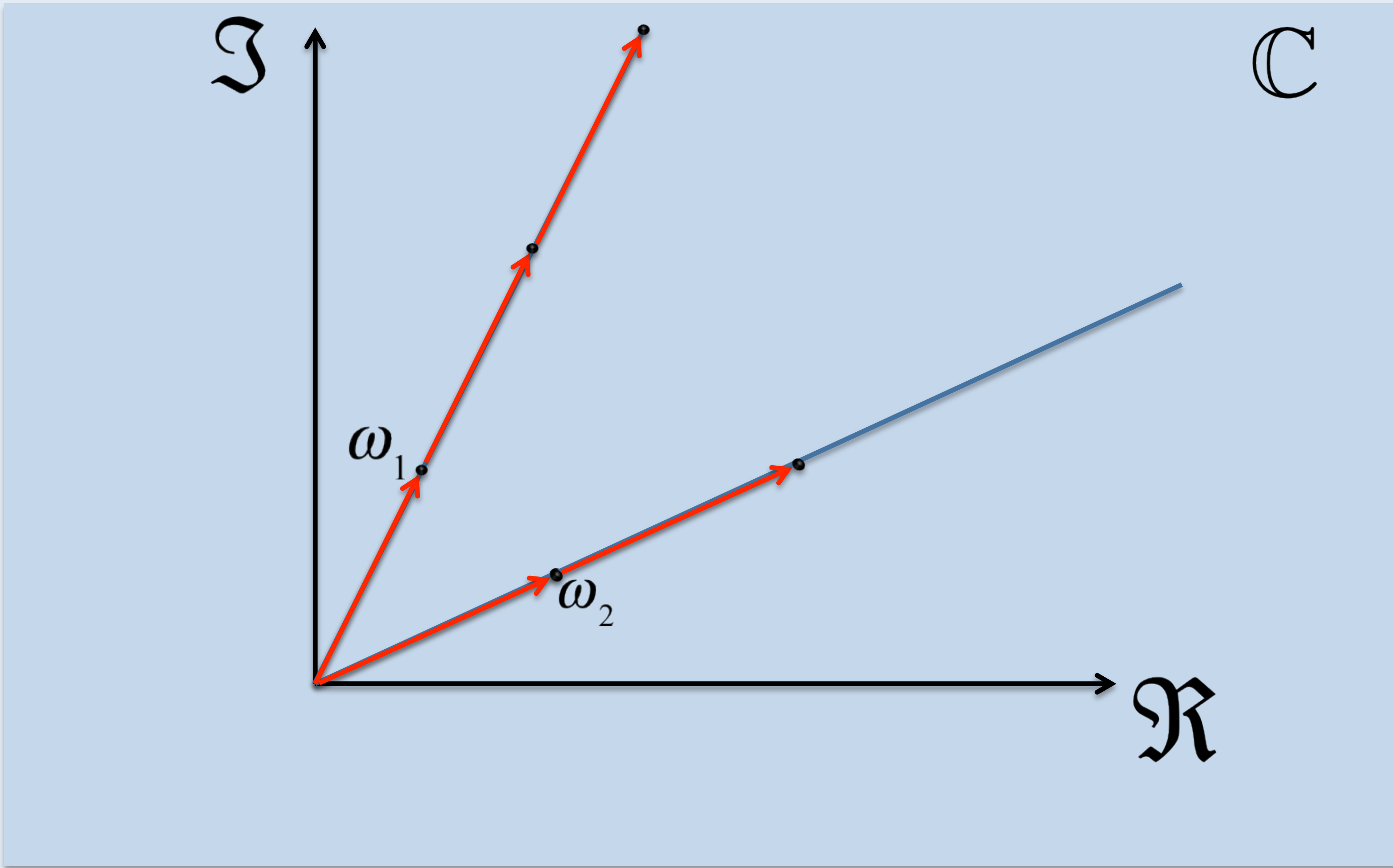

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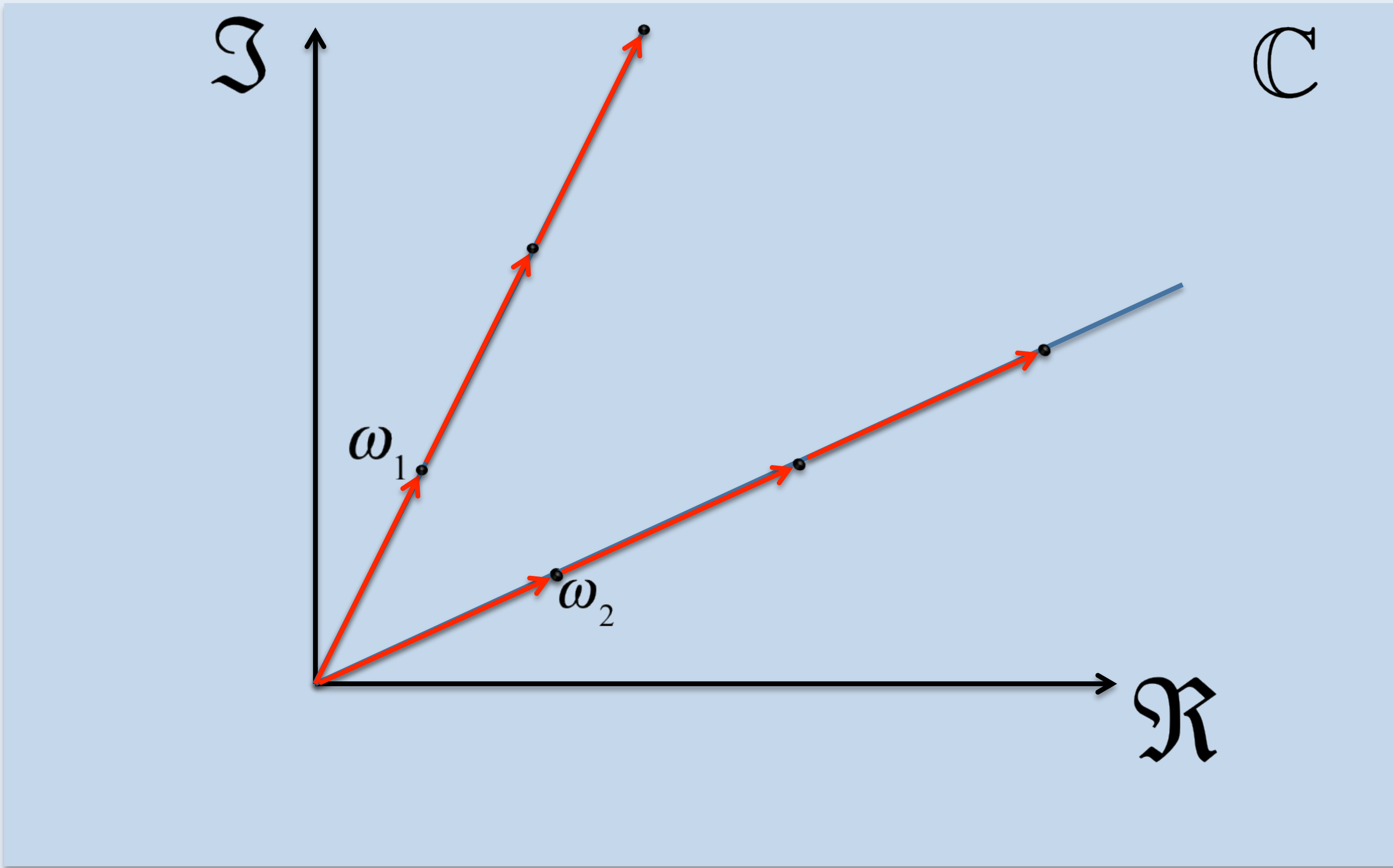

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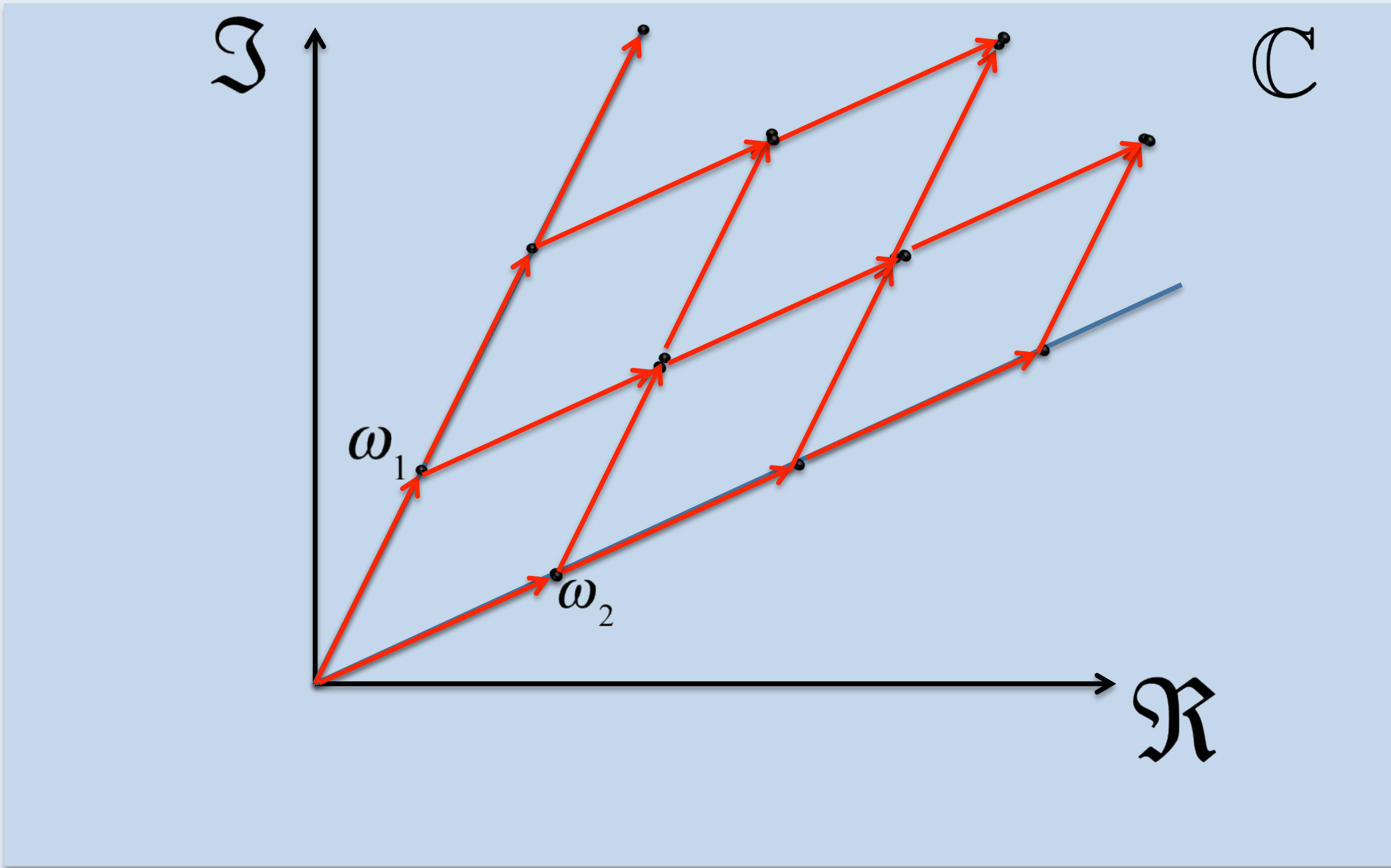

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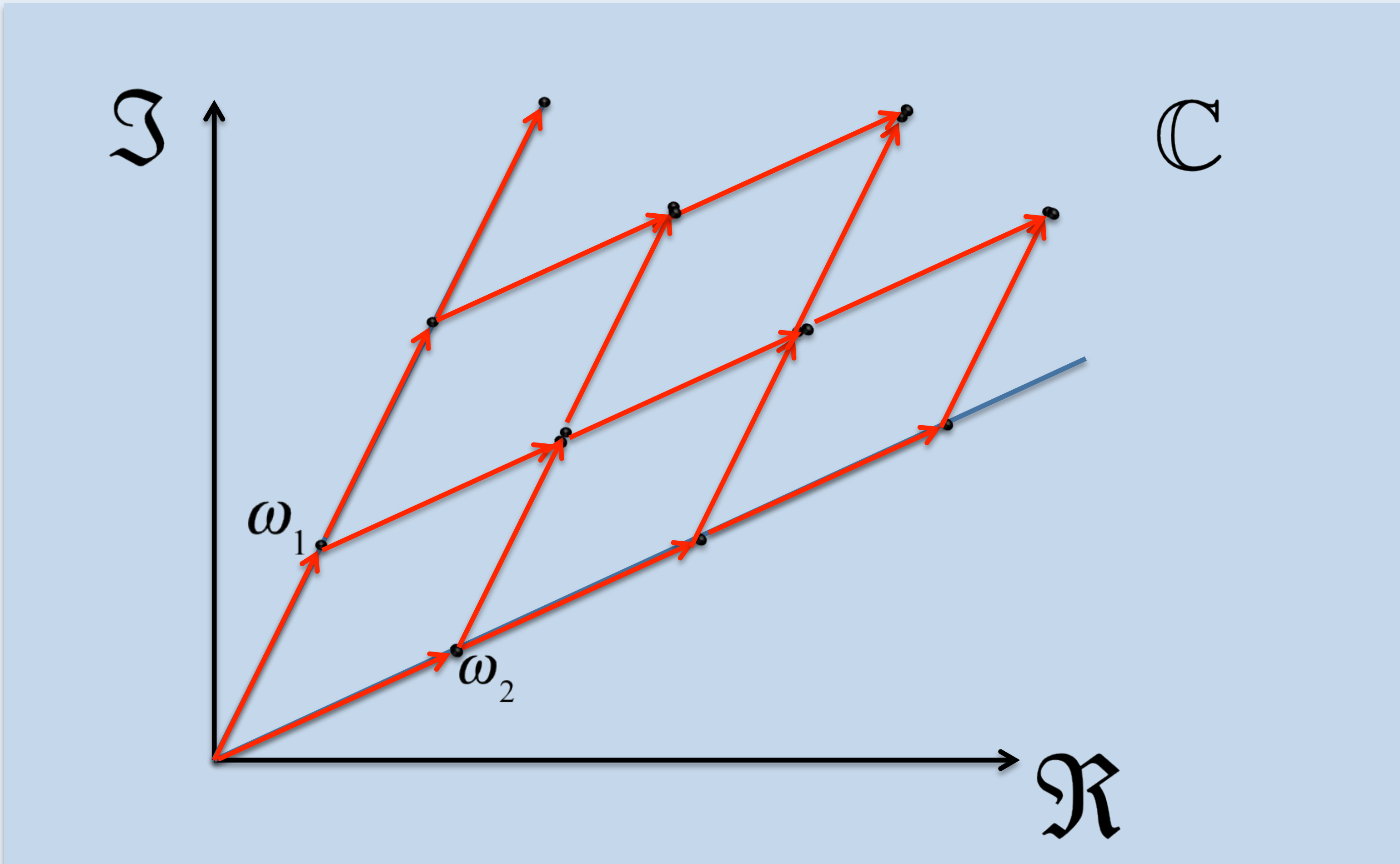
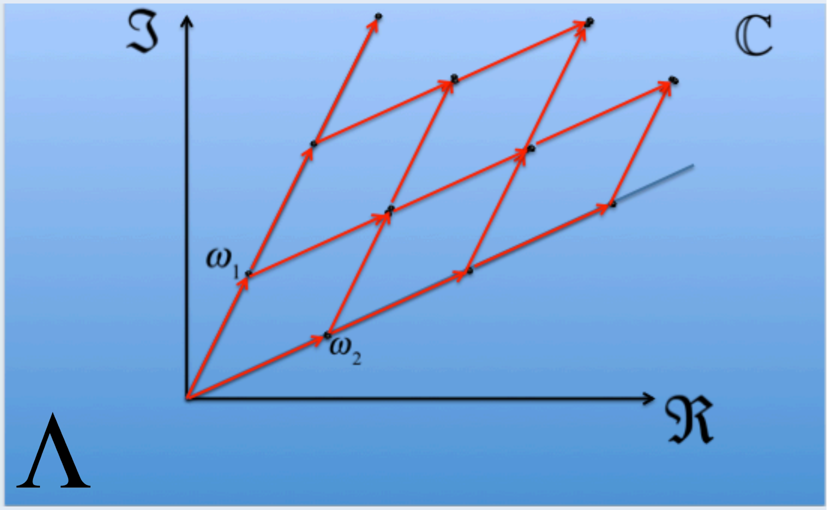


Lattice Λ

Elliptic curves over $\mathbb{C}$

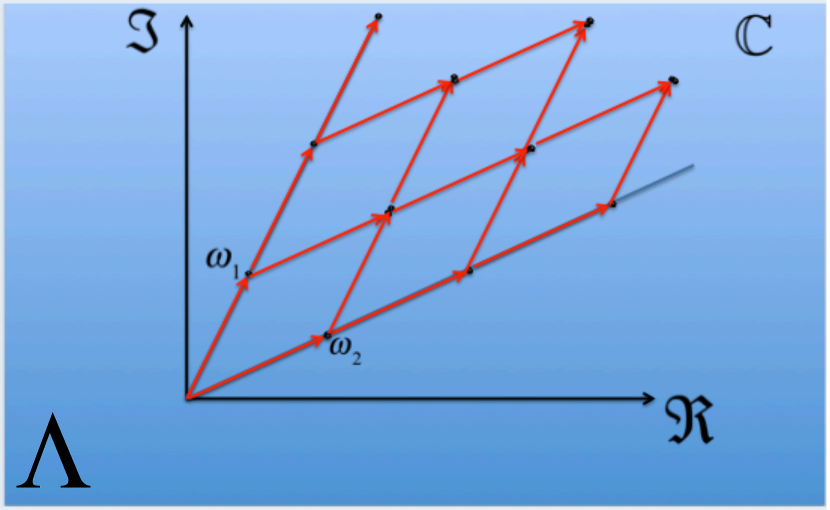


Lattice Λ

Weierstrass p-function

$$p(z,\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Elliptic curves over $\mathbb{C}$



Weierstrass p-function

$$p(z,\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$p(z+\omega) = p(z), \forall z \in \mathbb{C}, \omega \in \Lambda$$

Elliptic curves over $\mathbb{C}$

Up to appropriate scaling

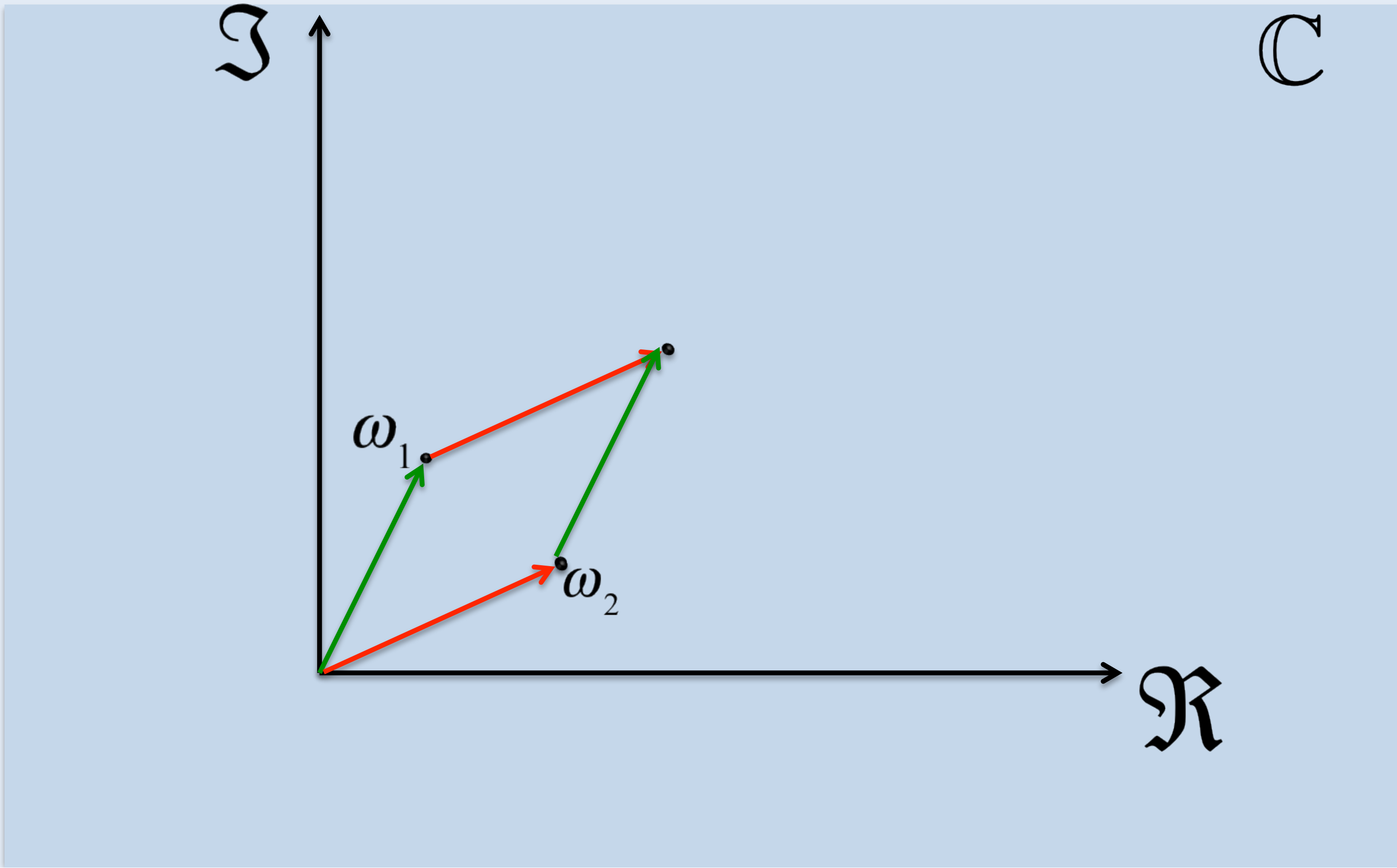
$$\exists \Lambda \subset \mathbb{C}$$

$$\mathbb{C} / \Lambda \simeq E(\mathbb{C})$$

as complex Lie groups

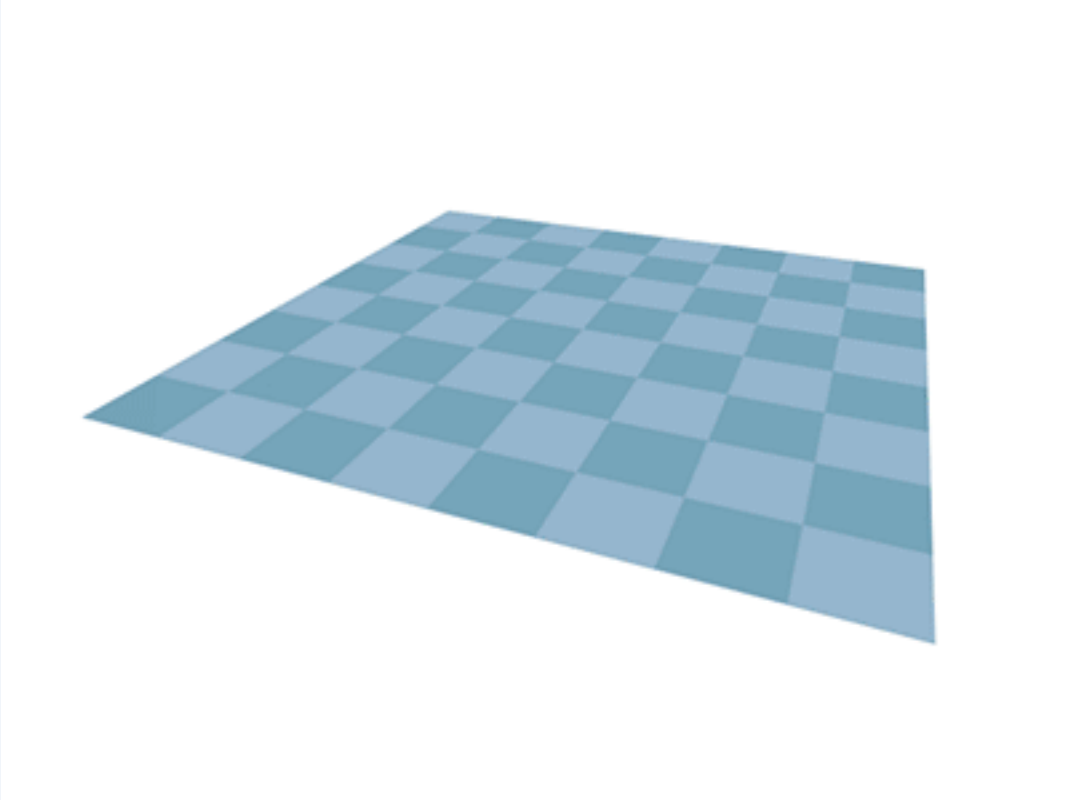
Elliptic curves over $\mathbb{C}$

$$\mathbb{C} / \Lambda$$



Elliptic curves over $\mathbb{C}$

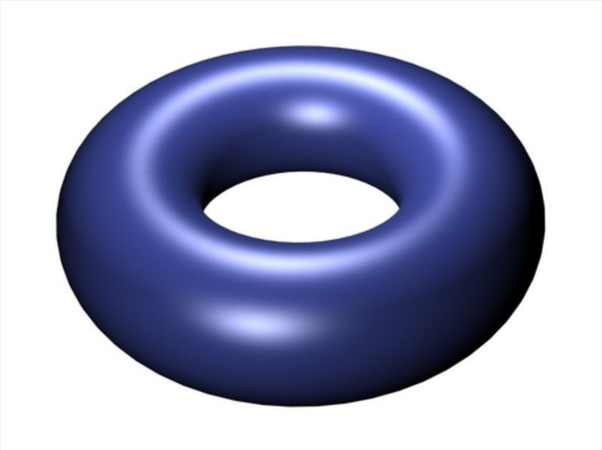
$$\mathbb{C} / \Lambda$$



Elliptic curves over $\mathbb{C}$

$$\mathbb{C}/\Lambda \simeq E(\mathbb{C})$$

$$z \mapsto (p(z,\Lambda), p'(z,\Lambda))$$



period of E ω

Example : E2478g3

$$y^2 = x^3 - 887912220251x - 1018364262838266$$

$r = 1$

$P(31511, 5361297)$

$|E_{tors}| = 2$

$Reg(E(\mathbb{Q}))$

9.9051782343077795324557391664963729405700

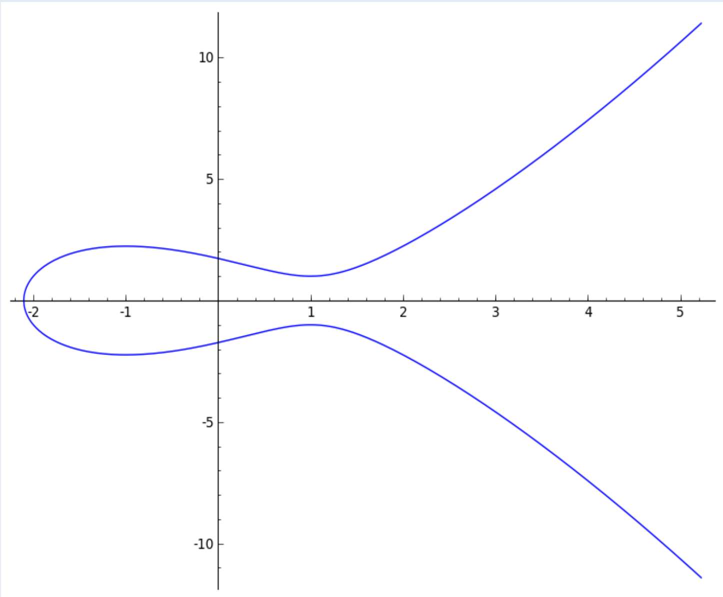
ω

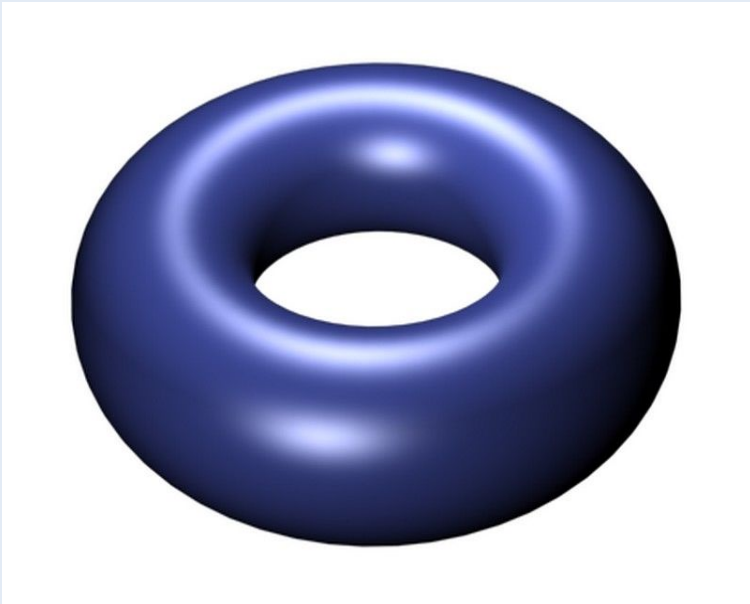
0.0524757070525734

Elliptic curves over $\mathbb{C}$

Over $\mathbb{R}$

What happens over $\mathbb{C}$





What happens locally ?

Elliptic curves mod p

Fix a prime p

$$\tilde{E} : y^2 = x^3 + \tilde{a}x + \tilde{b}$$

$$\tilde{a}, \tilde{b} \in \mathbb{Z} / p\mathbb{Z}$$

$$E(\mathbb{Z} / p\mathbb{Z}) := \{P(x, y) \mid (x, y) \in (\mathbb{Z} / p\mathbb{Z})^2\}$$

Elliptic curves mod p

Fix a prime p

$$\tilde{E} \ : y^2 = x^3 + \tilde{a}x + \tilde{b}$$

$$\sharp E(\mathbb{Z} / p\mathbb{Z})$$

Hasse's estimate :

$$| p + 1 - \sharp E(\mathbb{Z} / p\mathbb{Z}) | \leq 2\sqrt{p}$$

Elliptic curves mod p

Fix a prime p

$$\sharp E(\mathbb{Z}/p\mathbb{Z})$$

$$a_p = p + 1 - \sharp E(\mathbb{Z}/p\mathbb{Z})$$

Elliptic curves mod p

Collect local data

L-Series of E :

$$L(E,s) = \prod_{p \ prime} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \qquad s \in \mathbb{C}$$

Elliptic curves mod p

Collect local data

L-Series of E :

$$L(E,s) = \prod_{p \ prime} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \qquad s \in \mathbb{C}$$

$$|a_p| \leq 2\sqrt{p}$$

converges for $\Re(s) > \frac{3}{2}$

Elliptic curves mod p

L-Series of E :

$$L(E,s) = \prod_{p\ \ prime} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

converges for $\Re(s) > \frac{3}{2}$

Theorem (Wiles): L(E,s) extends to an analytic continuation function on all of $\mathbb{C}$

Elliptic curves mod p

L-Series of E :

$$L(E,s) = \prod_{p \ prime} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

$$L(E,1) = 0 \quad ?$$

Elliptic curves mod p

L-Series of E :

$$L(E,s) = \prod_{p \ \ prime} (1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}})^{-1}$$

$$L(E,1) = 0 \ ?$$

$$ord_{s=1} L(E,s) = k \qquad\qquad L^{(k)}(E,1) \neq 0$$

Example : E2478g3

$$y^2 = x^3 - 887912220251x - 1018364262838266 6$$

$$L(E,1) = 0$$

$$L'(E,1) = 3.63846861928943$$

Example : E2478g3

$$y^2 = x^3 - 88791220251x - 10183642628382666$$

$r = 1$ | $P(31511, 5361297)$

$|E_{tors}| = 2$

$Reg(E(\mathbb{Q}))$ | 9.9051782343077795324557391664963729405700

ω | 0.0524757070525734

$$ord_{s=1} L(E,s) = 1$$

Example : E2478g3

$r = 1$

$$ord_{s=1} L(E,s) = 1$$

Example : E2478g3

$r = 1$

Coincidence ?

$$ord_{s=1} L(E,s) = 1$$

Birch and Swinnerton-Dyer Conjecture 1 :

$$ord_{s=1} L(E, s) = rank(E(\mathbb{Q}))$$

Millennium Prize Problem, Clay Mathematics Institute

Birch and Swinnerton-Dyer Conjecture 1 :

$$ord_{s=1} L(E,s) = rank(E(\mathbb{Q}))$$

Local behavior

Existence of rational points

Birch and Swinnerton-Dyer Conjecture 1 :

$$ord_{s=1} L(E,s) = rank(E(\mathbb{Q}))$$

Local behavior

Existence of rational points

Local global principle for Elliptic Curve ?

Birch and Swinnerton-Dyer Conjecture 1 :

$$ord_{s=1} L(E,s) = rank(E(\mathbb{Q}))$$

Local global principle for Elliptic Curve ?

More complicated

Shafarevich-Tate Group of E denoted $Ш(E)$

Birch and Swinnerton-Dyer Conjecture 1 :

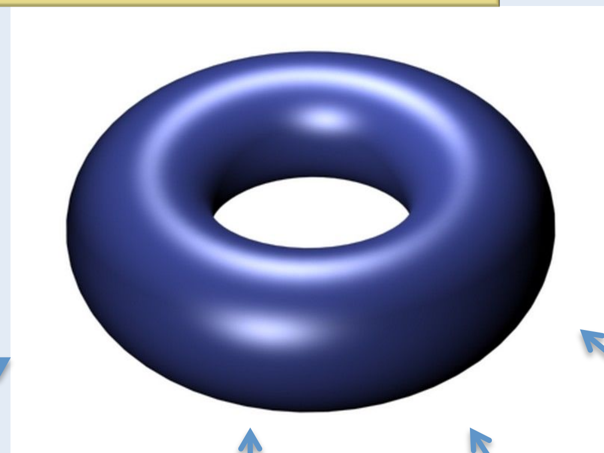
Shafarevich-Tate Group of E denoted $Ш(E)$

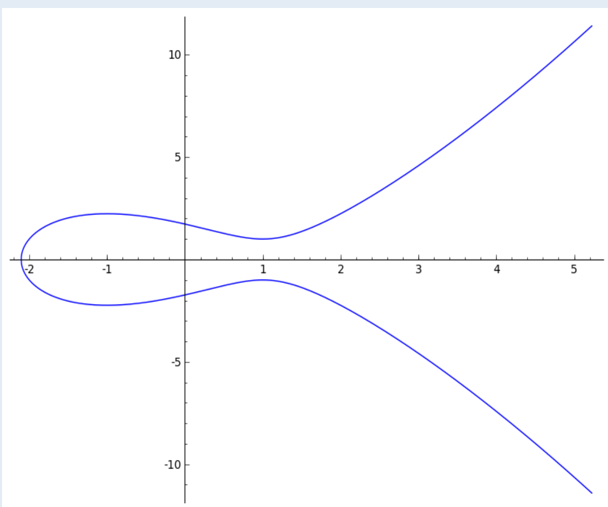
When we have to study a number field or an elliptic curve defined over $\mathbb{Q}$, some groups may appear which make the explicit computations more complicated and which are, in a way, not very "welcome". These groups are the class groups of number fields and the Tate-Shafarevich groups of elliptic curves. A direct study of their general behavior is a very difficult problem. In [3], Cohen and Lenstra explained how to obtain precise conjectures for this pur-

3 Heuristics on Tate-Shafarevich groups of elliptic curves

As with class groups, we are annoyed by Tate-Shafarevich groups of elliptic curves. Thus, we use the analogy described in the first section and sketch the work in [6] which shows how the Cohen-Lenstra philosophy can be adapted to our case. To do this, we take into account the particular structure of Tate-Shafarevich groups, i.e., the structure of groups of type S.

Shafarevich-Tate group $Ш(E)$





C1

C2

C3

No rational points but points everywhere locally

Shafarevich-Tate group $\text{Ш}(E)$

carries information about the failure of the local global principle for E

conjectured to be finite

if so, its order is a square

its order is an invariant for E $|\text{Ш}(E)|$

Example : E2478g3

$$y^2 = x^3 - 88791220251x - 1018364262838266$$

$r = 1$

$|E_{tors}| = 2$

$|Ш(E)| = 4$

$Reg(E(\mathbb{Q}))$ = 9.905178234307779532455739166496372940570

ω = 0.0524757070525734

$ord_{s=1} L(E,s) = 1$

$L'(E,1) = 3.63846861928943$

What if ?

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r \quad \omega \quad Reg(E(\mathbb{Q}))} = ?$$

What if ?

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r \quad \omega \quad Reg(E(\mathbb{Q}))} = ?$$

$$\frac{3.63846861928943 \times (2)^2}{1 \times 0.0524757070525734 \times 9.90517823430777953 2}$$

Birch and Swinnerton-Dyer Conjecture 2 :

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r\quad \omega\quad Reg(E(\mathbb{Q}))} = |Ш(E)|$$

$$\frac{3.63846861928943\times(2)^2}{1\times 0.0524757070525734\times 9.90517823430779532}$$

$$= 4.00000000000000000$$

Birch and Swinnerton-Dyer Conjecture 2 :

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r\quad \omega\quad Reg(E(\mathbb{Q}))\ c_p} = |Ш(E)|$$

$$\frac{3.63846861928943 \times (2^2)}{1 \times 0.0524757070525734 \times 9.9051782343077795 32 \times 7}$$

$$= 4.0000000000000000$$

Birch and Swinnerton-Dyer Conjecture 2 :

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r!\,\omega\, c_p\, Reg(E(\mathbb{Q}))} = |Ш(E)|$$

c_p linked to behavior over $\mathbb{Q}_p$

Birch and Swinnerton-Dyer Conjecture 1 & 2 :

$$ord_{s=1} L(E,s) = rank(E(\mathbb{Q}))$$

$$\frac{L^{(k)}(E,1)\ |E_{tors}|^2}{r!\,\omega\, c_p\, Reg(E(\mathbb{Q}))} = |Ш(E)|$$

Birch and Swinnerton-Dyer Conjecture 1 & 2 :

ON THE CONJECTURES OF BIRCH AND SWINNERTON-DYER

§ 2. <u>The evidence</u>

The numerical evidence for the conjectures is very impressive. Most of it is contained in [4], where Birch and Swinnerton-Dyer discuss the case $K = \mathbf{Q}$, and A an elliptic curve of the form $y^2 = x^3 - Dx$. In that case, as Weil [22] has shown, the function $L^*(s) = L_D^*(s)$ is, essentially, a Hecke L-series associated with the Gaussian field $\mathbf{Q}(i)$ of complex multiplications of A, and Birch and Swinnerton-Dyer were able to find a finite expression for $L_D^*(1)$. This expression is a sum of Δ^2 terms, where Δ is the product of the odd primes dividing D , each term involving a quartic residue symbol and a division value of a Weierstrass $\wp$-function associated with A . Their electronic computer could compute $L_D^*(1)$ for all D's corresponding to a given Δ (D is fourth-power free) in about $\Delta^2/20$ seconds. It computed the quantity

Birch and Swinnerton-Dyer Conjecture 1 & 2 :

Thank you for your attention